

Structural Dynamics

Theory and Computation

Fifth Edition

Updated with SAP2000®

Mario Paz
William Leigh



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CD-ROM



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Fifth Edition

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*I am my beloved's
and my beloved is mine*
The Song of Songs

to

Annis

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PREFACE TO THE FIFTH EDITION

The basic structure of the four previous editions is maintained in this fifth edition, although numerous revisions and additions have been introduced. The three chapters on Earthquake Engineering have been rewritten to present the most recent versions of the Uniform Building Code (UBC-97) and of the new International Building Code (IBC-2000) as in the fourth edition. A new chapter to serve as an introduction for the dynamic analysis of structures using the Finite Element Method has been incorporated in Part III, Structures Modeled as Discrete Multidegree-of-Freedom Systems. The chapter on Random Vibration has been extended to include the response of structures modeled as a multidegree-of-freedom system, subjected to several random forces or to a random motion at the base of the structure. The concept of damping is discussed more thoroughly, including the evaluation of equivalent viscous damping. The constant acceleration method to determine the response of nonlinear dynamic systems is presented in addition to the linear acceleration method presented in past editions. Chapter 5 on Response Spectra now includes the development of seismic response spectra with consideration of local soil conditions at the site of the structure. The secondary effect resulting from the lateral displacements of the building, commonly known as the P- Δ effect, is explicitly considered through the calculation of the geometric stiffness matrix. Finally, a much larger number of solved illustrative examples using the educational computer programs developed by the author or using the professional program SAP2000 have been incorporated in various chapters of the book.

The use of the professional computer program SAP2000 for the analysis and

solution of structural dynamics problems is introduced in this new edition. This program was selected from among the various professional programs available because of its capability in solving complex problems in structures as well as its wide use in professional practice by structural engineers. SAP2000 includes routines for the analysis and design of structures with linear or nonlinear behavior subjected to static or dynamics loads; (material non-linearity or large displacements non-linearities) and may be used most efficiently in the microcomputer. The larger versions of SAP2000 have the capability for the analysis of structures modeled with virtually any large number of nodes. This new fifth edition of the book uses, almost exclusively, the introductory version of SAP2000 which has a capability limited to 25 nodes or 25 elements. A CD-ROM containing the introductory version of SAP2000 as well as the educational set of the program developed by the author is included in this 5th edition of *Structural Dynamics: Theory and Computation*.

The set of educational programs in Structural Dynamics includes programs to determine the response in the time domain or in the frequency domain using the FFT (Fast Fourier Transform) of structures modeled as a single oscillator. Also included is a program to determine the response of an inelastic system with elastoplastic behavior, and another program for the development of seismic response spectral charts. A set of seven computer programs is included for modeling structures as two or three-dimensional frames and trusses. Finally, other programs, incorporating modal superposition or a step-by-step time-history solution, are provided for calculation of the responses to forces or motions exciting the structure. This fifth edition also includes a program to determine the response of single or multiple-degree-of-freedom systems subjected to random excitations. .

The book is organized in six parts. Part I deals with structures modeled as single-degree-of-freedom systems. It introduces basic concepts in Structural Dynamics and presents important methods for the solution of such dynamic systems. Part II introduces important concepts and methodology for multi-degree-of-freedom systems through the use of structures modeled as shear buildings. Part III describes in detail the Matrix Structural Analysis for modeling skeletal type of structures (beams, frames, and trusses) as discrete systems in preparation for dynamic analysis. Part III also includes a chapter to serve as an introduction to the Finite Element Method (F.E.M.) for modeling continuous structures such as plates for dynamic analysis. Part IV presents the mathematical solution for some simple structures, such as beams, modeled as systems with distributed properties, thus having an infinite number of degrees of freedom. Part V on Special Topics presents: an introduction to the magnificent Fourier Method and the use of the Fast Fourier Transform; an extension of the modeling complex structures as one degree-of-freedom systems through the use of Generalized Coordinates and of Rayleigh Method; and methods to evaluate absolute damping in structures from estimated modal damping coefficients. Part VI, which contains one chapter, introduces the reader to the complex but fascinating topic of Random Vibrations for the analysis of single degree of freedom systems, as well as for the analysis of structures modeled as multi-degree of freedom systems. Finally, Part VII presents the important current topic of Earthquake Engineering with applications to earthquake-resistant design of buildings

following the provisions of the Uniform Building Code and of the new International Building Code in use in The United States of America.

The author believes that a combination of knowledge on the subjects of applied mathematics, theory of structures, and computer programs is needed today for successful professional practice of engineering, just as knowledge of a combination of numbers and turns is needed to open a safe. To provide the reader with such a combination of knowledge has been the primary objective of this book. The reader may wish to inform the author on the extent to which this objective has been fulfilled.

Many students, colleagues, and practicing professionals have suggested improvements, identified typographical errors, and recommended additional topics for inclusion. In this new edition all these suggestions were carefully considered and have been included in this fifth edition whenever possible.

During the preparation of this fifth edition, I became indebted to many people to whom I wish to express my appreciation: First of all I am most grateful to many of my students who helped me through their inquisitive discussions in class to improve and clarify my presentation of the various topics in this book. It is now with great pain that I wish to recognize posthumously the preliminary work done by my student Elaine Fonseca, who prepared changes to some drawings from the Fourth Edition. Her tragic death was most unfortunately a great loss of a most promising engineering student. She will be sorely missed by her family, friends, fellow students and this instructor. I wish also to recognize and thank my graduate students, Xiaobing Cui and Zhiyong Zhao, for their diligent collaboration and expert use of scanning equipment to retrieve text and figures from the previous edition of this textbook. I am most grateful to my former colleague; Dr. Michael A. Cassaro, who diligently checked the chapter on the Finite Element Method and to Dr. Julius Wong, of the Department of Mechanical Engineering, whose comments and discussions helped me to refine my exposition. I am also grateful to my friend Dr. Farzad Naeim who has collaborated with me on the chapter Seismic Response Spectra in the "International Handbook of Earthquake Engineering: Codes, Programs and Examples" of which I am the editor. I have incorporated some of the material from the Handbook in updating the chapter on Response Spectra. I also wish to acknowledge Dr. Luis E. Suarez from the University of Puerto Rico in Mayaguez, who provided me with copies of his work in Random Vibration and a copy of his class notes on the Finite Element Method.

I also like to take this opportunity to thank my colleague, Dr Joseph Hagerty for his past help of many years ago, in the 1970s, at the time when I was just playing with the plan of writing a textbook in Structural Dynamics, without my knowledge, he approached a publishing company and initiated a contract in my name for the publication of the first edition of this textbook in 1980.

A special acknowledgement of gratitude is extended to my friend Dr. Assraf Habibullah, president of Computers and Structures Inc., who most kindly authorized me to include in this volume the introductory version of SAP2000. In addition, Dr.

Habibulla provided me with the full version of SAP 2000 so I could solve problems beyond the capability of the introductory version. I am also most grateful to two other computer scientists of that company, Drs. Syed Hasnain and Bob Morris who most patiently tutored me and clarified many of the intricacies in the use of SAP2000. The senior author is certainly very grateful to the co-author, Dr. William Leigh for his contribution in reviewing and editing this volume, especially those sections which used the computer programs. To those people whom I recognized in the prefaces to the previous editions for their help, I again express my wholehearted appreciation.

Finally, I thank my wife, Annis, who most diligently helped me with great proficiency in the final preparation of this new edition to be camera ready for publication. Her dedication to the work as well as her continuous support and encouragement is deeply appreciated. In recognition of her indispensable help, this new edition is duly dedicated to her.

Mario Paz
September 2003

Preface to the First Edition

Natural phenomena and human activities impose forces of time-dependent variability on structures as simple as a concrete beam or a steel pile, or as complex as a multistory building or a nuclear power plant constructed from different materials. Analysis and design of such structures subjected to dynamic loads involve consideration of time-dependent inertial forces. The resistance to displacement exhibited by a structure may include forces which are functions of the displacement and the velocity. As a consequence; the governing equations of motion of the dynamic system are generally nonlinear partial differential equations which are extremely difficult to solve in mathematical terms. Nevertheless, recent developments in the field of structural dynamics enable such analysis and design to be accomplished in a practical and efficient manner. This work is facilitated through the use of simplifying assumptions and mathematical models, and of matrix methods and modern computational techniques.

In the process of teaching courses on the subject of structural dynamics, the author came to the realization that there was a definite need for a text which would be suitable for the advanced undergraduate or the beginning graduate engineering student

being introduced to this subject. The author is familiar with the existence of several excellent texts of an advanced nature but generally these texts are, in his view, beyond the expected comprehension of the student. Consequently, it was his principal aim in writing this book to incorporate modern methods of analysis and techniques adaptable to computer programming in a manner as clear and easy as the subject permits. He felt that computer programs should be included in the book in order to assist the student in the application of modern methods associated with computer usage. In addition, the author hopes that this text will serve the practicing engineer for purposes of self-study and as a reference source.

In writing this text, the author also had in mind the use of the book as a possible source for research topics in structural dynamics for students working toward an advanced degree in engineering who are required to write a thesis. At Speed Scientific School, University of Louisville, most engineering students complete a fifth year of study with a thesis requirement leading to a Master in Engineering degree. The author's experience as a thesis advisor leads him to believe that this book may well serve the students in their search and selection of topics in subjects currently under investigation in structural dynamics.

Should the text fulfill the expectations of the author in some measure, particularly the elucidation of this subject, he will then feel rewarded for his efforts in the preparation and development of the material in this book.

MARIO PAZ

December, 1979

PART I

Structures Modeled as a Single-Degree of Freedom System

1 Undamped Single Degree-of-Freedom System

The analysis and design of structures to resist the effect produced by time dependent forces or motions requires conceptual idealizations and simplifying assumptions through which the physical system is represented by an idealized system known as the analytical or mathematical model. These idealizations or simplifying assumptions may be classified in the following three groups:

1. Material assumptions. These assumptions or simplifications include material properties such as homogeneity or isotropy and material behaviors such as linearity or elasticity.
2. Loading assumptions. Some common loading assumptions are to consider concentrated forces to be applied at a geometric point, to assume forces suddenly applied, or to assume external forces to be constant or periodic.
3. Geometric Assumptions. A general assumption for beams, frames and trusses is to consider these structures to be formed by unidirectional elements. Another common assumption is to assume that some structures such as plates are two-dimensional systems with relatively small thicknesses. Of greater importance is to assume that continuous structures may be analyzed as discrete systems by specifying locations (nodes) and directions for displacements (nodal coordinates) in the structures as described in the following section.

1.1 Degrees of Freedom

In structural dynamics the number of independent coordinates necessary to specify the configuration or position of a system at any time is referred to as the number of degrees of freedom. In general, a continuous structure has an infinite number of degrees of freedom. Nevertheless, the process of idealization or selection of an appropriate mathematical model permits the reduction to a discrete number of degrees of freedom. Figure 1.1 shows some examples of structures that may be represented for dynamic analysis as one-degree-of-freedom-systems, that is, structures modeled as systems with a single displacement coordinate.

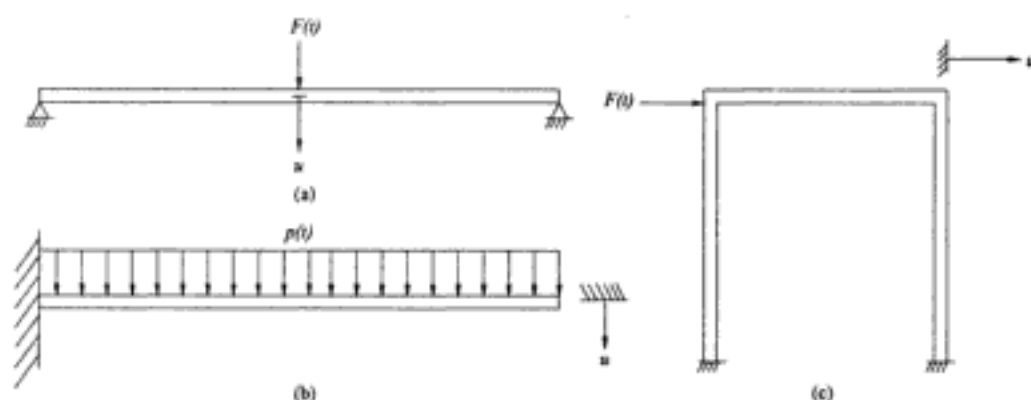


Fig. 1.1 Examples of structures modeled as one-degree-of-freedom systems.

These one-degree-of-freedom systems may be conveniently described by the analytical model shown in Fig. 1.2 which has the following elements:

1. A mass element m representing the mass and inertial characteristic of the structure.
2. A spring element k representing the elastic restoring force and potential energy storage of the structure.
3. A damping element c representing the frictional characteristics and energy dissipation of the structure.
4. An excitation force $F(t)$ representing the external forces acting on the structural system.

The force $F(t)$ is written this way to indicate that it is a function of time. In adopting the analytical model shown in Fig. 1.2, it is assumed that each element in the system represents a single property; that is, the mass m represents only the property of inertia and not elasticity or energy dissipation, whereas the spring k represents exclusively elasticity and not inertia or energy dissipation. Finally, the damper c only dissipates energy. The reader certainly realizes that such “pure” elements do not exist in our physical world and that analytical models are only conceptual idealizations of real structures. As such, analytical models may provide complete and accurate knowledge of the behavior of the model itself, but only limited or approximate information on the behavior of the real physical system. Nevertheless, from a practical point of view, the information acquired from the analysis of the analytical model may very well be

sufficient for an adequate understanding of the dynamic behavior of the physical system, including design and safety requirements.

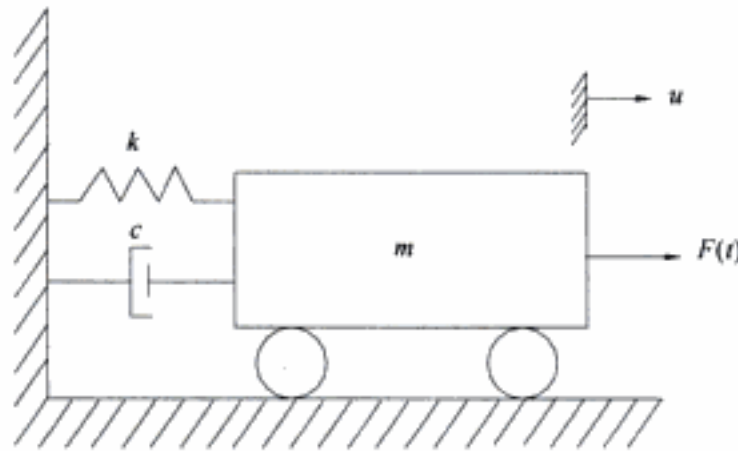


Fig. 1.2 Analytical model for one-degree-of-freedom systems.

1.2 Undamped System

We start the study of structural dynamics with the analysis of a fundamental and simple system, the one-degree-of-freedom system in which we disregard or “neglect” frictional forces or damping. In addition, we consider the system, during its motion or vibration, to be free from external actions or forces. Under these conditions, the system is said to be in free vibration and it is in motion governed only by the influence of the so-called initial conditions, that is, the given displacement and velocity at time $t = 0$ when the study of the system is initiated. This undamped, one-degree-of-freedom system is often referred to as the simple undamped oscillator. It is usually represented as shown in Fig. 1.3(a) or Fig. 1.3(b) or any other similar arrangement. These two figures represent analytical models that are dynamically equivalent. It is only a matter of preference to adopt one or the other. In these models the mass m is restrained by the spring k and is limited to rectilinear motion along one coordinate axis, designated in these figures by the letter u .

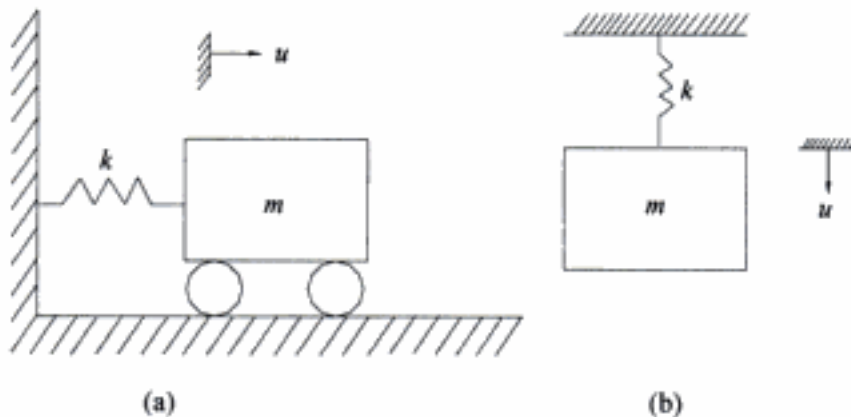


Fig. 1.3 Alternate representations of analytical models for one-degree-of-freedom systems.

The mechanical characteristic of a spring is described by the relationship between the magnitude of the force F_s applied to its free end and the resulting end displacement u as shown graphically in Fig 1.4 for three different springs.

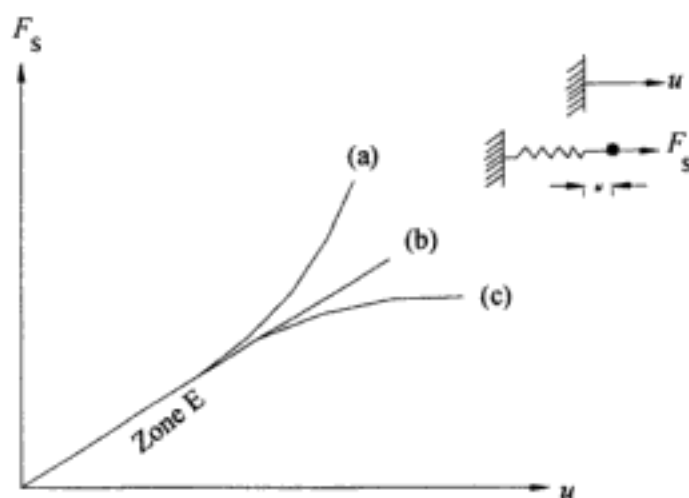


Fig. 1.4 Force-displacement relationship: (a) Hard spring, (b) Linear spring, (c) Soft spring.

The curve labeled (a) in Fig. 1.4 represents the behavior of a hard spring in which the force required to produce a given displacement becomes increasingly greater as the spring is deformed. The second spring (b) is designated a linear spring because the deformation is directly proportional to the force and the graphical representation of its characteristic is a straight line. The constant of proportionality between the force and displacement [slope of the line (b)] of a linear spring is referred to as the stiffness or the spring constant, usually designated by the letter k . Consequently, we may write the relationship between force and displacement for a linear spring as

$$F_s = ku \quad (1.1)$$

A spring with characteristics shown by curve (c) in Fig. 1.4 is known as a soft spring. For such a spring the incremental force required to produce additional deformation decreases as the spring deformation increases. Undoubtedly, the reader is aware from his previous exposure to analytical modeling of physical systems that the linear spring is the simplest type to manage mathematically. It should not come as a surprise to learn that most of the technical literature on structural dynamics deals with models using linear springs. In other words, either because the elastic characteristics of the structural system are, in fact, essentially linear, or simply because of analytical expediency, it is usually assumed that the force-deformation properties of the system are linear. In support of this practice, it should be noted that in many cases the displacements produced in the structure by the action of external forces or disturbances are small in magnitude (Zone E in Fig. 1.4), thus rendering the linear approximation close to the actual structural behavior.

1.3 Springs in parallel or in series

Sometimes it is necessary to determine the equivalent spring constant for a system in which two or more springs are arranged in parallel as shown in Fig. 1.5(a) or in series as in Fig. 1.5(b).

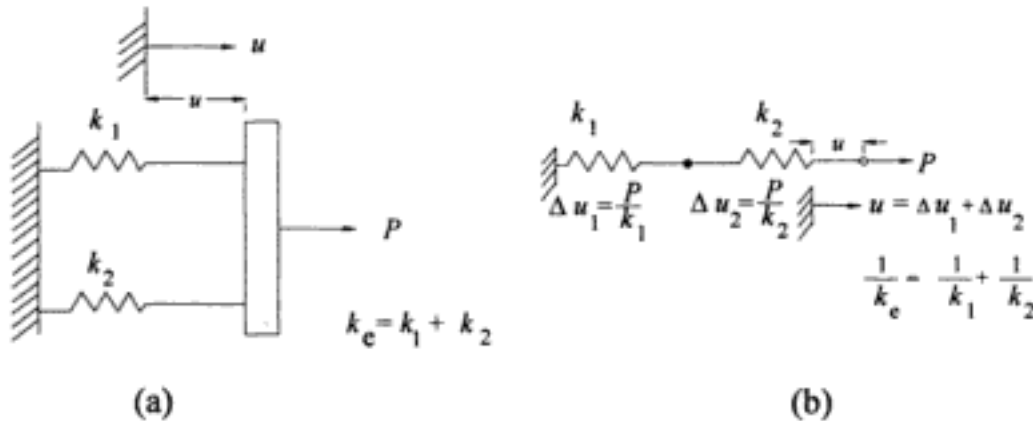


Fig. 1.5 Combination of springs: (a) Springs in parallel (b) Springs in series.

For two springs in parallel the total force required to produce a relative displacement of their ends of one unit is equal to the sum of their spring constants. This total force is by definition the equivalent spring constant k_e and is given by

$$k_e = k_1 + k_2 \quad (1.2)$$

In general for n springs in parallel

$$k_e = \sum_{i=1}^n k_i \quad (1.3)$$

For two springs assembled in series as shown in Fig. 1.5(b), the force P produces the relative displacements in the springs

$$\Delta u_1 = \frac{P}{k_1}$$

and

$$\Delta u_2 = \frac{P}{k_2}$$

Then, the total displacement u of the free end of the spring assembly is equal to $u = \Delta u_1 + \Delta u_2$, or substituting Δu_1 and Δu_2

$$u = \frac{P}{k_1} + \frac{P}{k_2} \quad (1.4)$$

Consequently, the force k_e necessary to produce one unit displacement (equivalent spring constant) is given by

$$k_e = \frac{P}{u}$$

Substituting u from this last relation into eq.(1.4), we may conveniently express the reciprocal value of the equivalent spring constant as

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \quad (1.5)$$

In general for n springs in series the equivalent spring constant may be obtained from

$$\frac{1}{k_e} = \sum_{i=1}^n \frac{1}{k_i} \quad (1.6)$$

1.4 Newton's Law of Motion

We continue with the study of the simple oscillator depicted in Fig. 1.3. The objective is to describe its motion, that is, to predict the displacement or velocity of the mass m at any time t , for a given set of initial conditions at time $t = 0$. The analytical relation between the displacement u , and time t , is given by Newton's Second Law of Motion, which in modern notation may be expressed as

$$F = ma \quad (1.7)$$

Where F is the resultant force acting on a particle of mass m and a its resultant acceleration. The reader should recognize that eq.(1.7) is a vector relation and as such it can be written in equivalent form in terms of its components along the coordinate axes x , y , and z , namely,

$$\sum F_x = ma_x \quad (1.8a)$$

$$\sum F_y = ma_y \quad (1.8b)$$

$$\sum F_z = ma_z \quad (1.8c)$$

The acceleration is defined as the second derivative of the position vector with respect to time; it follows that eqs.(1.8) are indeed differential equations. The reader should also be reminded that these equations as stated by Newton are directly applicable only to bodies idealized as particles, that is, bodies assumed to possess mass but no volume. However, as is proved in elementary mechanics, Newton's Law of Motion is also directly applicable to bodies of finite dimensions undergoing translatory motion.

For plane motion of a rigid body that is symmetric with respect to the reference plane of motion (x - y plane), Newton's Law of Motion yields the following equations:

$$\sum F_x = m(a_G)_x \quad (1.9a)$$

$$\sum F_y = m(a_G)_y \quad (1.9b)$$

$$\sum M_G = I_G \alpha \quad (1.9c)$$

In the above equations $(a_G)_x$ and $(a_G)_y$ are the acceleration components, along the x and y axes, of the center of mass G of the body; α is the angular acceleration; I_G is the mass moment of inertia of the body with respect to an axis through G , the center of mass; and $\sum M_G$ is the sum with respect to an axis through G , perpendicular to the x - y plane of the moments of all the forces acting on the body. Equations (1.9) are certainly also applicable to the motion of a rigid body in pure rotation about a fixed axis, alternatively, for this particular type of plane motion, eq.(1.9c) may be replaced by

$$\sum M_0 = I_0 \alpha \quad (1.9d)$$

in which the mass moment of inertia I_0 and the moment of the forces M_0 are determined with respect to the fixed axis of rotation. The general motion of a rigid body is described by two vector equations, one expressing the relation between the forces and the acceleration of the mass center, and another relating the moments of the forces and the angular motion of the body. This last equation expressed in its scalar components is rather complicated, but seldom needed in structural dynamics.

1.5 Free Body Diagram

At this point, it is advisable to follow a method conducive to an organized and systematic analysis in the solution of dynamics problems. The first and probably the most important practice to follow in any dynamic analysis is to draw a free body diagram of the system, prior to writing a mathematical description of the system.

10 Free Body Diagram

The free body diagram (FBD), as the reader may recall, is a sketch of the body isolated from all other bodies, in which all the forces external to the body are shown. For the case at hand, Fig. 1.6(b) depicts the FBD of the mass m of the oscillator, displaced in the positive direction with reference to coordinate u and acted upon by the spring force $F_s = ku$ (assuming a linear spring). The weight of the body mg and the normal reaction N of the supporting surface are also shown for completeness, though these forces, acting in the vertical direction, do not enter into the equation of motion written for the u direction. The application of Newton's Law of Motion gives

$$-ku = m\ddot{u} \quad (1.10)$$

where the spring force acting in the negative direction has a minus sign, and where the acceleration has been indicated by \ddot{u} . In this notation, double overdots denote the second derivative with respect to time and obviously a single overdot denotes the first derivative with respect to time, that is, the velocity.

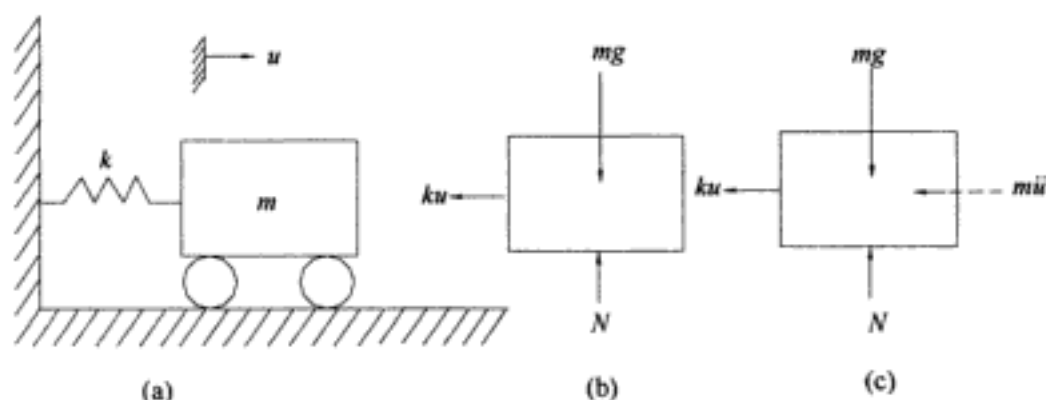


Fig. 1.6 Alternate free body diagrams: (a) Single degree-of-freedom system. (b) Showing only external forces. (c) Showing external and inertial forces.

1.6 D'Alembert's Principle

An alternative approach to obtain eq.(1.10) is to make use of D'Alembert's Principle which states that a system may be set in a state of dynamic equilibrium by adding to the external forces a fictitious force that is commonly known as the inertial force.

Figure 1.6(c) shows the FBD with inclusion of the inertial force $m\ddot{u}$. This force is equal to the mass multiplied by the acceleration, and should always be directed negatively with respect to the corresponding coordinate. The application of D'Alembert's Principle allows us to use equations of equilibrium in obtaining the equation of motion. For example, in Fig. 1.6(c), the summation of forces in the u direction gives directly

$$m\ddot{u} + ku = 0 \quad (1.11)$$

which obviously is equivalent to eq.(1.10).

The use of D'Alembert's Principle in this case appears to be trivial. This will not be the case for a more complex problem, in which the application of D'Alembert's Principle, in conjunction with the Principle of Virtual Work, constitutes a powerful tool of analysis. As will be explained later, the Principle of Virtual Work is directly applicable to any system in equilibrium. It follows then that this principle may also be applied to the solution of dynamic problems, provided that D'Alembert's Principle is used to establish the dynamic equilibrium of the system.

Illustrative Example 1.1

Show that the same differential equation is obtained for a body vibrating along a horizontal axis or for the same body moving vertically, as shown in Figs. 1.7(a) and 1.7(b).

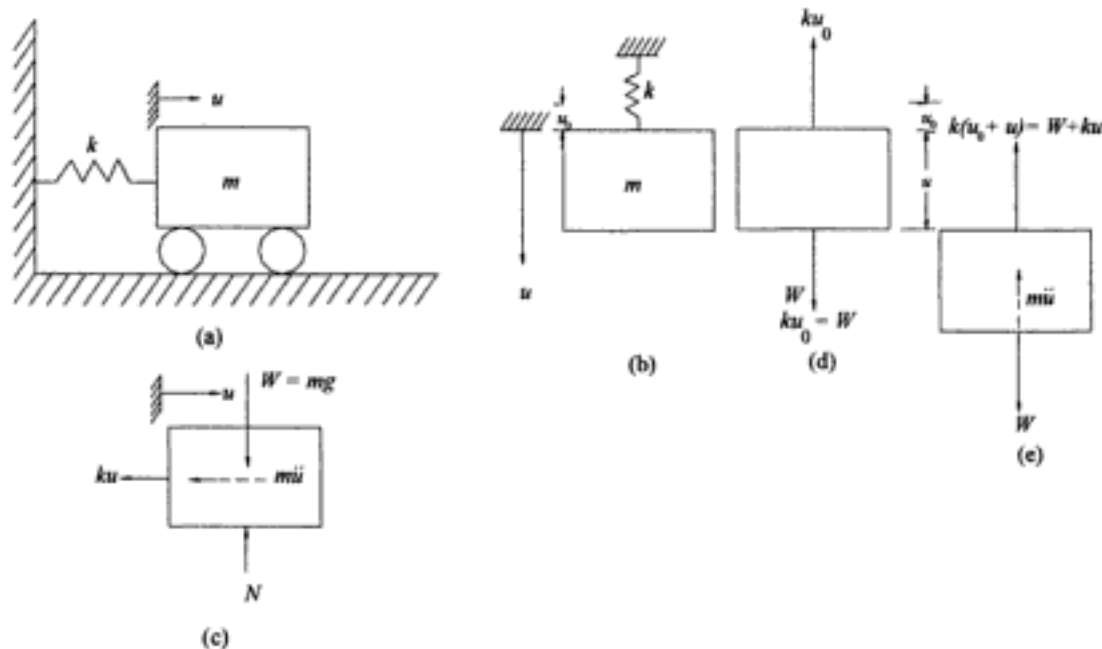


Fig. 1.7 Two representations of the simple oscillator and corresponding free body diagrams.

Solution:

The FBDs for these two representations of the simple oscillator are shown in Figs. 1.7(c) and 1.7(e), in which the inertial forces have been included. Equating to zero in Fig. 1.7(c) the sum of the forces along the direction u , we obtain

$$m\ddot{u} + ku = 0 \quad (a)$$

12 D'Alembert's Principle

When the body in Fig. 1.7(d) is in the static equilibrium position, the spring is stretched u_0 units and exerts a force $ku_0 = W$ upward on the body, where W is the weight of the body. When the body is displaced a distance u downward from this position of equilibrium the magnitude of the spring force is given by $F_s = k(u_0 + u)$ or $F_s = W + ku$ since $ku_0 = W$. Using this result and applying it to the body in Fig. 1.7(e), we obtain from Newton's Second Law of Motion

$$-(W + ku) + W - m\ddot{u} = 0 \quad (b)$$

or

$$m\ddot{u} + ku = 0$$

which is identical to eq. (a).

1.7 Solution of the Differential Equation of Motion

The next step toward our objective is to find the solution of the differential equation (1.11). We should again adopt a systematic approach and proceed first to classify this differential equation. Since the dependent variable u and second derivative \ddot{u} appear in the first degree in eq.(1.11), this equation is classified as linear and of second order. The fact that the coefficients of u and of \ddot{u} (k and m , respectively) are constants and that the second member (right-hand side) of the equation is zero further classifies this equation as homogenous with constant coefficients. We should recall, probably with a certain degree of satisfaction, that a general procedure exists for the solution of linear differential equations (homogenous or non-homogenous) of any order. For this simple, second-order differential equation we may proceed directly by assuming a trial solution given by

$$u = A \cos \omega t \quad (1.12)$$

or by

$$u = B \sin \omega t \quad (1.13)$$

where A and B are constants depending on the initiation of the motion while ω is a quantity denoting a physical characteristic of the system as it will be shown next. The substitution of eq. (1.12) into eq. (1.11) gives

$$(-m\omega^2 + k)A \cos \omega t = 0 \quad (1.14)$$

If this equation is to be satisfied at any time, the factor in parentheses must be equal to zero, or

$$\omega^2 = \frac{k}{m} \quad (1.15)$$

The reader should verify that eq.(1.13) is also a solution of the differential equation (1.11), with ω also satisfying eq.(1.15).

The positive root of eq.(1.15,

$$\omega = \sqrt{\frac{k}{m}} \quad (1.16a)$$

is known as the natural frequency of the system for reasons that will soon be apparent.

The quantity ω in equation (1.16a) may be expressed in terms of the static displacement resulting from the weight $W = mg$ applied to the spring. The substitution into eq.(1.16a) of $m = W / g$ results in

$$\omega = \sqrt{\frac{kg}{W}} \quad (1.16b)$$

Hence

$$\omega = \sqrt{\frac{g}{u_{st}}} \quad (1.16c)$$

where $u_{st} = W/k$ is the static displacement of the spring due to the weight W .

Since either eq.(1.12) or eq.(1.13) is a solution of eq.(1.11), and since this differential equation is linear, the superposition of these two solutions, indicated by eq.(1.17) below, is also a solution. Furthermore, eq.(1.17), having two constants of integration, A and B , is, in fact, the general solution for this linear second-order differential equation.

$$u = A \cos \omega t + B \sin \omega t \quad (1.17)$$

The expression for velocity, \dot{u} , is simply found by differentiating eq.(1.17) with respect to time, that is,

$$\dot{u} = -A\omega \sin \omega t + B\omega \cos \omega t \quad (1.18)$$

Next, we should determine the constants of integration A and B . These constants are determined from known values for the motion of the system which almost invariably are the displacement u_0 and the velocity v_0 at the initiation of the motion, that is, at time $t = 0$. These two conditions are referred to as initial conditions, and

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the problem of solving the differential equation for the initial conditions is called an initial value problem.

After substituting, for $t=0$, $u = u_0$, and $\dot{u} = v_0$ into eqs.(1.17) and (1.18) we find that

$$u_0 = A \quad (1.19a)$$

$$v_0 = B\omega \quad (1.19b)$$

Finally, the substitution of A and B from eqs. (1.19) into eq.(1.17) gives

$$u = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t \quad (1.20)$$

which is the expression of the displacement u of the simple oscillator as a function of the time variable t . Thus, we have accomplished our objective of describing the motion of the simple undamped oscillator modeling structures with a single degree of freedom.

1.8 Frequency and Period

An examination of eq.(1.20) shows that the motion described by this equation is *harmonic* and therefore periodic, that is, it can be expressed by a sine or cosine function of the same frequency ω . The period may easily be found since the functions sine and cosine both have a period of 2π . The period T of the motion is determined from

$$\omega T = 2\pi$$

or

$$T = \frac{2\pi}{\omega} \quad (1.21)$$

The period is usually expressed in seconds per cycle or simply in seconds, with the tacit understanding that it is "per cycle". The reciprocal value of the period is the natural frequency f . From eq.(1.21)

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad (1.22)$$

The natural frequency f is usually expressed in hertz or cycles per second (cps). Because the quantity ω differs from the natural frequency f only by the constant factor 2π , ω also is sometimes referred to as the natural frequency. To distinguish

between these two expressions for natural frequency, ω may be called the circular or angular natural frequency. Most often, the distinction is understood from the context or from the units. The natural frequency f is measured in cps as indicated, while the circular frequency ω should be given in radians per second (rad/sec).

Illustrative Example 1.2

Determine the natural frequency of the beam-spring system shown in Fig. 1.8 consisting of a weight of $W = 50.0$ lb attached to a horizontal cantilever beam through the coil spring k_2 . The cantilever beam has a thickness $h = 1/4$ in, a width $b = 1$ in, modulus of elasticity $E = 30 \times 10^6$ psi, and length $L = 12.5$ in. The coil spring has a stiffness $k_2 = 100$ (lb/in)

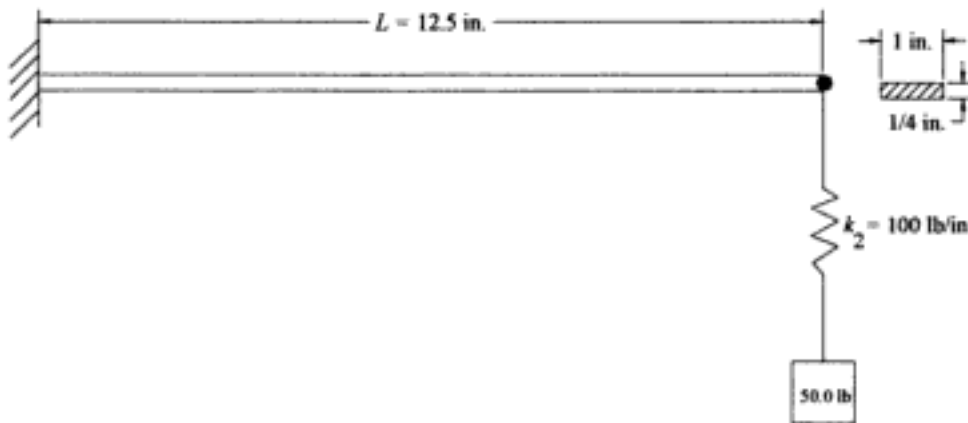


Fig. 1.8 System for Illustrative Example 1.2.

Solution:

The deflection Δ at the free end of a uniform cantilever beam acted upon by a static force P at the free end is given by

$$\Delta = \frac{PL^3}{3EI}$$

The corresponding spring constant k_1 is then

$$k_1 = \frac{P}{\Delta} = \frac{3EI}{L^3}$$

where the cross-section moment of inertia $I = \frac{1}{12}bh^3$ (for a rectangular section).

Now, the cantilever and the coil spring of this system are connected as springs in series. Consequently, the equivalent spring constant as given from eq.(1.5) is

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \quad (1.5) \text{ repeated}$$

Substituting corresponding numerical values, we obtain

$$I = \frac{1}{12} \times 1 \times \left(\frac{1}{4}\right)^3 = \frac{1}{768} \text{ (in)}^4$$

$$k_1 = \frac{3 \times 30 \times 10^6}{(12.5)^3 \times 768} = 60 \text{ lb/in}$$

and

$$\frac{1}{k_e} = \frac{1}{60} + \frac{1}{100}$$

$$k_e = 37.5 \text{ lb/in}$$

The natural frequency for this system is then given by eq.(1.16a) as

$$\omega = \sqrt{k_e / m} \quad (m = W / g \text{ and } g = 386 \text{ in/sec}^2)$$

$$\omega = \sqrt{37.5 \times 386 / 50.0}$$

$$\omega = 17.01 \text{ rad/sec}$$

or using eq.(1.22)

$$f = 2.71 \text{ cps} \quad (\text{Ans.})$$

1.9 Amplitude of Motion

Let us now examine in more detail eq.(1.20), the solution describing the free vibratory motion of the undamped oscillator. A simple trigonometric transformation may show us that we can rewrite this equation in the equivalent forms, namely

$$u = C \sin(\omega t + \alpha) \quad (1.23)$$

or

$$u = C \cos(\omega t - \beta) \quad (1.24)$$

where

$$C = \sqrt{u_0^2 + (v_0 / \omega)^2} \quad (1.25)$$

$$\tan \alpha = \frac{u_0}{v_0/\omega} \quad (1.26)$$

and

$$\tan \beta = \frac{v_0/\omega}{u_0} \quad (1.27)$$

The simplest way to obtain eq.(1.23) or eq.(1.24) is to multiply and divide eq.(1.20) by the factor C defined in eq.(1.25) and to define α (or β) by eq.(1.26) [or eq.(1.27)]. Thus

$$u = C \left(\frac{u_0}{C} \cos \omega t + \frac{v_0/\omega}{C} \sin \omega t \right) \quad (1.28)$$

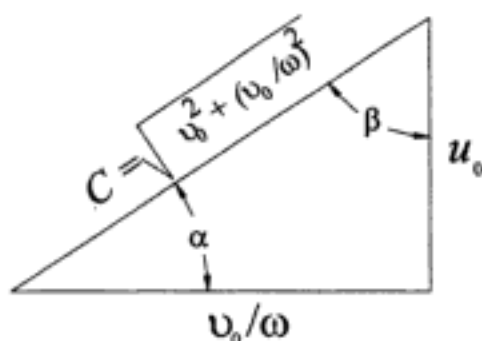


Fig. 1.9 Definition of angle α or angle β .

With the assistance of Fig. 1.9, we recognize that

$$\sin \alpha = \frac{u_0}{C} \quad (1.29)$$

and

$$\cos \alpha = \frac{v_0/\omega}{C} \quad (1.30)$$

The substitution of eqs.(1.29) and (1.30) into eq.(1.28) gives

$$u = C(\sin \alpha \cos \omega t + \cos \alpha \sin \omega t) \quad (1.31)$$

The expression within the parentheses of eq.(1.31) is identical to $\sin(\omega t + \alpha)$, which yields eq.(1.23). Similarly, the reader should verify without difficulty, the form of solution given by eq.(1.24).

The value of C in eq.(1.23) [or eq.(1.24)] is referred to as the amplitude of motion and the angle α (or β) as the phase angle. The solution for the motion of the simple oscillator is shown graphically in Fig. 1.10.

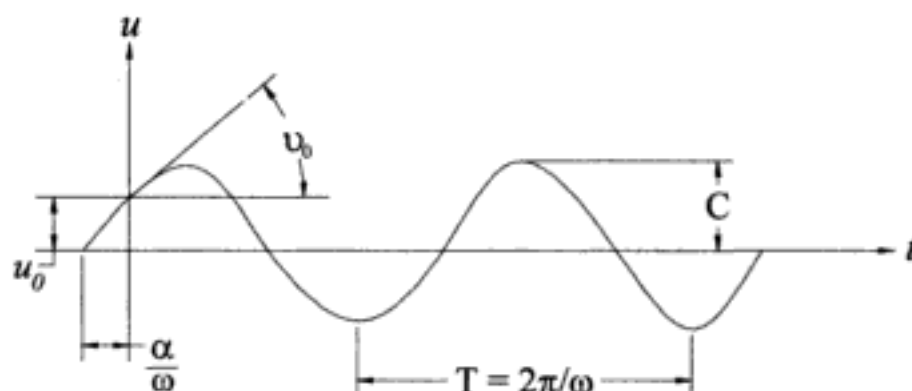


Fig. 1.10 Undamped free-vibration response

Illustrative Example 1.3

Consider the steel frame shown in Fig. 1.11(a) having a rigid horizontal member to which a horizontal dynamic force is applied. As part of the overall structural design it is required to determine the natural frequency of this structure. Two assumptions are made:

1. The masses of the columns are neglected.
2. The horizontal members are sufficiently rigid to prevent rotation at the tops of the columns.

These assumptions are not mandatory for the solution of the problem, but they serve to simplify the analysis. Under these conditions, the frame may be modeled by the spring-mass system shown in Fig. 1.11(b).

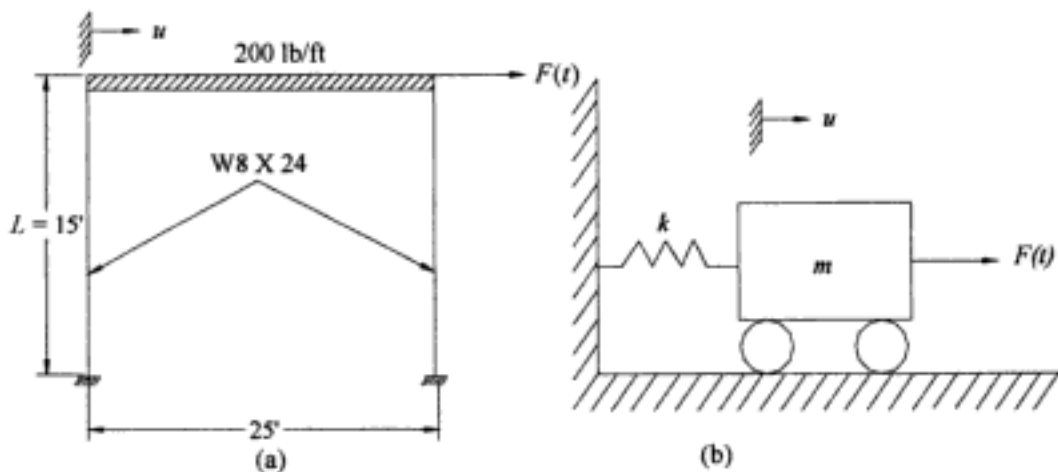


Fig. 1.11 One-degree-of-freedom frame and corresponding analytical model for Illustrative Example 1.3.

Solution:

The parameters of this model may be computed as follows:

$$W = 200 \times 25 = 5000 \text{ lb}$$

$$I = 82.5 \text{ in}^4$$

$$E = 30 \times 10^6 \text{ psi}$$

$$k = \frac{12E(2I)}{L^3} = \frac{12 \times 30 \times 10^6 \times 165}{(15 \times 12)^3}$$

$$k = 10,185 \text{ lb/in} \quad (\text{Ans.})$$

Note: A unit displacement of the top of a fixed column requires a force equal to $12EI/L^3$.

Therefore, the natural frequency from eqs.(1.16b) and (1.22) is

$$f = \frac{1}{2\pi} \sqrt{\frac{kg}{W}} = \frac{1}{2\pi} \sqrt{\frac{10,185 \times 386}{5000}}$$

$$f = 4.46 \text{ cps} \quad (\text{Ans.})$$

Illustrative Example 1.4

The elevated water tower tank with a capacity for 5000 gallons of water shown in Fig. 1.12(a) has a natural period in lateral vibration of 1.0 sec when empty. When the tank is full of water, its period lengthens to 2.2 sec. Determine the lateral stiffness k of the tower and the weight W of the tank. Neglect the mass of the supporting columns (one gallon of water weighs approximately 8.34 lb).

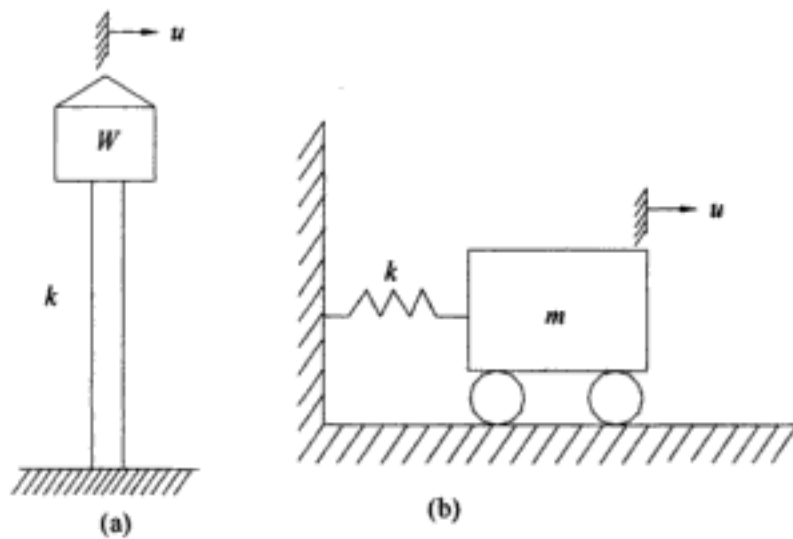


Fig. 1.12 (a) Water tower tank of Illustrative Example 1.4. (b) Analytical model.

Solution:

In its lateral motion, the water tower is modeled by the simple oscillator shown in Fig. 1.12(b) in which k is the lateral stiffness of the tower and m is the vibrating mass of the tank.

- a) Natural frequency ω_E (tank empty):

$$\omega_E = \frac{2\pi}{T_E} = \frac{2\pi}{1.0} = \sqrt{\frac{kg}{W}} \quad (a)$$

- b) Natural frequency ω_F (tank full of water)

Weight of water W_w :

$$W_w = 5000 \times 8.34 = 41,700 \text{ lb}$$

$$\omega_F = \frac{2\pi}{T_F} = \frac{2\pi}{2.2} = \sqrt{\frac{kg}{W + 41,700}} \quad (b)$$

Squaring eqs.(a) and (b) and dividing correspondingly the left and right sides of these equations, results in

$$\frac{(2.2)^2}{(1.0)^2} = \frac{W + 41,700}{W}$$

and solving for W

$$W = 10,860 \text{ lb}$$

(Ans.)

Substituting in eq.(a), $W = 10,860$ lb and $g = 386$ in/sec², yields

$$\frac{2\pi}{1.00} = \sqrt{\frac{k386}{10,860}}$$

and

$$k = 1110 \text{ lb/in} \quad (\text{Ans.})$$

Illustrative Example 1.5

The steel frame shown in Fig. 1.13(a) is fixed at the base and has a rigid top W that weighs 1000 lb. Experimentally, it has been found that its natural period in lateral vibration is equal to 1/10 of a second. It is required to shorten or lengthen its period by 20% by adding weight or strengthening the columns. Determine needed additional weight or additional stiffness (neglect the weight of the columns).

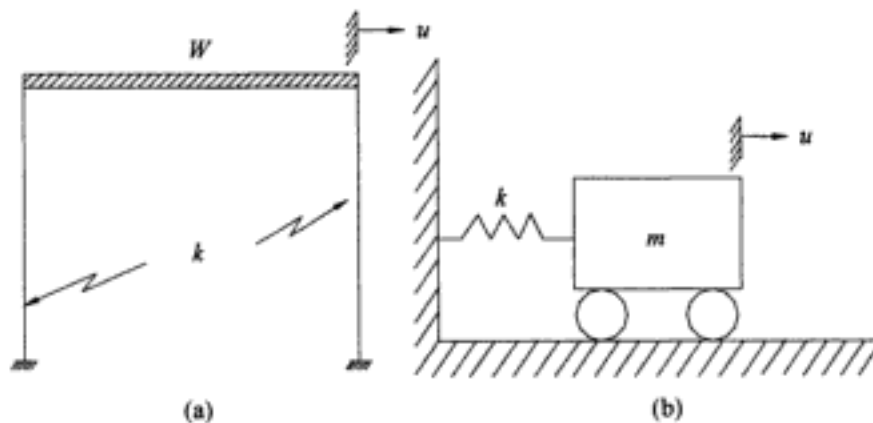


Fig. 1.13 (a) Frame of Illustrative Example 1.5. (b) Analytical model.

Solution:

The frame is modeled by the spring-mass system shown in Fig. 1.13(b). Its stiffness is calculated from

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{kg}{W}}$$

as

$$\frac{2\pi}{0.1} = \sqrt{\frac{kg}{1000}} \quad (g = 386 \text{ in/sec}^2)$$

or

$$k = 10,228 \text{ lb/in}$$

- a) Lengthen the period to $T_L = 1.2 \times 0.10 = 0.12 \text{ sec}$ by adding weight ΔW :

$$\omega = \frac{2\pi}{0.12} = \sqrt{\frac{10,228 \times 386}{1000 + \Delta W}}$$

Solve for ΔW :

$$\Delta W = 440 \text{ lb} \quad (\text{Ans.})$$

- b) Shorten the period to $T_s = 0.8 \times 0.1 = 0.08 \text{ sec}$ by strengthening columns in Δk :

$$\omega = \frac{2\pi}{0.08} = \sqrt{\frac{(10,228 + \Delta K)(386)}{1000}}$$

Solve for Δk :

$$\Delta k = 5753 \text{ lb/in} \quad (\text{Ans.})$$

1.10 Summary

Several basic concepts were introduced in this chapter:

1. The analytical or mathematical model of a structure is an idealized representation for its analysis.
2. The number of degrees of freedom of a structural system is equal to the number of independent coordinates necessary to describe its position.
3. The free body diagram (FBD) for dynamic equilibrium (to allow application of D'Alembert's Principle) is a diagram of the system isolated from all other bodies, showing all the external forces on the system, including the inertial force.
4. The stiffness or spring constant of a linear system is the force necessary to produce a unit displacement.
5. The differential equation of the undamped simple oscillator in free motion is

$$m\ddot{u} + ku = 0$$

and its general solution is

$$u = A \cos \omega t + B \sin \omega t$$

where A and B are constants of integration determined from initial conditions of the displacement u_0 and of the velocity v_0 :

$$A = u_0$$

$$B = v_0 / \omega$$

$$\omega = \sqrt{k/m} \text{ is the natural frequency in rad/sec}$$

$$f = \frac{\omega}{2\pi} \text{ is the natural frequency in cps}$$

$$T = \frac{1}{f} \text{ is the natural period in seconds}$$

6. The equation of motion may be written in the alternate forms:

$$u = C \sin(\omega t + \alpha)$$

or

$$u = C \cos(\omega t - \beta)$$

where

$$C = \sqrt{u_0^2 + (v_0 / \omega)^2}$$

and

$$\tan \alpha = \frac{u_0}{v_0 / \omega}$$

$$\tan \beta = \frac{v_0 / \omega}{u_0}$$

1.11 Problems

Problem 1.1

Determine the natural period for the system in Fig. P1.1. Assume that the beam and springs supporting the weight W are massless.

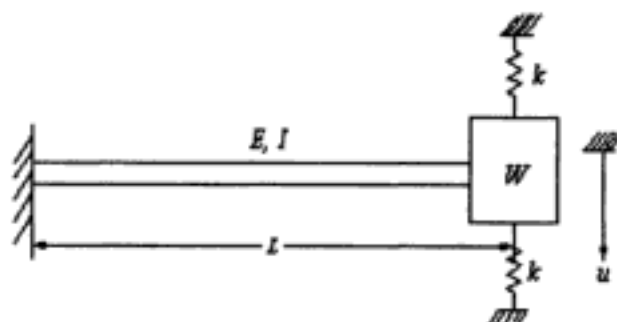


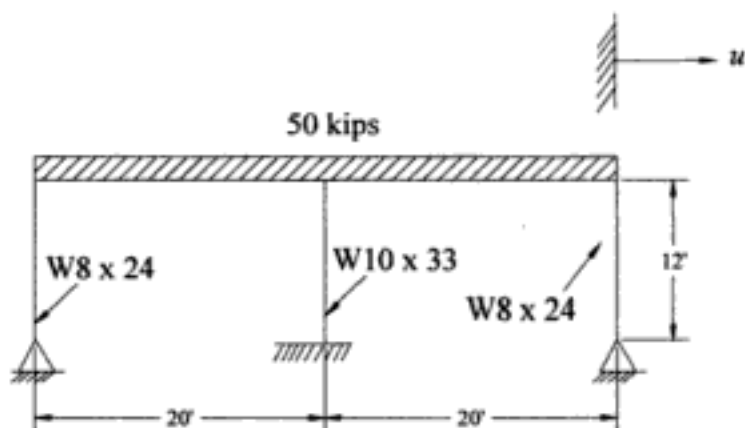
Fig. P1.1

Problem 1.2

The following numerical values are given in Problem 1.1: $L = 100$ in, $EI = 10^8$ (lb.in²), $W = 3000$ lb, $k = 2000$ lb/in. If the weight W has an initial displacement of $u_0 = 1.0$ in and an initial velocity of $v_0 = 20$ in/sec, determine the displacement and the velocity 1 sec later.

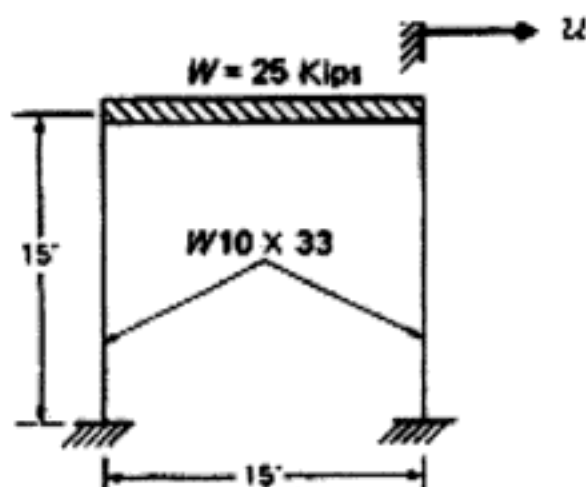
Problem 1.3

Determine the natural frequency for horizontal motion of the steel frame in Fig. P1.3. Assume the horizontal girder to be infinitely rigid and neglect the mass of the columns.

**Fig. P1.3****Problem 1.4**

Calculate the natural frequency in the horizontal mode of the steel frame in Fig. P1.4 for the following cases:

- The horizontal member is assumed to be rigid.
- The horizontal member is flexible and made of steel sections-- W 8 x 24.

**Fig. P1.4**

Problem 1.5

Determine the natural frequency of the fixed beam in Fig. P1.5 carrying a concentrated weight W at its center. Neglect the mass of the beam.

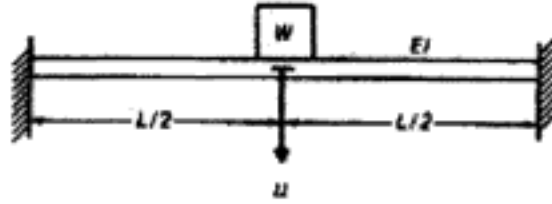


Fig. P1.5

Problem 1.6

The numerical values for Problem 1.5 are given as: $L = 120$ in, $EI = 10^9$ (lb.in²), $W = 5000$ lb, with initial conditions $u_0 = 0.5$ in and $v_0 = 15$ in/sec. Determine the displacement, velocity, and acceleration of W at $t = 2$ sec later.

Problem 1.7

Consider the simple pendulum of weight W illustrated in Fig. P1.7. If the cord length is L , determine the motion of the pendulum. The initial angular displacement and initial angular velocity are θ_0 and $\dot{\theta}_0$, respectively. (Assume the angle θ is small)

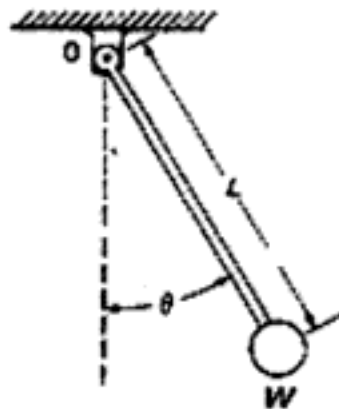


Fig. P1.7

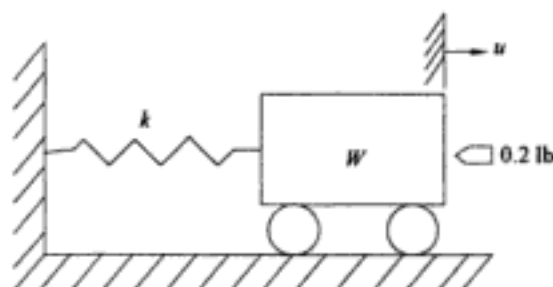
Note: A simple pendulum is a particle of concentrated weight that oscillates in a vertical arc and is supported by a weightless cord. The only forces acting are those of gravity and the cord tension (i.e., frictional resistance is neglected)

Problem 1.8

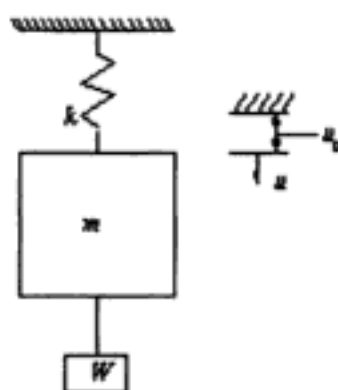
A diver standing at the end of a diving board that cantilevers 2 ft oscillates at a frequency 2 cps. Determine the flexural rigidity EI of the diving board. The weight of the diver is 180 lb. (Neglect the mass of the diving board).

Problem 1.9

A bullet weighing 0.2 lb is fired at a speed of 100 ft/sec into a wooden block weighing $W = 50$ lb and supported by a spring of stiffness 300 lb/in (Fig. P1.9). Determine the displacement $u(t)$ and velocity $v(t)$ of the block after t sec.

**Fig. P1.9****Problem 1.10**

An elevator weighing 500 lb is suspended from a spring having a stiffness of $k = 600$ lb/in. A weight of 300 lb is suspended through a cable to the elevator as shown schematically in Fig. P1.10. Determine the equation of motion of the elevator if the cable of the suspended weight suddenly breaks.

**Fig. P1.10****Problem 1.11**

Write the differential equation of motion for the inverted pendulum shown in Fig. P1.11 and determine its natural frequency. Assume small oscillations, and neglect the mass of the rod.

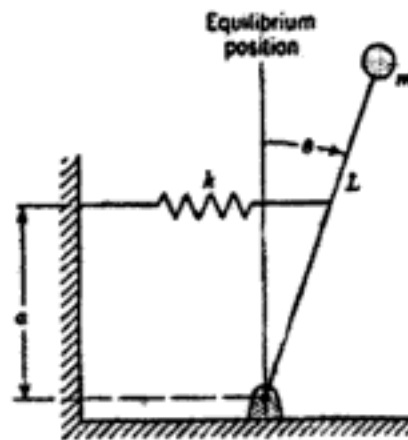


Fig. P1.11

Problem 1.12

Show that the natural frequency for the system of Problem 1.11 may be expressed as

$$f = f_0 \sqrt{1 - \frac{W}{W_{cr}}}$$

where $W = mg$, W_{cr} is the critical buckling weight, and f_0 is the natural frequency neglecting the effect of gravity.

Problem 1.13

A vertical pole of length L and flexural rigidity EI carries a mass m at its top, as shown in Fig. P1.13. Neglecting the weight of the pole, derive the differential equation for small horizontal vibrations of the mass, and find the natural frequency. Assume that the effect of gravity is small and neglect nonlinear effects.



Fig. P1.13

Problem 1.14

Show that the natural frequency for the system in Problem 1.13 may be expressed as:

$$f = f_0 \sqrt{1 - \frac{W}{W_{cr}}}$$

where f_0 is the natural frequency calculated neglecting the effect of gravity and W_{cr} is the critical buckling weight.

Problem 1.15

Determine an expression for the natural frequency of the weight W in each of the cases shown in Fig. P1.15. The beams are uniform of cross-sectional moment of inertia I and modulus of elasticity E . Neglect the mass of the beams.

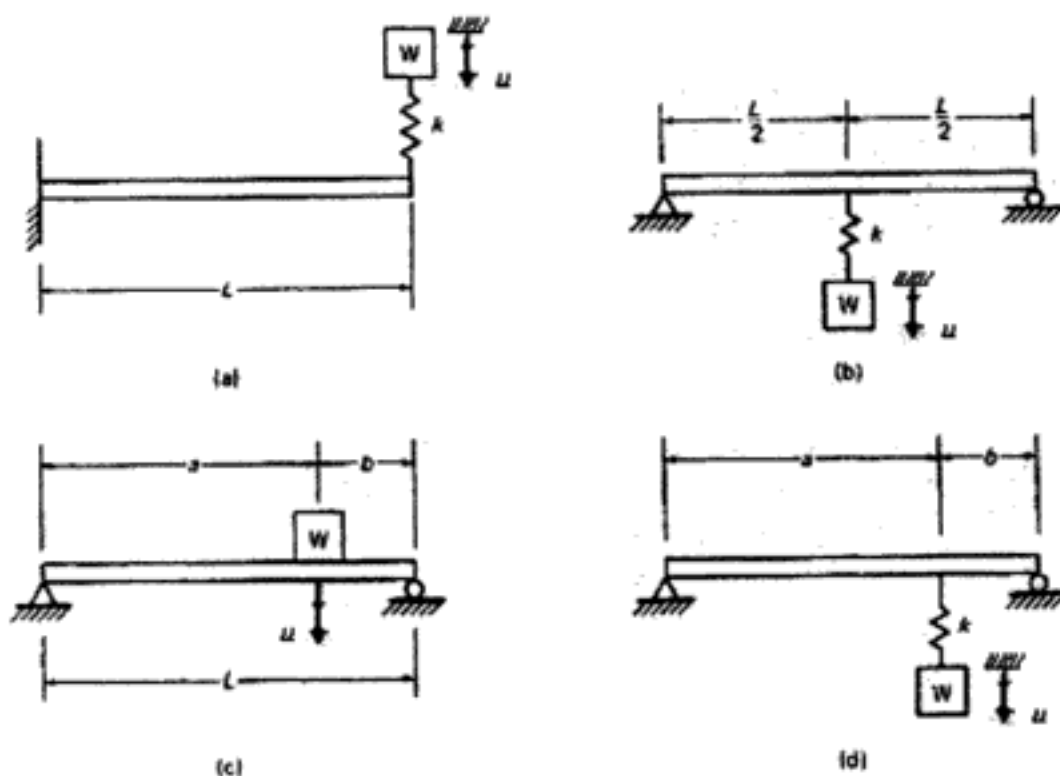


Fig. P1.15

Problem 1.16

A system is modeled by two freely vibrating masses m_1 and m_2 interconnected by a spring having a constant k as shown in Fig. P1.6. Determine for this system the differential equation of motion for the relative displacement $u_r = u_2 - u_1$ between the two masses. Also determine the corresponding natural frequency of the system.

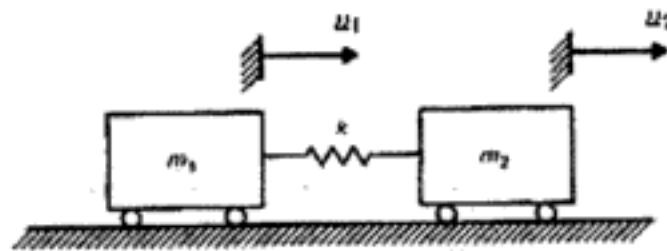


Fig. P1.16

Problem 1.17

Calculate the natural frequency for the vibration of the mass m shown in Fig. P1.17. Member AE is rigid with a hinge at C and a supporting spring of stiffness k at D . (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

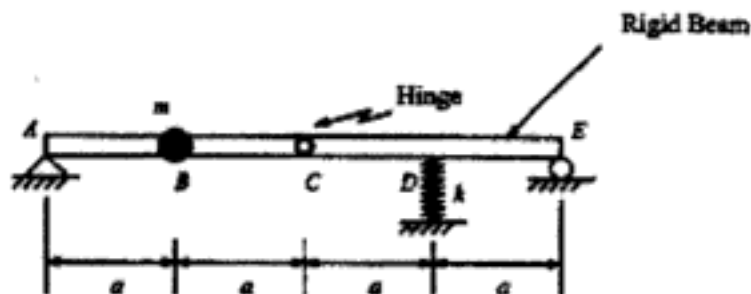


Fig. P1.17

Problem 1.18

Determine the natural frequency of vibration in the vertical direction for the rigid foundation (Fig. P1.18) transmitting a uniformly distributed pressure on the soil having a resultant force $Q = 2,000$ kN. The area of the foot of the foundation is $A = 10$ m². The coefficient of elastic compression of the soil is $k = 25,000$ kN/m³. (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

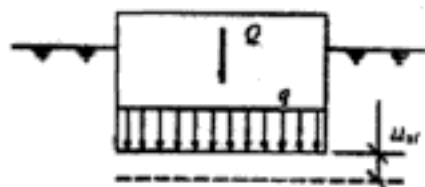


Fig. P1.18

Problem 1.19

Calculate the natural frequency of free vibration of the chimney on elastic foundation (Fig. P1.19), permitting the rotation of the structure as a rigid body about

the horizontal axis x - x . The total weight of the structure is W with its center of gravity at a height h from the base of the foundation. The mass moment of inertia of the structure about the axis x - x is I and the rotational stiffness of the soil is k (resisting moment of the soil per unit rotation). (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

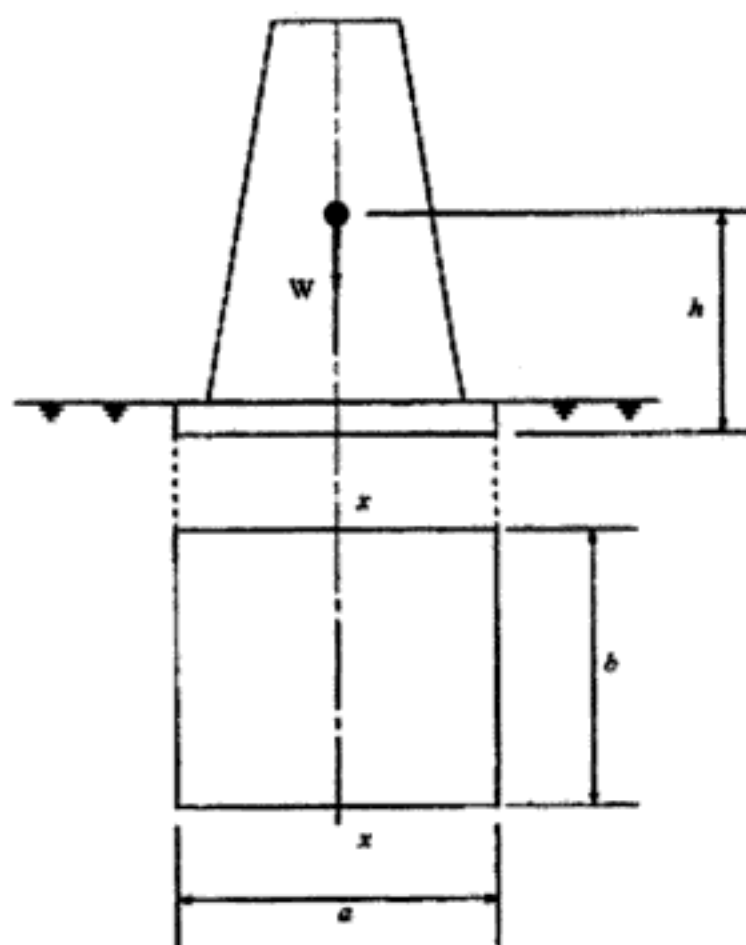


Fig. P1.19

2 Damped Single Degree-of-Freedom System

We have seen in the preceding chapter that the simple oscillator under idealized conditions of no damping, once excited, will oscillate indefinitely with a constant amplitude at its natural frequency. However, experience shows that it is not possible to have a device that vibrates under these ideal conditions. Forces designated as frictional or damping forces are always present in any physical system undergoing motion. These forces dissipate energy; more precisely, the unavoidable presence of these frictional forces constitute a mechanism through which the mechanical energy of the system, kinetic or potential energy, is transformed to other forms of energy such as heat. The mechanism of this energy transformation or dissipation is quite complex and is not completely understood at this time. In order to account for these dissipative forces in the analysis of dynamic systems, it is necessary to make some assumptions about these forces, on the basis of experience.

2.1 Viscous Damping

In considering damping forces in the dynamic analysis of structures, it is usually assumed that these forces are proportional to the magnitude of the velocity, and opposite to the direction of motion. This type of damping is known as viscous damping; it is the type of damping force that could be developed in a body restrained in its motion by a surrounding viscous fluid.

There are situations in which the assumption of viscous damping is realistic and in which the dissipative mechanism is approximately viscous. Nevertheless, the assumption of viscous damping is often made regardless of the actual dissipative characteristics of the system. The primary reason for such wide use of this assumed type of damping is that it leads to a relatively simple mathematical analysis.

2.2 Equation of Motion

Let us assume that we have modeled a structural system as a simple oscillator with viscous damping, as shown in Fig. 2.1(a). In this figure m and k are, respectively, the mass and spring constant of the oscillator and c is the viscous damping coefficient.

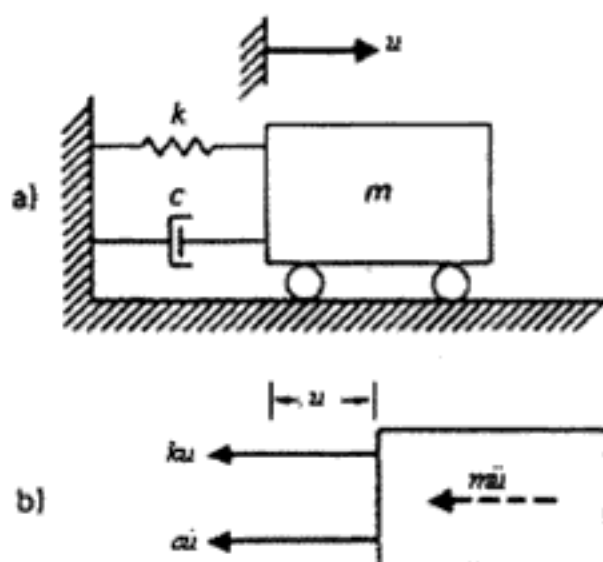


Fig. 2.1 (a) Viscous damped oscillator. (b) Free body diagram.

We proceed, as in the case of the undamped oscillator, to draw the free body diagram (FBD) and apply Newton's Law to obtain the differential equation of motion. Figure 2.1(b) shows the FBD of the damped oscillator in which the inertial force $m\ddot{u}$ is also shown, so that we can use D'Alembert's Principle. The summation of forces in the u direction gives the differential equation of motion

$$m\ddot{u} + c\dot{u} + ku = 0 \quad (2.1)$$

The reader may verify that a trial solution $u = A \sin \omega t$ or $u = B \cos \omega t$ will not satisfy eq.(2.1). However, the exponential function $u = Ce^{pt}$ does satisfy this equation. Substitution of this function into eq.(2.1) results in the equation

$$mCp^2 e^{pt} + cCpe^{pt} + kCe^{pt} = 0$$

which, after cancellation of the common factors, reduces to an equation called the characteristic equation for the system, namely

$$mp^2 + cp + k = 0 \quad (2.2)$$

The roots of this quadratic equation are

$$\begin{matrix} p_1 \\ p_2 \end{matrix} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad (2.3)$$

thus the general solution of eq.(2.1) is given by the superposition of the two possible solutions, namely

$$u(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} \quad (2.4)$$

where C_1 and C_2 are constant of integration to be determined from the initial conditions.

The final form of eq.(2.4) depends on the sign of the expression under the radical in eq.(2.3). Three distinct cases may occur; the quantity under the radical may either be zero, positive or negative. The limiting case in which the quantity under the radical is zero is treated first. The damping present in this case is called critical damping.

2.3 Critically Damped System

For a system oscillating with critical damping ($c = c_{cr}$), the expression under the radical in eq.(2.3) is equal to zero, that is

$$\left(\frac{c_{cr}}{2m}\right)^2 - \frac{k}{m} = 0 \quad (2.5)$$

or

$$c_{cr} = 2\sqrt{km} \quad (2.6)$$

where c_{cr} designates the critical damping value. Since the natural frequency of the undamped system is given by $\omega = \sqrt{\frac{k}{m}}$, the critical damping coefficient given by eq. (2.6) may also be expressed in alternative expressions as

$$c_{cr} = 2m\omega \quad \text{or} \quad c_{cr} = \frac{2k}{\omega} \quad (2.7)$$

In a critically damped system the roots of the characteristic equation are equal, and from eq.(2.3), they are

$$p_1 = p_2 = -\frac{c_{cr}}{2m} \quad (2.8)$$

34 Critically Damped System

Since the two roots are equal, the general solution given by eq.(2.4) would provide only one independent constant of integration, hence, one independent solution, namely

$$u_1(t) = C_1 e^{-(c_{cr}/2m)t} \quad (2.9)$$

Another independent solution may be found by using the function

$$u_2(t) = C_2 t u_1(t) = C_2 t e^{-(c_{cr}/2m)t} \quad (2.10)$$

$u_2(t)$, as the reader may verify, also satisfies the differential equation (2.1). The general solution for a critically damped system is then given by the superposition of these two solutions,

$$u(t) = (C_1 + C_2 t) e^{-(c_{cr}/2m)t} \quad (2.11)$$

2.4 Overdamped System

In an overdamped system, the damping coefficient is greater than the value for critical damping, namely

$$c > c_{cr} \quad (2.12)$$

Therefore, the expression under the radical of eq.(2.3) is positive; thus the two roots of the characteristic equation are real and distinct, and consequently the solution is given directly by eq.(2.4). It should be noted that for the overdamped or the critically damped system, the resulting motion is not oscillatory; the magnitude of the oscillations decays exponentially with time to zero. Figure 2.2 depicts graphically the response for the simple oscillator with critical damping. The response of the overdamped system is similar to the motion of the critically damped system of Fig. 2.2, but in the return toward the neutral position requires more time as the damping is increased.

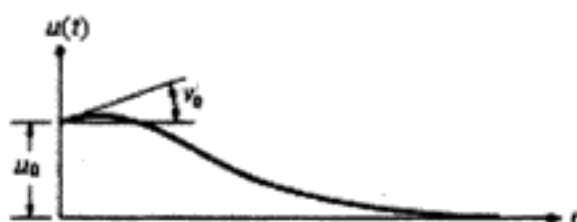


Fig. 2.2 Free-vibration response with critical damping

2.5 Underdamped System

When the value of the damping coefficient is less than the critical value ($c < c_{cr}$), which occurs when the expression under the radical is negative, the roots of the characteristic eq.(2.3) are complex conjugates, so that

$$\begin{aligned} P_1 &= -\frac{c}{2m} \pm i \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \\ P_2 & \end{aligned} \quad (2.13)$$

where $i = \sqrt{-1}$ is the imaginary unit.

For this case, it is convenient to make use of Euler's equations which relate exponential and trigonometric functions, namely,

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned} \quad (2.14)$$

The substitution of the roots p_1 and p_2 from eq.(2.13) into eq.(2.4) together with the use of eq.(2.14) gives the following convenient form for the general solution of the underdamped system:

$$u(t) = e^{-(c/2m)t} (A \cos \omega_D t + B \sin \omega_D t) \quad (2.15)$$

where A and B are redefined constants of integration and ω_D , the damping frequency of the system, is given by

$$\omega_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \quad (2.16)$$

or

$$\omega_D = \omega \sqrt{1 - \xi^2} \quad (2.17)$$

This last result is obtained after substituting, in eq.(2.16), the expression for the undamped natural frequency

$$\omega = \sqrt{\frac{k}{m}} \quad (2.18)$$

and defining the damping ratio of the system as

$$\xi = \frac{c}{c_{cr}} \quad (2.19)$$

where the critical damping coefficient c_{cr} is given by eq.(2.6).

Finally, when the initial conditions of displacement and velocity, u_0 and v_0 , are introduced, the constants of integration can be evaluated and substituted into eq.(2.15), giving

$$u(t) = e^{-\xi\omega t} \left(u_0 \cos \omega_D t + \frac{v_0 + u_0 \xi \omega}{\omega_D} \sin \omega_D t \right) \quad (2.20)$$

Alternatively, this expression can be written as

$$u(t) = C e^{-\xi\omega t} \cos(\omega_D t - \alpha) \quad (2.21)$$

where

$$C = \sqrt{u_0^2 + \frac{(v_0 + u_0 \xi \omega)^2}{\omega_D^2}} \quad (2.22)$$

and

$$\tan \alpha = \frac{(v_0 + u_0 \xi \omega)}{\omega_D u_0} \quad (2.23)$$

A graphical record of the response of an underdamped system with initial displacement u_0 but starting with zero velocity ($v_0 = 0$) is shown in Fig. 2.3). It may be seen in this figure that the motion is oscillatory, but not periodic. The amplitude of vibration is not constant during the motion but decreases for successive cycles; nevertheless, the oscillations occur at equal intervals of time. This time interval is designated as the damped period of vibration and is given from eq.(2.17) by

$$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega \sqrt{1 - \xi^2}} \quad (2.24)$$

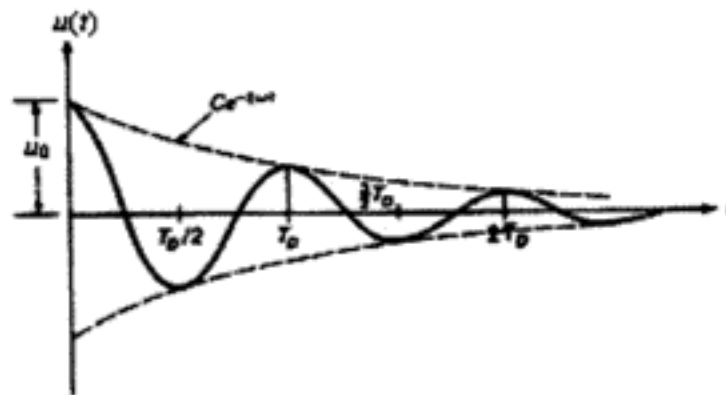


Fig. 2.3 Free-vibration response for undamped system.

The value of the damping coefficient for real structures is much less than the critical damping coefficient and usually ranges between 2 and 10% of the critical damping value. Substituting for the extreme value $\xi = 0.10$ into eq.(2.17),

$$\omega_D = 0.995\omega \quad (2.25)$$

It can be seen that the frequency of vibration for a system with as much as a 10% damping ratio is essentially equal to the undamped natural frequency. Thus in practice, the natural frequency for a damped system may be taken to be equal to the undamped natural frequency.

2.6 Logarithmic Decrement

A practical method for determining experimentally the damping coefficient of a system is to initiate free vibration, obtain a record of the oscillatory motion, such as the one shown in Fig. 2.4, and measure the rate of decay of the amplitude of motion. The decay may be conveniently expressed by the logarithmic decrement δ which is defined as the natural logarithm of the ratio of any two successive peak amplitudes, u_1 and u_2 , in the free vibration, that is

$$\delta = \ln \frac{u_1}{u_2} \quad (2.26)$$

The evaluation of damping from the logarithmic decrement follows. Consider the damped vibration motion represented graphically in Fig. 2.4 and given analytically by eq.(2.21) as

$$u(t) = Ce^{-\xi\omega t} \cos(\omega_D t - \alpha) \quad (2.21) \text{ repeated}$$

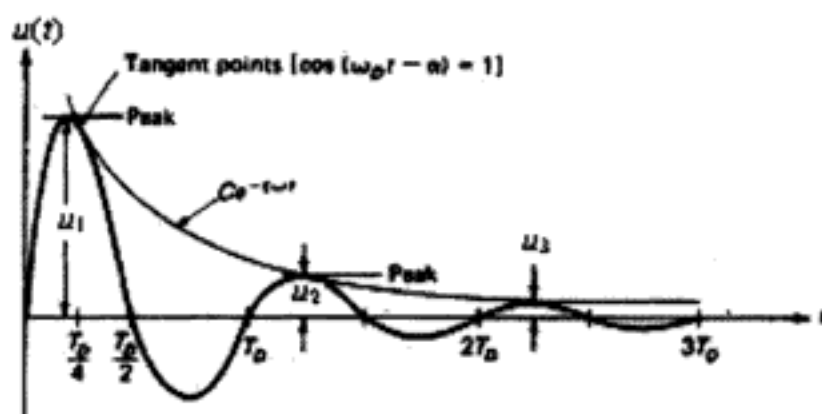


Fig. 2.4 Curve showing peak displacements and displacements at the points of tangency.

We note from this equation that when the cosine factor is unity, the displacement is on points of the exponential curve $u(t) = Ce^{-\xi\omega t}$ as shown in Fig. 2.4. However, these points are near but not equal to the positions of maximum displacement. The points on the exponential curve appear slightly to the right of the points of peak or maximum amplitude. For most practical problems, the discrepancy is negligible and the displacement curve at points where the cosine is equal to one may be assumed to coincide at the peak amplitude with the curve $u(t) = Ce^{-\xi\omega t}$, so that we may write for two consecutive peaks u_1 at time t_1 and u_2 at time T_D seconds later.

$$u_1 = Ce^{-\xi\omega t_1}$$

and

$$u_2 = Ce^{-\xi\omega(t_1 + T_D)}$$

Dividing these two peak amplitudes and taking the natural logarithm, we obtain

$$\delta = \ln \frac{u_1}{u_2} = \xi\omega T_D \quad (2.27)$$

or by substituting T_D , the damped period, from eq.(2.24),

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \quad (2.28)$$

As we can see, the damping ratio ξ can be calculated from eq.(2.28) after determining experimentally the amplitudes of two successive peaks of the system in free vibration. For small values of the damping ratio, eq.(2.28) can be approximated by

$$\delta = 2\pi\xi \quad (2.29)$$

Alternatively, the logarithmic decrement may be calculated as the ratio of two consecutive peak accelerations, which are easier to measure experimentally than displacements. In this case, taking the first and the second derivatives in eq.(2.21), we obtain

$$\begin{aligned}\dot{u}(t) &= C e^{-\xi \omega t} [-\xi \omega \cos(\omega_D t - \alpha) - \omega_D \sin(\omega_D t - \alpha)] \\ \ddot{u}(t) &= C e^{-\xi \omega t} \{ [-\xi \omega \cos(\omega_D t - \alpha) - \omega_D \sin(\omega_D t - \alpha)](-\xi \omega) \\ &\quad + [\xi \omega \omega_D \sin(\omega_D t - \alpha) - \omega_D^2 \cos(\omega_D t - \alpha)] \}\end{aligned}$$

At a time t_1 , when $\cos(\omega_D t_1 - \alpha) = 1$ and $\sin(\omega_D t_1 - \alpha) = 0$

$$\ddot{u}_1 = C e^{-\xi \omega t_1} \{ \xi^2 \omega^2 - \omega_D^2 \}$$

and at time $t_2 = t_1 + T_D$, corresponding to a period later, when again the cosine function is equal to one and the sine function is equal to zero,

$$\ddot{u}_2 = C e^{-\xi \omega (t_1 + T_D)} \{ \xi^2 \omega^2 - \omega_D^2 \}$$

The ratio of the acceleration at times t_1 and t_2 is then

$$\frac{\ddot{u}_1}{\ddot{u}_2} = e^{\xi \omega T_D}$$

and taking natural logarithmic results in the logarithmic decrement in terms of the accelerations as

$$\delta = \ln \frac{\ddot{u}_1}{\ddot{u}_2} = \xi \omega T_D$$

which is identical to the expression for the logarithmic decrement given by eq.(2.27) in terms of displacement.

Illustrative Example 2.1

A vibrating system consisting of a weight of $W = 10$ lb and a spring with stiffness $k = 20$ lb/in is viscously damped so that the ratio of two consecutive amplitudes is 1.00 to 0.85. Determine:

1. The natural frequency of the undamped system.
2. The logarithmic decrement.
3. The damping ratio.
4. The damping coefficient.
5. The damped natural frequency.

Solution:

1. The undamped natural frequency of the system in radians per second is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{(20 \text{ lb/in} \times \frac{386 \text{ in/sec}^2}{10 \text{ lb}})} = 27.78 \text{ rad/sec}$$

or in cycles per second

$$f = \frac{\omega}{2\pi} = 4.42 \text{ cps}$$

2. The logarithmic decrement is given by eq.(2.26) as

$$\delta = \ln \frac{u_1}{u_2} = \ln \frac{1.00}{0.85} = 0.163$$

3. The damping ratio from eq.(2.29) is approximately equal to

$$\xi = \frac{\delta}{2\pi} = \frac{0.163}{2\pi} = 0.026$$

4. The damping coefficient is obtained from eqs.(2.6) and (2.19) as

$$c = \xi c_{cr} = 2 \times 0.026 \sqrt{(10 \times 20) / 386} = 0.037 \frac{\text{lb} \cdot \text{sec}}{\text{in}}$$

5. The natural frequency of the damped system is given by eq.(2.17), so that

$$\begin{aligned} \omega_D &= \omega \sqrt{1 - \xi^2} \\ \omega_D &= 27.78 \sqrt{1 - (0.026)^2} = 27.77 \text{ rad/sec} \end{aligned}$$

Illustrative Example 2.2

A platform of weight $W = 4000 \text{ lb}$ is being supported by four equal columns that are clamped to the foundation as well as to the platform. Experimentally it has been determined that a static force of $F = 1000 \text{ lb}$ applied horizontally to the platform produces a displacement of $\Delta = 0.10 \text{ in}$. It is estimated that damping in the structures is of the order of 5% of the critical damping. Determine for this structure the following:

1. Undamped natural frequency.
2. Absolute damping coefficient
3. Logarithmic decrement.

4. The number of cycles and the time required for the amplitude of motion to be reduced from an initial value of 0.1 to 0.01 in.

Solution:

1. The stiffness coefficient (force per unit displacement) is computed as

$$k = \frac{F}{\Delta} = \frac{1000}{0.1} = 10,000 \text{ lb/in}$$

and the undamped natural frequency

$$\omega = \sqrt{\frac{kg}{W}} = \sqrt{\frac{10,000 \times 386}{4000}} = 31.06 \text{ rad/sec}$$

2. The critical damping is

$$c_{cr} = 2\sqrt{km} = 2\sqrt{10,000 \times 4000 / 386} = 643.8 \frac{\text{lb}\cdot\text{sec}}{\text{in}}$$

and the absolute damping

$$c = \xi c_{cr} = 0.05 \times 643.8 = 32.19 \frac{\text{lb}\cdot\text{sec}}{\text{in}}$$

3. Approximately, the logarithmic decrement is

$$\delta = \ln\left(\frac{u_0}{u_1}\right) = 2\pi\xi = 2\pi(0.05) = 0.314$$

and the ratio of two consecutive amplitudes

$$\frac{u_0}{u_1} = 1.37$$

4. The ratio between the first amplitude u_0 and the amplitude u_k after k cycles may be expressed as

$$\frac{u_0}{u_k} = \frac{u_0}{u_1} \cdot \frac{u_1}{u_2} \cdots \frac{u_{k-1}}{u_k}$$

Then taking the natural logarithm, we obtain

$$\ln \frac{u_0}{u_k} = \delta + \delta + \dots + \delta = k\delta$$

$$\ln \frac{0.1}{0.01} = 0.314k$$

$$k = \frac{\ln 10}{0.314} = 7.73 \rightarrow 8 \text{ cycles}$$

The damped frequency ω_D is given by

$$\omega_D = \omega \sqrt{1 - \xi^2} = 31.06 \sqrt{1 - (0.05)^2} = 31.02 \text{ rad/sec}$$

and the damped period T_D by

$$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{31.02} = 0.2025 \text{ sec}$$

Then the time for eight cycles is

$$t(8 \text{ cycles}) = 8T_D = 1.62 \text{ sec}$$

Illustrative Example 2.3

A machine weighing 1000 lb is mounted through springs having a total stiffness $k = 2000 \text{ lb/in}$ to a simple supported beam as shown in Fig. 2.5(a). Determine using the analytical model shown in Fig. 2.5(b) the equivalent mass m_E , the equivalent spring constant k_E , and the equivalent damping coefficient c_E for the system assumed to have 10% of the critical damping. Neglect the mass of the beam.

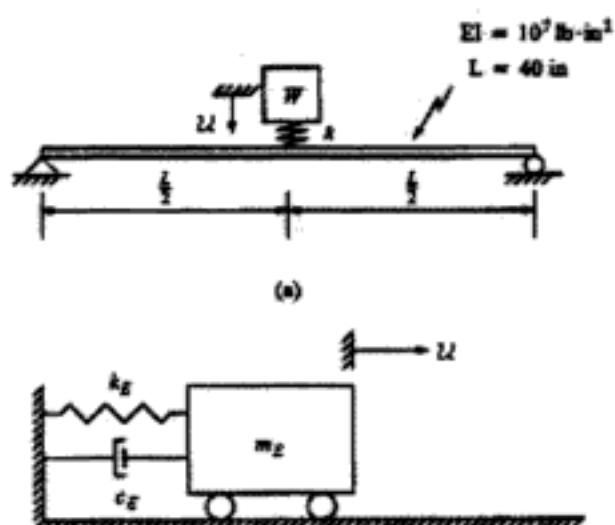


Fig. 2.5 (a) System for Illustrative Example 2.3. (b) Analytical model.

Solution:

The spring constant k_b for a uniform simple supported beam is obtained from the deflection δ resulting for a force P applied at the center of the beam:

$$\delta = \frac{PL^3}{48EI}$$

Hence,

$$\begin{aligned} k_b &= \frac{P}{\delta} = \frac{48EI}{L^3} \\ &= \frac{48 \times 10^7}{40^3} = 7500 \text{ lb/in} \end{aligned}$$

The equivalent spring constant is then calculated using eq.(1.5) for two springs in a series:

$$\begin{aligned} \frac{1}{k_E} &= \frac{1}{k} + \frac{1}{k_b} \\ &= \frac{1}{2000} + \frac{1}{7500} \\ k_E &= 1579 \text{ lb/in} \end{aligned} \quad (\text{Ans})$$

The equivalent mass is:

$$m_E = \frac{W}{g} = \frac{1000}{386} = 2.59 \left(\frac{\text{lb} \cdot \text{sec}^2}{\text{in}} \right)$$

The critical damping is calculated from eq.(2.6):

$$\begin{aligned} c_c &= 2\sqrt{k_E m_E} \\ &= 2\sqrt{1579 \times 2.59} = 127.92 \left(\frac{\text{lb} \cdot \text{sec}}{\text{in}} \right) \end{aligned} \quad (\text{Ans.})$$

The damping coefficient c_E is then calculated from eq.(2.19):

$$\xi = \frac{c_E}{c_{cr}} \quad (\text{Ans.})$$

$$c_E = \xi c_{cr} = 0.10 \times 127.92 = 12.79 \left(\frac{\text{lb. sec}}{\text{in}} \right)$$

2.7 Summary

Real structures dissipate energy while undergoing vibratory motion. The most common and practical method for considering this dissipation of energy is to assume that it is due to viscous damping forces. These forces are assumed to be proportional to the magnitude of the velocity but acting in the direction opposite to the motion. The factor of proportionality is called the viscous damping coefficient. It is expedient to express this coefficient as a fraction of the critical damping in the system ($\xi = c/c_{cr}$). The critical damping may be defined as the least value of the damping coefficient for which the system will not oscillate when disturbed initially, but it will simply return to the equilibrium position.

The differential equation of motion for the free vibration of a damped single degree-of-freedom system is given by

$$m\ddot{u} + c\dot{u} + ku = 0 \quad \text{eq.(2.1) repeated}$$

The analytical expression for the solution of this equation depends on the magnitude of the damping ratio. Three cases are possible:

1. Critically damped system ($\xi = 1$).
2. Overdamped system ($\xi > 1$).
3. Underdamped system ($\xi < 1$).

For the underdamped system ($\xi < 1$), the solution of the differential equation of motion may be written as

$$u(t) = e^{-\xi\omega t} \left[u_0 \cos \omega_D t + \frac{v_0 + u_0 \xi \omega}{\omega_D} \sin \omega_D t \right] \quad \text{eq.(2.20) repeated}$$

in which

$\omega = \sqrt{k/m}$ is the undamped frequency

$\omega_D = \omega \sqrt{1 - \xi^2}$ is the damped frequency

$\xi = \frac{c}{c_{cr}}$ is the damping ratio

$c_{cr} = 2\sqrt{km}$ is the critical damping

and u_0 and v_0 are, respectively, the initial displacement and velocity.

A common method of determining the damping present in a system is to evaluate experimentally the logarithmic decrement, which is defined as the natural logarithm of the ratio of two consecutive peaks for the displacement or acceleration, in free vibration, that is,

$$\delta = \ln \frac{u_1}{u_2} \quad \text{or} \quad \delta = \ln \frac{\ddot{u}_1}{\ddot{u}_2} \quad \text{eq.(2.26) repeated}$$

The damping ratio in structural systems is usually less than 10% of the critical damping ($\xi < 0.1$). For such systems, the damped frequency is approximately equal to the undamped frequency.

2.8 Problems

Problem 2.1

Repeat Problem 1.2 assuming that the system has 15% of critical damping.

Problem 2.2

Repeat problem 1.6 assuming that the system has 10% of critical damping.

Problem 2.3

The amplitude of vibration of the system shown in Fig. P2.3 is observed to decrease 5% on each consecutive cycle of motion. Determine the damping coefficient c of the system $k = 200 \text{ lb/in}$ and $m = 10 \text{ lb}\cdot\text{sec}^2/\text{in}$.

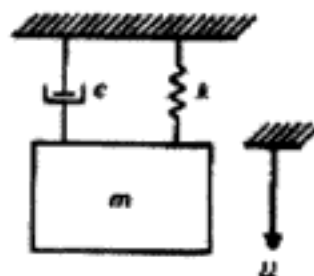


Fig P2.3

Problem 2.4

It is observed experimentally that the amplitude of free vibration of a certain structure, modeled as a single degree-of-freedom system, decreases in 10 cycles from 1 in to 0.4 in. What is the percentage of critical damping?

Problem 2.5

Show that the displacement for critical and overcritical damped systems with initial displacement u_0 and velocity v_0 may be written as

$$u = e^{-\alpha t} [u_0(1 + \omega t) + v_0 t] \quad \text{for } \xi = 1$$

$$u = e^{-\xi \omega t} \left[u_0 \cosh \omega_D' t + \frac{v_0 + u_0 \xi \omega}{\omega_D'} \sinh \omega_D' t \right] \quad \text{for } \xi > 1$$

where $\omega_D' = \omega \sqrt{\xi^2 - 1}$.

Problem 2.6

A structure is modeled as a damped oscillator having a spring constant $k = 30$ kip/in and undamped natural frequency $\omega = 25$ rad/sec. Experimentally it was found that a force of 1 kip produced a relative velocity of 1.0 in/sec in the damping element. Determine:

- The damping ratio ξ .
- The damped period T_D .
- The logarithmic decrement δ .
- The ratio between two consecutive amplitudes.

Problem 2.7

In Fig. 2.4 it is indicated that the tangent points to the displacement curve corresponds to $\cos(\omega_D t - \alpha) = 1$. Therefore the difference in $\omega_D t$ between any two consecutive tangent points is 2π . Show that the difference in $\omega_D t$ between any two consecutive peaks is also 2π .

Problem 2.8

Show that for a damped system in free vibration the logarithmic decrement may be written as

$$\delta = \frac{1}{k} \ln \frac{u_i}{u_{i+k}}$$

where k is the number of cycles separating the two measured peak amplitudes u_i and u_{i+k} .

Problem 2.9

It has been estimated that damping in the system of Problem 1.11 is 10% of the critical value. Determine the damped frequency f_D of the system and the absolute value of the damped coefficient c .

Problem 2.10

A single degree-of-freedom system consists of a mass with a weight of 386 lb and a spring of stiffness $k = 3000$ lb/in. By testing the system it was found that a force of 100 lb produces a relative velocity of 12 in/sec. Find:

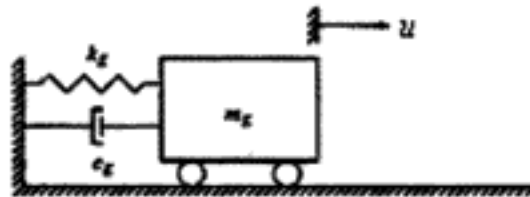
- a) The damping ratio, ξ .
- b) The damped frequency of vibration, f_D .
- c) Logarithmic decrement, δ .
- d) The ratio of two consecutive amplitudes.

Problem 2.11

Solve Problem 2.10 when the damping coefficient is $c = 2 \text{ lb}\cdot\text{sec}/\text{in}$.

Problem 2.12

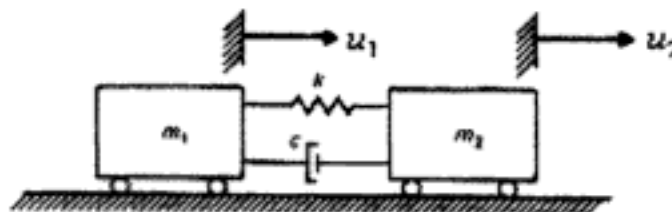
For each of the systems considered in Problem 1.15, determine the equivalent spring constant k_E and the equivalent damping coefficient c_E in the analytical model shown in Fig. P2.12. Assume that the damping in these systems is equal to 10% of critical damping.

**Fig. P2.12****Problem 2.13**

A vibration generator with two weights each of 30 lb with an eccentricity of 10 in rotating about vertical axis in opposite directions is mounted on the roof of a one-story building with a roof that weighs 300 kips. It is observed that the maximum lateral acceleration of 0.05 g occurs when the vibrator generator is rotating at 400 rpm. Determine the equivalent viscous damping in the structure.

Problem 2.14

A system modeled by two freely vibrating masses m_1 and m_2 is interconnected by a spring and a damper element as shown in Fig. P2.14. Determine for this system the differential equation of motion in terms of relative motion of the masses $u_r = u_2 - u_1$.

**Fig. P2.14**

Problem 2.15

Determine the relative motion $u_r = u_2 - u_1$ for the system shown in Fig. P2.14 in terms of the natural frequency ω , damped frequency, ω_D and relative damping, ζ .

Hint: Define the equivalent mass as $M = m_1 m_2 / (m_1 + m_2)$.

3 Response of One-Degree-of-Freedom System to Harmonic Loading

In this chapter, we will study the motion of structures idealized as single-degree-of-freedom systems excited harmonically, that is, structures subjected to forces or displacements whose magnitudes may be represented by a sine or cosine function of time. This type of excitation results in one of the most important motions in the study of mechanical vibrations as well as in applications to structural dynamics. Structures are very often subjected to the dynamic action of rotating machinery which produces harmonic excitations due to the unavoidable presence of mass eccentricities in the rotating parts of such machinery. Furthermore, even in those cases when the excitation is not a harmonic function, the response of the structure may be obtained using the Fourier Method, as the superposition of individual responses to the harmonic components of external excitation. This approach will be dealt with in Chapter 20 as a special topic.

3.1 Harmonic Excitation: Undamped System

The impressed force $F(t)$ acting on the simple oscillator in Fig.3.1 is assumed to be harmonic and equal to $F_0 \sin \bar{\omega}t$ where F_0 is the peak amplitude and $\bar{\omega}$ is the frequency of the force in radians per second.

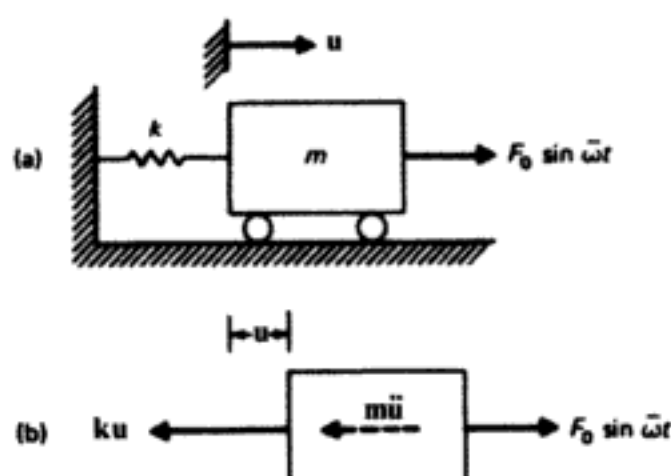


Fig. 3.1 (a) Undamped oscillator harmonically excited. (b) Free body diagram.

The differential equation obtained by summing all the forces in the free body diagram of Fig.3.1(b) is

$$m\ddot{u} + ku = F_0 \sin \bar{\omega} t \quad (3.1)$$

The solution of eq.(3.1) can be expressed as

$$u(t) = u_c(t) + u_p(t) \quad (3.2)$$

where $u_c(t)$ is the complementary solution satisfying the homogeneous equation, that is, eq.(3.1) with the left hand-side set equal to zero; and $u_p(t)$ is the particular solution based on the solution satisfying the nonhomogeneous differential equation (3.1). The complementary solution, $u_c(t)$, is given by eq.(1.17) as

$$u_c(t) = A \cos \omega t + B \sin \omega t \quad (3.3)$$

where

$$\omega = \sqrt{k/m}$$

The nature of the forcing function in eq.(3.1) suggests that the particular solution be taken as

$$u_p(t) = U \sin \bar{\omega} t \quad (3.4)$$

where U is the amplitude of the particular solution. The substitution of eq.(3.4) into eq.(3.1) followed by cancellation of common factors gives

$$-m\bar{\omega}^2 U + kU = F_0$$

or

$$U = \frac{F_0}{k - m\bar{\omega}^2} = \frac{F_0/k}{1 - r^2} \quad (3.5)$$

in which r represents the ratio (frequency ratio) of the applied forced frequency to the natural frequency of vibration of the system, that is,

$$r = \frac{\bar{\omega}}{\omega} \quad (3.6)$$

Combining eqs.(3.3) through (3.5) with eq.(3.2) yields

$$U(t) = A \cos \omega t + B \sin \omega t + \frac{F_0/k}{1 - r^2} \sin \bar{\omega} t \quad (3.7)$$

If the initial conditions for the displacement and for the velocity at time $t = 0$ are taken as zero ($u_0 = 0$, $v_0 = 0$), the constants of integration determined from eq.(3.7) are:

$$A = 0 \quad \text{and} \quad B = -r \frac{F_0/k}{1 - r^2}$$

which, upon substitution in eq.(3.7), results in

$$u(t) = \frac{F_0/k}{1 - r^2} (\sin \bar{\omega} t - r \sin \omega t) \quad (3.8)$$

As we can see from eq.(3.8), the response is given by the superposition of two harmonic terms of different frequencies. The resulting motion is not harmonic; however, in the practical case, damping forces will always be present in the system and will cause the last term, i.e., the free frequency term in eq. (3.8), to eventually vanish. For this reason, this term is said to represent the transient response. The forcing frequency term in eq.(3.8), namely

$$u(t) = \frac{F_0/k}{1 - r^2} \sin \bar{\omega} t \quad (3.9)$$

is referred to as the steady-state response. It is clear from eq.(3.8) that in the case of no damping in the system, the transient will not vanish and the response is then given by eq.(3.8). It can also be seen from eq.(3.8) or eq.(3.9) that when the forcing frequency is equal to natural frequency ($r = 1.0$), the amplitude of the motion becomes infinitely large. A system acted upon by an external excitation of frequency coinciding with the natural frequency is said to be at resonance. In this circumstance, the amplitude will increase gradually to infinite. However, materials that are

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commonly used in practice are subjected to strength limitations and in actual structures failures occur long before extremely large amplitudes can be attained.

3.2 Harmonic Excitation: Damped System

Now consider the case of the one-degree-of-freedom system in Fig.3.2(a) vibrating under the influence of viscous damping. The differential equation of motion is obtained by equating to zero the sum of the forces in the free body diagram of Fig. 3.2(b). Hence

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \bar{\omega} t \quad (3.10)$$

The complete solution of this equation again consists of the complementary solution $u_c(t)$ and the particular solution $u_p(t)$. The complementary solution is given for the underdamped case ($c < c_{cr}$) by eqs.(2.15) after using (2.19) as

$$u_c(t) = e^{-\xi \bar{\omega} t} (A \cos \omega_D t + B \sin \omega_D t) \quad (3.11)$$

The particular solution may be found by substituting u_p in this case assumed to be of the form

$$u_p(t) = C_1 \sin \bar{\omega} t + C_2 \cos \bar{\omega} t \quad (3.12)$$

into eq. (3.10) and equating the coefficients of the sine and cosine functions. However, here we follow a more elegant approach using Euler's relation, namely

$$e^{i\bar{\omega} t} = \cos \bar{\omega} t + i \sin \bar{\omega} t$$

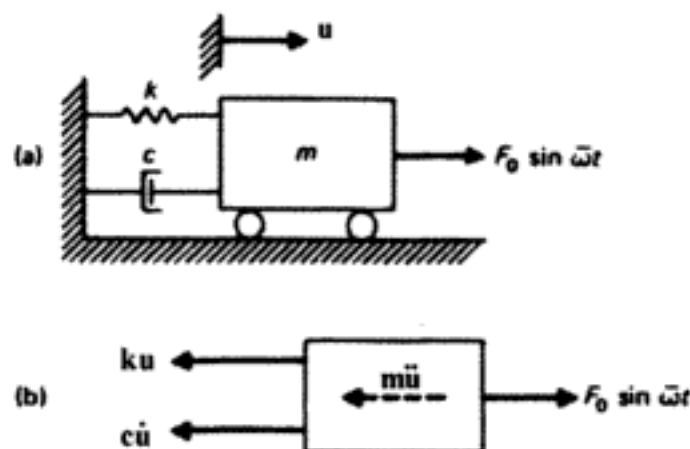


Fig. 3.2 (a) Damped oscillator harmonically excited. (b) Free body diagram.

For this purpose, the reader should realize that we can write eq.(3.10) as

$$m\ddot{u} + c\dot{u} + ku = F_0 e^{i\bar{\omega}t} \quad (3.13)$$

with the understanding that only the imaginary component of $F_0 e^{i\bar{\omega}t}$, i.e., the force component of $F_0 \sin \bar{\omega}t$, is acting and, consequently, the response will then consist only of the imaginary part of the total solution of eq.(3.13). In other words, we obtain the solution of eq.(3.13) which has real and imaginary components, and disregard the real component.

It is reasonable to expect that the particular solution of eq.(3.13) will be of the form

$$u_p = C e^{i\bar{\omega}t} \quad (3.14)$$

Substitution of eq.(3.14) into eq.(3.13) and cancellation of the factor $e^{i\bar{\omega}t}$ gives

$$-m\bar{\omega}^2 C + ic\bar{\omega}C + kC = F_0$$

or

$$C = \frac{F_0}{k - m\bar{\omega}^2 + ic\bar{\omega}}$$

and

$$u_p = \frac{F_0 e^{i\bar{\omega}t}}{k - m\bar{\omega}^2 + ic\bar{\omega}} \quad (3.15)$$

By using polar coordinate form, the complex denominator in eq.(3.15) may be written as

$$u_p = \frac{F_0 e^{i\bar{\omega}t}}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2} e^{i\theta}}$$

or

$$u_p = \frac{F_0 e^{i[\bar{\omega}t - \theta]}}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}} \quad (3.16)$$

where

$$\tan \theta = \frac{c\bar{\omega}}{k - m\bar{\omega}^2} \quad (3.17)$$

The response to the force in $F_0 \sin \bar{\omega} t$ (the imaginary component of $F_0 e^{i\bar{\omega} t}$) is then the imaginary component of eq.(3.16), namely

$$u_p = \frac{F_0 \sin(\bar{\omega} t - \theta)}{\sqrt{(k - m\bar{\omega})^2 + (c\bar{\omega})^2}} \quad (3.18)$$

or

$$u_p = U \sin(\bar{\omega} t - \theta) \quad (3.19)$$

where

$$U = \frac{F_0}{\sqrt{(k - m\bar{\omega})^2 + (c\bar{\omega})^2}}$$

is the amplitude of the steady-state motion. Equations (3.18) and (3.17) may conveniently be written in terms of dimensionless ratios as

$$u(t) = \frac{u_{st} \sin(\bar{\omega} t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad (3.20)$$

and

$$\tan \theta = \frac{2\xi r}{1 - r^2} \quad (3.21)$$

where $u_{st} = F_0/k$ is seen to be the static deflection of the spring acted upon by the force F_0 , $\xi = c/c_{cr}$ the damping ratio, and $r = \bar{\omega}/\omega$ the frequency ratio. The total response is then obtained by combining the complementary solution (transient response) from eq.(3.11) and the particular solution (steady-state response) from eq.(3.20), that is,

$$u(t) = e^{-\xi\omega t} (A \cos \omega_D t + B \sin \omega_D t) + \frac{u_{st} \sin(\bar{\omega} t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \quad (3.22)$$

The reader should be warned that the constants of integration A and B must be evaluated from initial conditions using the total response given by eq.(3.22) and not from just the transient component of the response given in eq.(3.11). By examining the transient component of response, it may be seen that the presence of the exponential factor $e^{-\xi\omega t}$ will cause this component to vanish, leaving only the steady-state motion which is given by eq. (3.20).

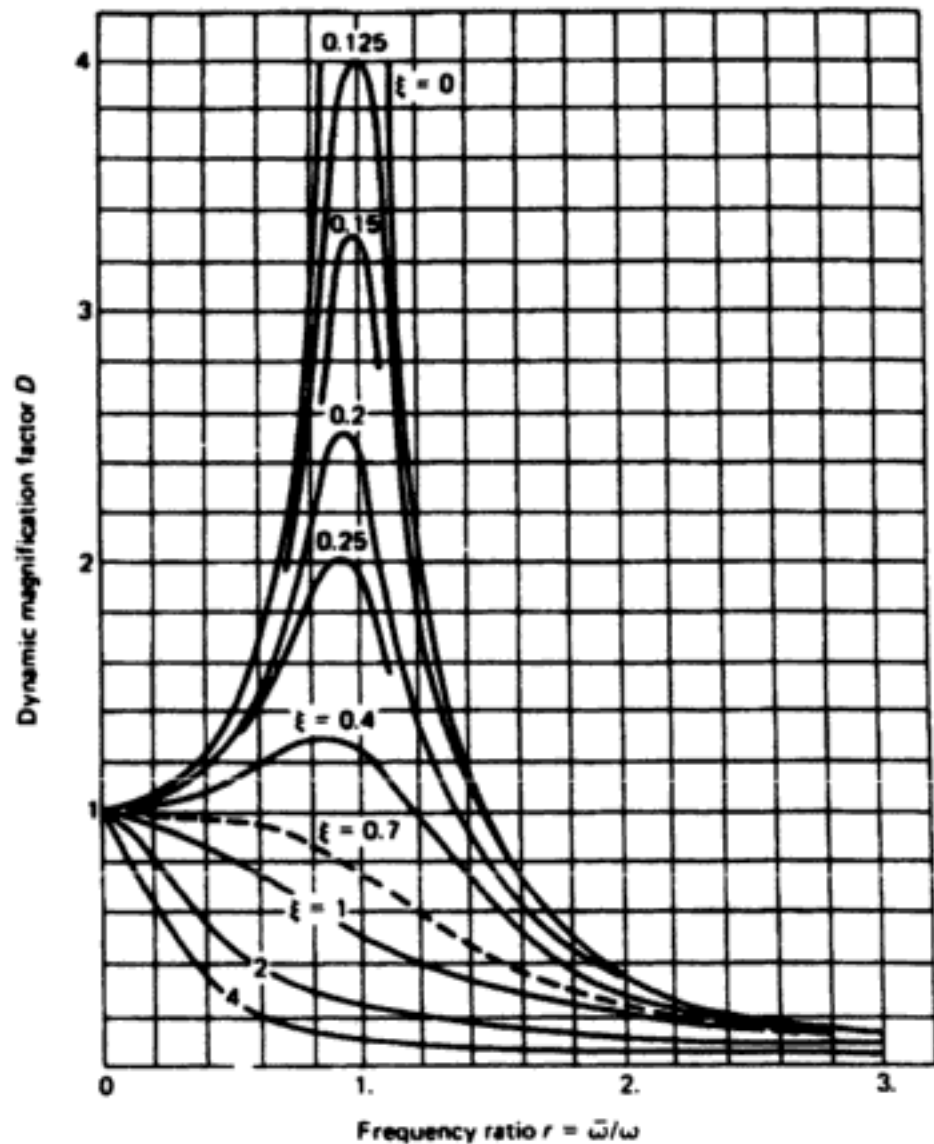


Fig. 3.3 Dynamic magnification factor as a function of the frequency ratio for various amounts of damping.

The ratio of the steady-state amplitude of $u_p(t)$ to the static deflection u_{st} defined above is known as the dynamic magnification factor D , and is given from eqs.(3.19) and (3.20) by

$$D = \frac{U}{u_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \quad (3.23)$$

It may be seen from eq.(3.23) that the dynamic magnification factor varies with the frequency ratio r and the damping ratio ξ . Parametric plots of the dynamic magnification factor are shown in Fig. 3.3. The phase angle θ , given in eq.(3.21), also varies with the same quantities as it is shown in the plots of Fig. 3.4. We note in Fig 3.3 that for a lightly damped system, the peak amplitude occurs at a frequency ratio very close to $r = 1$; that is, the dynamic magnification factor has its maximum value virtually at resonance ($r = 1$). It can also be seen from eq.(3.23) that

at resonance the dynamic magnification factor is inversely proportional to the damping ratio, that is,

$$D(r=1) = \frac{1}{2\xi} \quad (3.24)$$

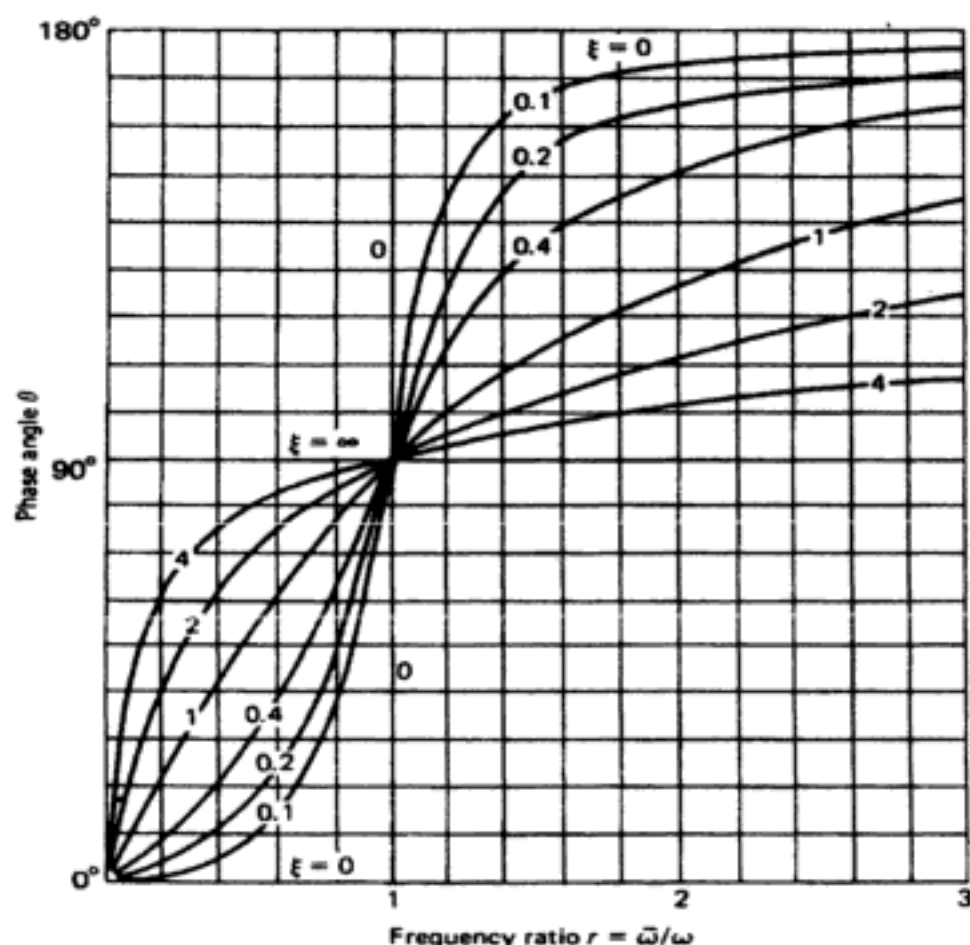


Fig. 3.4 Phase angle θ as a function of frequency ratio for various damping values.

Although the dynamic magnification factor evaluated at resonance is close to its maximum value, it is not exactly the maximum response for a damped system. However, for moderate amounts of damping, the difference between the approximate value of eq.(3.24) and the exact maximum is negligible.

Illustrative Example 3.1

A simple beam supports at its center a machine having a weight $W = 16,000$ lb. The beam is made of two standard S8 X 23 sections with a clear span $L = 12$ ft and total cross-sectional moment of inertia $I = 2 \times 64.2 = 128.4$ in⁴. The motor runs at 300 rpm, and its rotor is out of balance to the extent of $W' = 40$ lb at an eccentricity of $e_0 = 10$ in. What will be the amplitude of the steady-state response if the equivalent viscous damping for the system is assumed 10% of the critical?

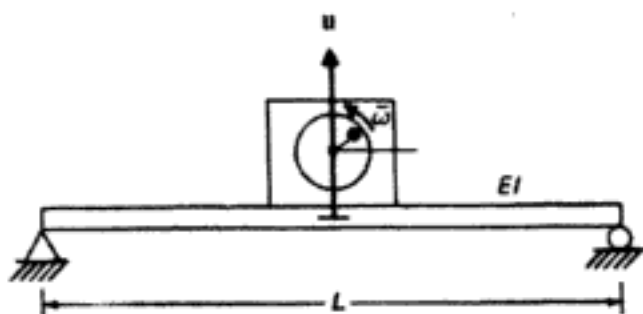


Fig. 3.5 Diagram for beam-machine system of Illustrative Example 3.1.

Solution:

This dynamic system may be modeled by the damped oscillator. The distributed mass of the beam will be neglected in comparison with the large mass of the machine. Figures 3.5 and 3.6 show, respectively, the schematic diagram of the beam-machine system and the adapted model.

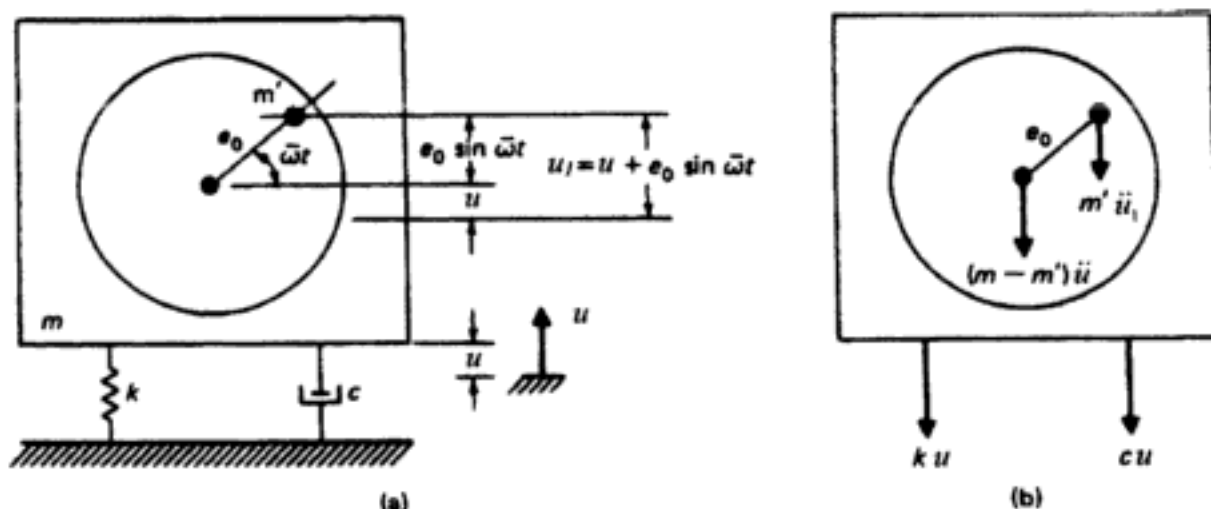


Fig. 3.6 (a) Analytical model for Illustrative Example 3.1; (b) Free body diagram.

The force at the center of a simply supported beam necessary to deflect this point one unit (i.e., the stiffness coefficient) is given by the formula

$$k = \frac{48EI}{L^3} = \frac{48 \times 30 \times 10^6 \times 128.4}{(144)^3} = 61,920 \text{ lb/in}$$

The natural frequency of the system (neglecting the mass of the beam) is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{61,920}{16,000/386}} = 38.65 \text{ rad/sec,}$$

The forced frequency is

$$\bar{\omega} = \frac{300 \times 2\pi}{60} = 31.41 \text{ rad/sec}$$

and the frequency ratio

$$r = \frac{\bar{\omega}}{\omega} = \frac{31.41}{38.65} = 0.813$$

Referring the Fig. 3.6, let m be the total mass of the motor and m' the unbalanced rotating mass. Then, if u is the vertical displacement from the equilibrium position of the non-rotating mass ($m - m'$), the displacement u_1 of the eccentric mass m' as shown in Fig. 3.6 is

$$u_1 = u + e_0 \sin \bar{\omega} t \quad (a)$$

The equation of motion is then obtained by summing forces along the vertical direction in the free body diagram of Fig. 3.6(b), where the inertial forces of both the nonrotating mass and of the eccentric mass are also shown. This summation yields

$$(m - m')\ddot{u} + m'\ddot{u}_1 + c\dot{u} + ku = 0 \quad (b)$$

in which $m' = W'/g$ is the eccentric mass.

Substitution of \ddot{u}_1 obtained from eq.(a) gives

$$(m - m')\ddot{u} + m'(\ddot{u} - e_0\bar{\omega}^2 \sin \bar{\omega} t) + c\dot{u} + ku = 0 \quad (b)$$

and with a rearrangement of terms

$$m\ddot{u} + c\dot{u} + ku = m'e_0\bar{\omega}^2 \sin \bar{\omega} t \quad (c)$$

This last equation is of the same form as the equation of motion (3.10) for the damped oscillator excited harmonically by a force of amplitude

$$F_0 = m'e_0\bar{\omega}^2 \quad (d)$$

Substituting in eq.(d) the numerical values for this example, we obtain

$$F_0 = (40)(10)(31.41)^2 / 386 = 1022 \text{ lb}$$

From eq.(3.20), the amplitude of the steady-state resulting motion is then

$$U = \frac{1022 / 61,920}{\sqrt{(1 - 0.813^2)^2 + (2 \times 0.813 \times 0.1)^2}}$$

$$U = 0.044 \text{ in} \quad (\text{Ans.})$$

Illustrative Example 3.2

The steel frame shown in Fig.3.7 supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 5.3t$ lb. Assuming 5% of critical damping, determine: (a) the steady-state amplitude of vibration and (b) the maximum dynamic stress in the columns. Assume the girder is rigid.

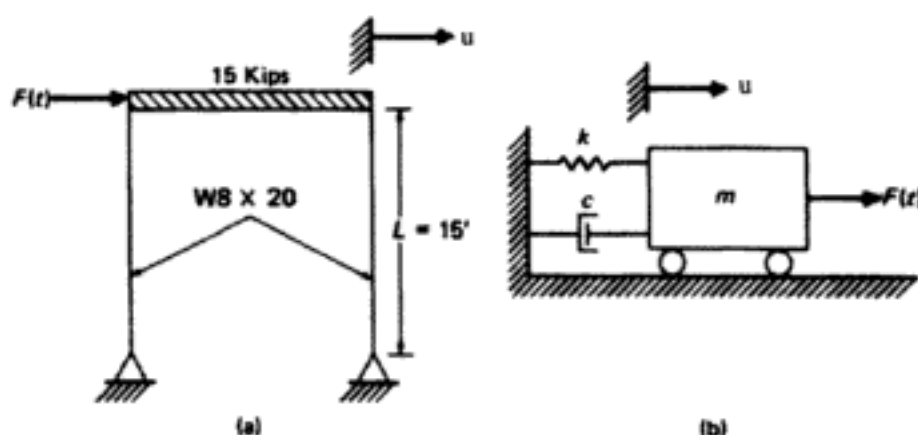


Fig. 3.7 (a) Diagram of the frame for Illustrative Example 3.2 (b) Analytical model.

Solution:

This structure may be modeled for dynamic analysis as the damped oscillator shown in Fig.3.7(b). The parameters in this model are computed as follows:

$$k^* = \frac{3E(2I)}{L^3} = \frac{3 \times 30 \times 10^6 \times 2 \times 69.2}{(12 \times 15)^3} = 2136 \text{ lb/in}$$

$$\xi = 0.05$$

$$u_{st} = \frac{F_0}{k} = \frac{200}{2136} = 0.0936 \text{ in}$$

- A unit displacement at the top of pinned supported columns requires a force equal to $3EI/L^3$.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2136 \times 386}{15,000}} = 7.41 \text{ rad/sec}$$

$$r = \frac{\bar{\omega}}{\omega} = \frac{5.3}{7.41} = 0.715$$

The steady-state amplitude from eqs.(3.19) and (3.20) is

$$U = \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.189 \text{ in}$$

Then, the maximum shear force is the columns is

$$V_{\max} = \frac{3EIU}{L^3} = 201.8 \text{ lb}$$

The maximum bending moment

$$M_{\max} = V_{\max} L = 36,324 \text{ lb}\cdot\text{in}$$

and the maximum stress

$$\sigma_{\max} = \frac{M_{\max}}{I/c} = \frac{36,324}{17} = 2136 \text{ psi} \quad (\text{Ans.})$$

in which I/c is the section modulus.

3.3 Evaluation of Damping at Resonance

We have seen in Chapter 2 that the free-vibration decay curve permits the evaluation of damping of a single-degree-of freedom system by simply calculating the logarithm decrement and using eq.(2.28) or eq.(2.29). Another technique for determining damping is based on observations of steady-state harmonic response, which requires harmonic excitations of the structure in a range of frequencies in the neighborhood of resonance. With the application of a harmonic force $F_0 \sin \bar{\omega}t$ at closely spaced values of frequencies, the response curve for the structure can be plotted, resulting in displacement amplitudes as a function of the applied frequencies. A typical response curve for such a moderately damped structure is shown in Fig.3.8. It is seen from eq.(3.24) that, at resonance, the damping ratio is given by

$$\xi = \frac{1}{2D(r=1)} \quad (3.25)$$

where $D(r=1)$ is the dynamic magnification factor evaluated at resonance. In practice, the damping ratio ξ is determined from the dynamic magnification factor evaluated at the maximum amplitude, namely

$$\xi = \frac{1}{2D_m} \quad (3.26)$$

where

$$D_m = \frac{U_m}{u_{st}}$$

and U_m is the maximum amplitude. The error involved in evaluating the damping ratio ξ using the approximate eq.(3.26) is not significant in ordinary structures. This method of determining the damping ratio requires only some simple equipment to vibrate the structure in a range of frequencies that span the resonance frequency and a transducer for measuring amplitudes; nevertheless, the evaluation of the static displacement $u_{st} = F_0 / k$ may present a problem since, frequently, it is difficult to apply a static lateral load to the structure.

3.4 Bandwidth Method (Half-Power) to Evaluate Damping

An examination of the response curves in Fig.3.3 shows that the shape of these curves is controlled by the amount of damping present in the system; in particular, the bandwidth, that is, the difference between two frequencies corresponding to the same response amplitude, is related to the damping in the system. A typical frequency amplitude curve obtained experimentally for a moderately damped structure is shown in Fig.3.9. In the evaluation of damping it is convenient to measure the bandwidth at $1/\sqrt{2}$ of the peak amplitude as shown in this figure. The frequencies corresponding in this bandwidth f_1 and f_2 are also referred to as half-power points and are shown in Fig. 3.9. The values of the frequencies for this bandwidth can be determined by setting the response amplitude in eq.(3.20) equal to $1/\sqrt{2}$ times the resonant amplitude given by eq.(3.24), that is

$$\frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = \frac{1}{\sqrt{2}} \frac{u_{st}}{2\xi}$$

Squaring both sides and solving for the frequency ratio results in

$$r^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1 + \xi^2}$$

or by neglecting ξ^2 in the square root term

$$r_1^2 \cong 1 - 2\xi^2 - 2\xi$$

$$r_2^2 \cong 1 - 2\xi^2 + 2\xi$$

$$r_1 \cong 1 - \xi - \xi^2$$

$$r_2 \cong 1 + \xi - \xi^2$$

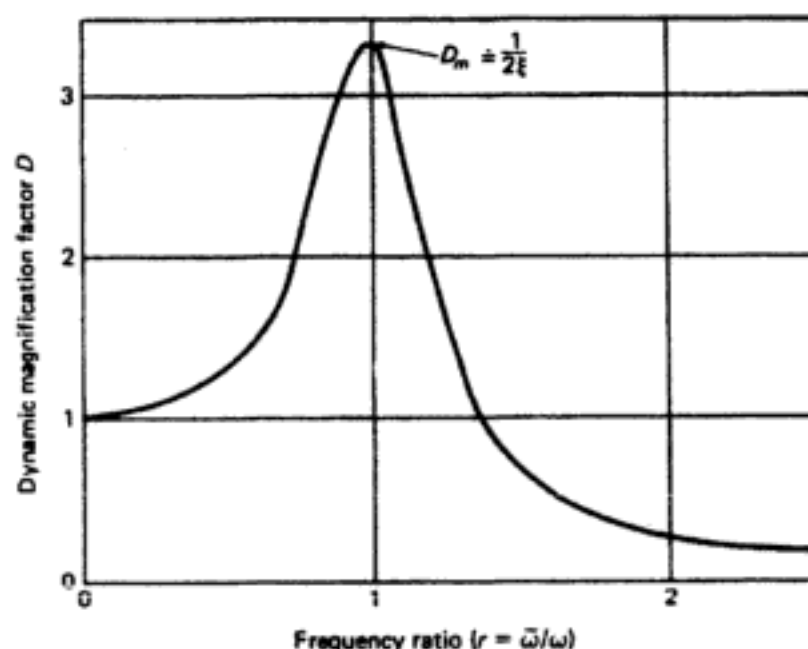


Fig. 3.8 Frequency response curve for moderately damped system.

Finally, the damping ratio is given approximately by half the difference between these half-power frequency ratios, namely

$$\xi = \frac{1}{2}(r_2 - r_1)$$

or

$$\xi = \frac{1}{2} \frac{\omega_2 - \omega_1}{\omega} = \frac{f_2 - f_1}{f_2 + f_1} \quad (3.27)$$

since

$$\frac{\omega_2 - \omega_1}{2\omega} = \frac{f_2 - f_1}{2f} \quad \text{and} \quad f \approx \frac{f_1 + f_2}{2}$$

Illustrative Example 3.3

Experimental data for the frequency response of a single degree-of-freedom system are plotted in Fig. 3.9. Determine the damping ratio of this system.

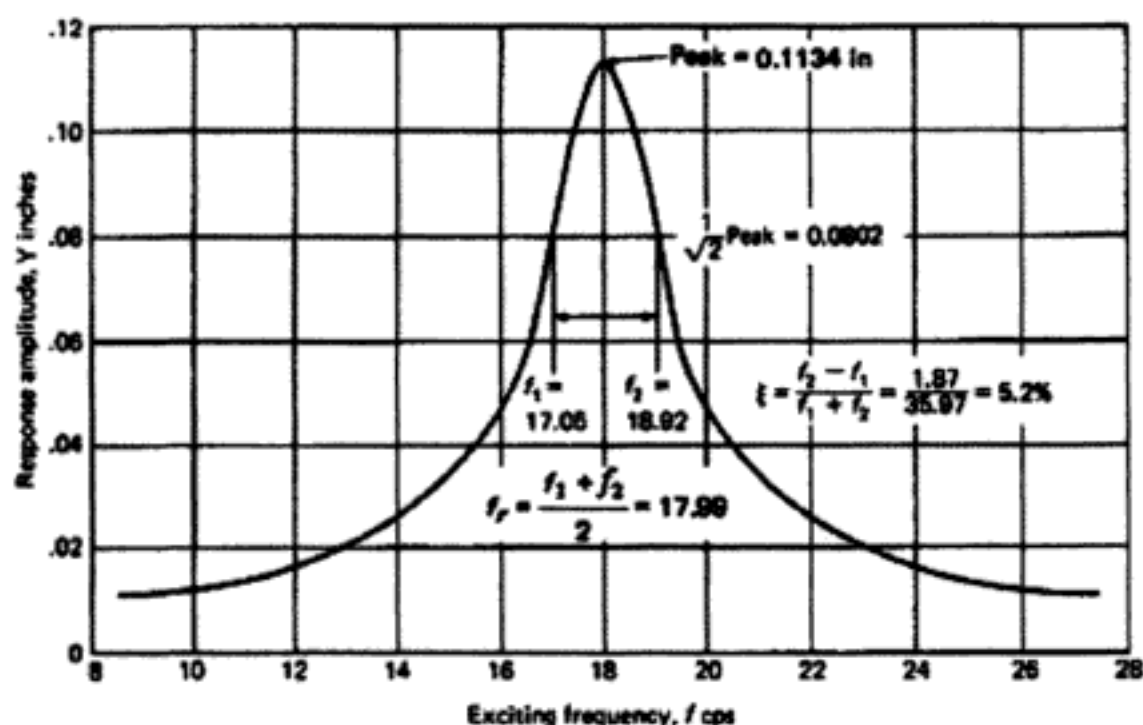


Fig. 3.9 Experimental frequency response curve of Illustrative Example 3.3.

Solution:

From Fig.3.9, the peak amplitude is 0.1134 in; hence the amplitude at half-power is equal to

$$0.1134 / \sqrt{2} = 0.0802 \text{ in}$$

The frequencies at this amplitude obtained from Fig.3.9 are

$$f_1 = 17.05 \text{ cps}$$

$$f_2 = 18.92$$

the damping ratio is then calculated from eq.(3.27) as

$$\xi \cong \frac{f_2 - f_1}{f_2 + f_1} \quad (\text{Ans.})$$

$$\xi \cong \frac{18.92 - 17.05}{18.92 + 17.05} = 5.2\%$$

3.5 Energy Dissipated by Viscous Damping

The energy E_D dissipated by viscous damping during one cycle of harmonic vibration of frequency $\bar{\omega}$ is equal to the work done by the damping force $c\dot{u}$ during

a differential displacement du integrated over one period of vibration $T = 2\pi / \bar{\omega}$. Hence,

$$E_D = \int_0^{2\pi/\bar{\omega}} (c\dot{u}) du = \int_0^{2\pi/\bar{\omega}} (c\dot{u}) \frac{du}{dt} dt = \int_0^{2\pi/\bar{\omega}} c\dot{u}^2 dt \quad (3.28)$$

The velocity $\dot{u} = \dot{u}(t)$ for the damped oscillator acted upon by the harmonic force, $F = F_0 \sin \bar{\omega} t$, is given by the derivative of eq.(3.19) as

$$\dot{u}(t) = U\bar{\omega} \cos(\bar{\omega} t - \theta) \quad (3.29)$$

which substituted in eq.(3.28) gives

$$E_D = cU^2\bar{\omega}^2 \int_0^{2\pi/\bar{\omega}} \cos^2(\bar{\omega} t - \theta) dt = \pi c\bar{\omega} U^2$$

$$E_D = 2\pi \xi r k U^2 \quad (3.30)$$

where as previously defined

$$\xi = \frac{c}{c_{cr}}, \quad r = \frac{\bar{\omega}}{\omega}, \quad \omega = \sqrt{\frac{k}{m}}, \quad \text{and} \quad c_{cr} = 2\sqrt{km} \quad (3.31)$$

Equation (3.30) shows that the energy dissipated by viscous damping is proportional to the square of the amplitude of the motion U . It can be shown (see Problem 3.1) that during one cycle, the work W_F of the external force $F = F_0 \sin \bar{\omega} t$ is precisely the equal to the energy E_D dissipated by the damping force as expressed by eq.(3.30).

3.6 Equivalent Viscous Damping

As mentioned in the introductory sections of Chapter 2, the mechanism by which structures dissipate energy during vibratory motion is usually assumed to be viscous. This assumption provides the enormous advantage that the differential equation of motion remains linear for damped dynamic systems vibrating in the elastic range. Only for some exceptionally situations such as the use of frictional devices installed in buildings to ameliorate damage resulting from strong motion earthquakes, viscous damping is usually assumed to account for frictional or damping forces in structural dynamics. The numerical value assigned to the damping coefficient is based on values obtained experimentally and the determination of an equivalent viscous damping.

The concept of equivalent viscous damping is based on test results obtained using harmonic forces. Thus, in reference to the experimental frequency response plot in Fig.3.9, the equivalent damping ξ_{eq} may be based on the maximum relative

amplitude of motion, $D_m = U_m / u_{st}$, or on the bandwidth corresponding to frequencies f_1 to f_2 at amplitude equal to $D_m / \sqrt{2}$. Thus, the equivalent viscous damping ξ_{eq} may be calculated from eq.(3.26) as

$$\xi_{eq} = \frac{u_{st}}{2U_m} \quad (3.32)$$

or from eq.(3.27) as

$$\xi_{eq} = \frac{f_2 - f_1}{f_2 + f_1} \quad (3.33)$$

Alternatively, the equivalent viscous damping ξ_{eq} may also be evaluated experimentally using the expression for the logarithmic decrement, δ , from eq.(2.28) or approximately from eq.(2.29) as

$$\xi_{eq} = \frac{\delta}{2\pi} \quad (3.34)$$

However, the most common definition of equivalent viscous damping is based on equating the energy dissipated, in a vibratory cycle of the actual structure, to the energy dissipated in an equivalent viscous system. Hence, equating the energy, E^* , dissipated in a cycle of harmonic vibration determined from experiment to the energy E_D , dissipated by an equivalent viscous system given by eq.(3.30) we have

$$E^* = 2\pi \xi_{eq} r k U^2$$

and

$$\xi_{eq} = \frac{E^*}{2\pi r k U^2}$$

or

$$\xi_{eq} = \frac{1}{4\pi r} \frac{E^*}{E_s} \quad (3.35)$$

in which E_s , the strain energy stored at maximum displacement if the system were elastic, which is given by

$$E_s = \frac{1}{2} k U^2 \quad (3.36)$$

In the determination of the energy, E^* , dissipated per cycle and the elastic energy, E_s , stored at the maximum displacement, an experiment is conducted by vibrating the structure at the resonant frequency for which $r = \bar{\omega} / \omega = 1$. At this frequency, damping in the system has a maximum effect. With appropriate test equipment and

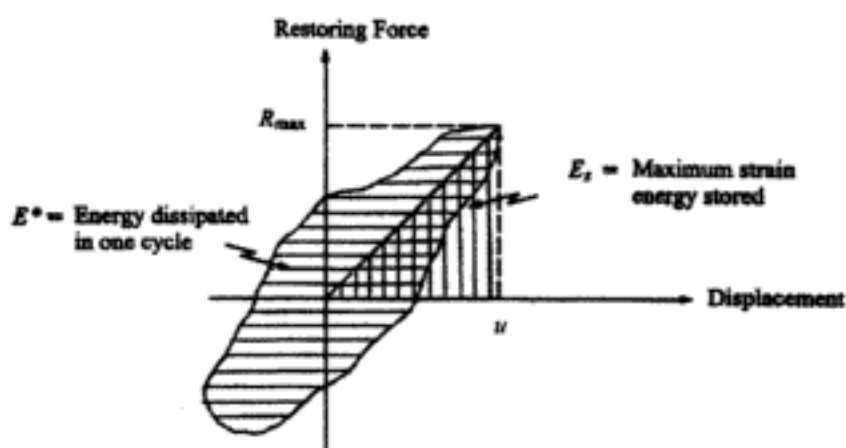


Fig. 3.10 Restoring force vs. displacement during a cycle of vibration showing the energy dissipated E^* (area within the loop) and the maximum energy stored (triangular area under maximum displacement)

measuring instrumentation, the restoring force and displacement during a cycle of vibration are measured to obtain a plot of the type shown in Fig. 3.10. The area enclosed in the loop during one cycle of vibration is equal to the energy dissipated, E^* , and the triangular area corresponding to the amplitude U is equal to the strain energy, E_s . Consequently, the equivalent viscous damping is evaluated by eq. (3.35) from the experimental results E^* and E_s , with $r = 1$, obtained from resisting force-displacement plot.

Illustrative Example 3.4

Laboratory tests on a structure modeled by the damped spring-mass system [Fig. 3.11(a)], are conducted to evaluate equivalent viscous damping using (a) peak amplitude and (b) energy dissipated. The experimental restoring force-displacement plot at resonance is shown in Fig. 3.11(b).

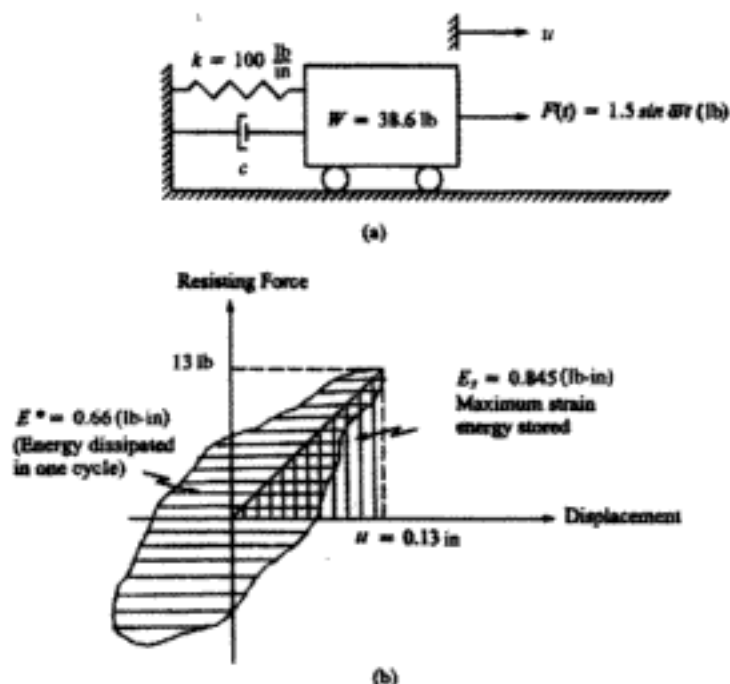


Fig. 3.11 (a) Analytical model for Illustrative Example 3.4; (b) Force-displacement plot.

Solution:

The static deflection, the mass, and the natural frequency are:

$$u_{st} = \frac{F_0}{k} = \frac{1.5}{100} = 0.015 \text{ in} \quad m = \frac{W}{g} = \frac{38.6}{386} = 0.1 \text{ (lb-sec}^2/\text{in)}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{0.1}} = 31.62 \text{ rad/sec}$$

a) Equivalent viscous damping calculated from peak amplitude:

$$\xi_{eq} = \frac{u_{st}}{2U_m} = \frac{0.015}{(2)(0.13)} = 0.0576 = 5.7\% \quad \text{by eq.(3.32)}$$

b) Equivalent viscous damping calculated from energy dissipated $E^* = 0.66$ (lb.in) and elastic energy at maximum displacement, $E_s = 0.845$ (lb.in), as shown in Fig. 3.11: (at resonance, $r = 1.0$)

$$\xi_{eq} = \frac{1}{4\pi r} \frac{E^*}{E_s} = \frac{0.66}{4\pi(1.0)0.845} = 0.0621 = 6.21\% \quad \text{by eq.(3.35)}$$

3.7 Response to Support Motion

3.7.1 Absolute Motion

There are many actual cases where the foundation or support of a structure is subjected to time varying motion. Structures subjected to ground motion by earthquakes or other excitations such as explosions or dynamic action of machinery are examples in which support motions may have to be considered in the analysis of dynamic response. Let us consider in Fig. 3.12 the case where the support of the simple oscillator modeling the structure is subjected to a harmonic motion given by the expression

$$u_s(t) = u_0 \sin \bar{\omega} t \quad (3.37)$$

where u_0 is the maximum amplitude and $\bar{\omega}$ is the frequency of the support motion. The differential equation of motion is obtained by setting equal to zero the sum of the forces (including the inertial force) in the corresponding free body diagram shown in Fig.3.12(b). The summation of the forces in the horizontal direction gives

$$m\ddot{u} + c(\dot{u} - \dot{u}_s) + k(u - u_s) = 0 \quad (3.38)$$

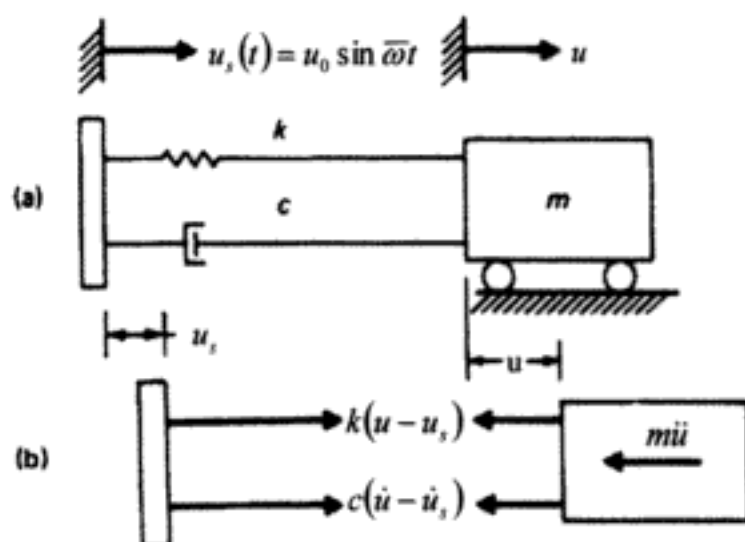


Fig. 3.12 (a) Damped simple oscillator harmonically excited through its support. (b) Free body diagram including inertial force.

The substitution of eq.(3.37) into eq.(3.38) and the rearrangement of terms result in

$$m\ddot{u} + c\dot{u} + ku = ku_0 \sin \bar{\omega} t + c\bar{\omega} u_0 \cos \bar{\omega} t \quad (3.39)$$

The two harmonic terms of frequency $\bar{\omega}$ in the right-hand side of this equation may be combined and eq.(3.39) rewritten [similarly to eqs.(1.20) and (1.23)] as

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin(\bar{\omega} t + \beta) \quad (3.40)$$

where

$$F_0 = u_0 \sqrt{k^2 + (c\bar{\omega})^2} = u_0 k \sqrt{1 + (2r\xi)^2} \quad (3.41)$$

and

$$\tan \beta = c\bar{\omega} / k = 2r\xi \quad (3.42)$$

It is apparent that the differential equation (3.40) is of the same form as eq.(3.10) for the oscillator excited by the harmonic force $F_0 \sin(\bar{\omega} t + \beta)$. Consequently, the steady-state solution of eq.(3.40) is given as before by eqs.(3.19) and (3.20), except for the addition of the angle β in the argument of the sine function, that is

$$u(t) = \frac{F_0 / k \sin(\bar{\omega} t + \beta - \theta)}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.43)$$

or substituting F_0 from eq.(3.41)

$$\frac{u(t)}{u_0} = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \sin(\bar{\omega}t + \beta - \theta) \quad (3.44)$$

Equation (3.44) is the expression for the relative transmission of the support motion to the oscillator. This is an important problem in vibration isolation in which equipment must be protected from harmful vibrations of the supporting structure. The degree of relative isolation is known as transmissibility and is defined as the ratio of the amplitude of motion U of the oscillator to the amplitude u_0 , the motion of the support. From eq.(3.44), the transmissibility T_r is then given by

$$T_r = \frac{U}{u_0} = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.45)$$

Analogously to the motion transmitted, we may find the acceleration transmitted from the foundation to the mass. The acceleration transmitted to the mass is given by the second derivative of $u(t)$ in eq.(3.44) as

$$\ddot{u}(t) = \frac{-\bar{\omega}^2 u_0 \sqrt{1 + (2r\xi)^2} \sin(\bar{\omega}t + \beta - \theta)}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.46)$$

while the acceleration $\ddot{u}_s(t)$ of the foundation is obtained from eq.(3.37)

$$\ddot{u}_s(t) = -u_0 \bar{\omega}^2 \sin \bar{\omega}t \quad (3.47)$$

The acceleration transmissibility, T_r , is then given by the ratio of the amplitudes of the acceleration in eqs.(3.46) and (3.47). Hence,

$$T_r = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.48)$$

It may be seen that the transmissibility of acceleration given by eq.(3.48) is identical to eq.(3.45), the transmissibility of displacements. Hence, the same expression will give either displacement or acceleration transmissibility.

A plot of transmissibility as a function of the frequency ratio and damping ratio is shown in Fig.3.13. The curves in this figure are similar to the curves in Fig. 3.3, representing the frequency response of the damped oscillator. The major difference between these two sets of curves is that all of the curves in Fig. 3.13 pass through the same point at a frequency ratio $r = \sqrt{2}$. It can be seen in Fig.3.13 that damping tends to reduce the effectiveness of vibration isolation for frequencies greater than this ratio, that is, for r greater than $\sqrt{2}$.

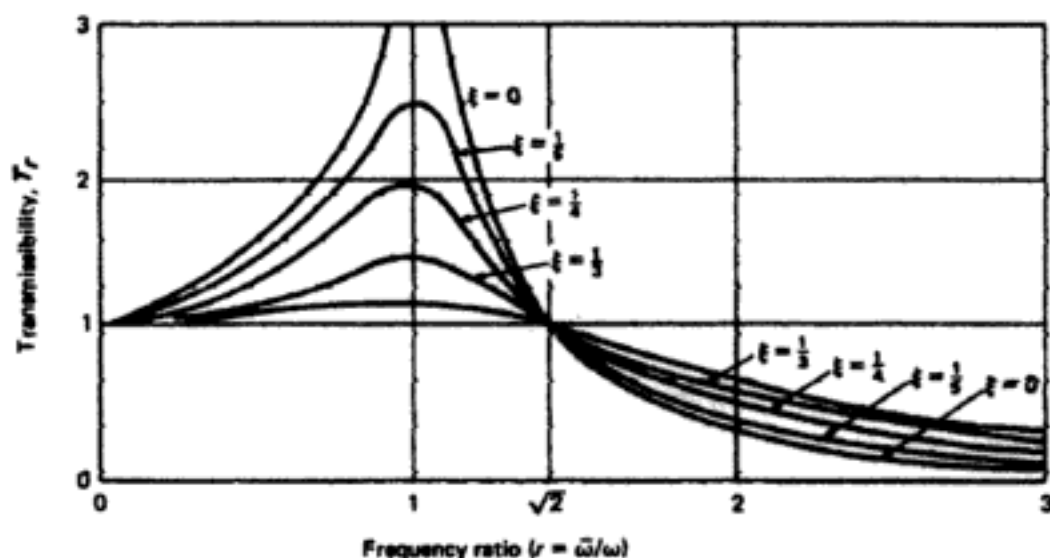


Fig. 3.13 Transmissibility versus frequency ratio for vibration isolation

Illustrative Example 3.5

A delicate instrument weighing 100 lb is to be mounted on a rubber pad to the floor of a test laboratory where the vertical acceleration is 0.1 g at a frequency of $f = 10$ cps. It has been determined experimentally that the ratio of the stiffness, k , to the damping coefficient, c , is equal to 100 (1/sec) for the type of rubber pad material used in the isolation. What is the stiffness of the isolation required to reduce to 0.01g the acceleration transmitted to the instrument?

Solution:

Setting the acceleration transmissibility given by eq.(3.48) equal to $0.01g / 0.1g = 0.1$, we have

$$T_r = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1-r)^2 + (2r\xi)^2}} = 0.1 \quad (a)$$

Beginning with an assumed value $\xi = 0.10$ for the damping ratio and squaring both sides of eq.(a):

$$\frac{1 + 0.04r^2}{(1-r^2)^2 + 0.04r^2} = 0.01$$

Then solving this resulting quadratic equation yields

$$\begin{aligned}
 r^2 &= 13.346 \\
 r &= \bar{\omega} / \omega = 3.653 \\
 \bar{\omega} &= 2\pi f = 2\pi 10 = 62.83 \text{ rad/sec} \\
 \omega &= \bar{\omega} / r = 17.20 \text{ rad/sec} \\
 m &= 100/386 = 0.259 \text{ lb}\cdot\text{sec}^2/\text{in} \\
 k &= m\omega^2 = 0.259 \times 17.20^2 = 76.64 \text{ lb/in}
 \end{aligned}$$

Now, we check the value of damping contained in the rubber spring:

$$k / c = 100$$

or

$$c = k / 100 = 76.22 / 100 = 0.766 \text{ (lb}\cdot\text{sec/in)}$$

Critical damping:

$$c_{cr} = 2\sqrt{km} = 2\sqrt{76.64 \times 0.259} = 8.91 \text{ (lb}\cdot\text{sec/in)}$$

Then, the calculated damping ratio is

$$\xi = c / c_{cr} = 0.766 / 8.91 = 0.086$$

which is somewhat less than the assumed value $\xi = 0.10$. If desired, an iterative cycle could be performed introducing $\xi = 0.086$ in eq.(a) and repeating the calculations.

3.7.2 Relative Motion

Equation (3.43) provides the absolute response of the damped oscillator to a harmonic motion of its base. Alternatively, we can solve the differential equation (3.38) in terms of the relative motion between the mass m and the support given by

$$u_r = u - u_s \quad (3.49)$$

which substituted into eq.(3.38) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff} \sin \bar{\omega} t \quad (3.50)$$

where $F_{eff} = -m\ddot{u}_s$ may be interpreted as the amplitude of the effective force acting on the mass of the oscillator with the displacement indicated by coordinate u_r . Using eq.(3.37) to obtain \ddot{u}_s and substituting in eq.(3.50) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = mu_0 \bar{\omega}^2 \sin \bar{\omega} t \quad (3.51)$$

Again, eq.(3.51) is of the same form as eq.(3.10) with $F_0 = mu_0\bar{\omega}^2$. Then, from eqs.(3.19) and (3.20), the steady-state response in terms of relative motion is given by

$$\frac{u_r(t)}{u_0} = \frac{r^2 \sin(\bar{\omega}t - \theta)}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \quad (3.52)$$

where θ is given in eq.(3.21),

$$r^2 = \frac{\bar{\omega}^2}{\omega^2} \quad \text{and} \quad \omega^2 = \frac{k}{m}.$$

Illustrative Example 3.6

If the frame of Illustrative Example 3.2 (Fig. 3.7) is subjected to a sinusoidal ground motion $u_s(t) = 0.2 \sin 5.3t$, determine: (a) the transmissibility of motion to the girder, (b) the maximum shearing force in the supporting columns, and (c) maximum stresses in the columns.

Solution:

- a) The parameters for this system are calculated in Illustrative Example 3.2 as

$$\begin{aligned} k &= 2136 \text{ lb/in} \\ \xi &= 0.05 \\ u_0 &= 0.2 \text{ in} \\ u_{st} &= 0.0936 \text{ in} \\ \omega &= 7.41 \text{ rad/sec} \\ \bar{\omega} &= 5.3 \text{ rad/sec} \\ r &= 0.715 \end{aligned}$$

The transmissibility from eq.(3.45) is

$$T_r = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 2.1 \quad (\text{Ans.})$$

- b) The maximum relative amplitude of the displacement U_r is from eq.(3.52)

$$U_r = \frac{u_0 r^2}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.206 \text{ in}$$

Then the maximum shear force in each column is

$$V_{\max} = \frac{kU_r}{2} = 219.8 \text{ lb} \quad (\text{Ans.})$$

- c) The maximum bending moment

$$M_{\max} = V_{\max} L = 39,507 \text{ lb-in}$$

and the corresponding stress

$$\sigma_{\max} = \frac{M_{\max}}{I/c} = \frac{39,567}{17} = 2327 \text{ psi} \quad (\text{Ans.})$$

in which I/c is the section modulus.

Illustrative Example 3.7

A machine having a total weight of 1800 lb, including its foundation, is to be isolated from the vibration of the ground, which is $f = 22.8$ cps, due to other machines operating nearby. Determine the stiffness of a rubber isolation spring to limit the transmitted vibration to 1/10: (a) neglect damping and (b) consider damping given by the expression $c = k/170$ obtained experimentally [units of c (lb.sec/in) and of k (lb/in)].

Solution:

- a) $\xi = 0$

By eq.(3.45) with $\xi = 0$

$$T_r = \frac{U}{u_0} = \frac{1}{\pm(1-r^2)} = 0.1$$

$$-1+r^2 = 10 \quad r^2 = 11$$

$$r = 3.3166 = \frac{\bar{\omega}}{\omega}$$

$$\bar{\omega} = 2\pi f = 143.24 \text{ rad/sec}$$

$$m = \frac{1800}{386} = 4.663 \text{ lb-sec}^2/\text{in}$$

$$\omega = \frac{\bar{\omega}}{r} = \frac{143.24}{3.3166} = 43.188 \text{ rad/sec}$$

$$\text{Then,} \quad k = \omega^2 m = (43.188)^2 (4.663) = 8698 \text{ lb/in} \quad (\text{Ans.})$$

- b) Assume $\xi = 0.10$

c) Squaring eq. (3.45) and substituting $\xi = 0.10$ gives

$$T_r^2 = \frac{1 + (0.2r)^2}{(1 - r^2)^2 + (0.2r)^2} = \left(\frac{1}{10}\right)^2$$

which results in the following quadratic equation in r^2 :

$$r^4 - 5.96r^2 - 99 = 0$$

$$r^2 = 13.366 \quad r = \frac{\bar{\omega}}{\omega} = 3.65$$

and

$$\omega = \frac{\bar{\omega}}{r} = \frac{143.24}{3.65} = 39.244 \text{ (rad/sec)}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{4.663}} = 39.244 \text{ (rad/sec)}$$

giving

$$k = 7181.3 \text{ lb/in}$$

The damping is then determined from

$$c = k / 170 = 7181.3 / 170$$

as

$$c = 42.24 \text{ lb-sec/in}$$

and the damping ratio as

$$c_{cr} = 2\sqrt{km} = 2\sqrt{(7181.3)(4.663)} = 348.34 \text{ (lb-sec/in)}$$

$$\xi = \frac{42.24}{348.34} = 0.1212 = 12\%$$

which is slightly higher than the value $\xi = 0.10$, initially assumed. Now, repeating the calculations for $\xi = 0.11$ gives

$$r^4 - 6.7916 - 99 = 0$$

$$r^2 = 13.909 \quad r = \frac{\bar{\omega}}{\omega} = 3.73$$

$$\omega = \frac{143.24}{3.73} = 38.40 = \sqrt{\frac{k}{m}}$$

$$k = 6876 \text{ lb/in} \quad \text{and} \quad c = \frac{k}{170} = 40.45 \text{ lb-sec/in} \quad (\text{Ans.})$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{6876 \times 4.663} = 358.14 \text{ lb-sec/in}$$

and

$$\xi = \frac{40.45}{358.14} = 0.113 = 11\%$$

This calculated value, $\xi = 11\%$ for the damping ratio, agrees with the last value tried. Therefore, the required spring constant for the damped isolation is $k = 6876$ lb/in as calculated above.

3.8 Force Transmitted to the Foundation

In the preceding section, we determined the response of the structure to a harmonic motion of its foundation. In this section we shall consider a similar problem of vibration isolation; the problem now, however, is to find the force transmitted to the foundation. Consider again the damped oscillator with a harmonic force $F(t) = F_0 \sin \bar{\omega} t$ acting on its mass as shown in Fig. 3.2. the differential equation of motion is

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \bar{\omega} t$$

with the steady-state solution, eq.(3.19),

$$u = U \sin(\bar{\omega} t - \theta)$$

where U and θ , are given, respectively, by eqs.(3.20) and (3.21) as

$$U = \frac{F_0 / k}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.53)$$

and

$$\tan \theta = \frac{2\xi r}{1 - r^2}$$

The force transmitted to the support through the spring is ku and through the damping element is $c\dot{u}$. Hence the total force transmitted F_T is

$$F_T = ku + c\dot{u} \quad (3.54)$$

Differentiating eq.(3.19) and substituting in eq.(3.54) yields

$$F_T = U[k \sin(\bar{\omega} t - \theta) + c\bar{\omega} \cos(\bar{\omega} t - \theta)]$$

or

$$F_T = U\sqrt{k^2 + c^2\bar{\omega}^2} \sin(\bar{\omega} t - \theta + \beta) \quad (3.55)$$

$$F_T = Uk\sqrt{1 + (2r\xi)^2} \sin(\bar{\omega}t - \phi) \quad (3.56)$$

in which

$$\tan \beta = \frac{c\bar{\omega}}{k} = 2\xi r \quad (3.57)$$

and

$$\phi = \theta - \beta \quad (3.58)$$

Then, from eqs.(3.53) and (3.56), the maximum force A_T transmitted to the foundation is

$$A_T = F_0 \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.59)$$

In this case, the transmissibility T_r is defined as the ratio between the amplitude of the force transmitted to the foundation and the amplitude of the applied force. Hence from eq.(3.59)

$$T_r = \frac{A_T}{F_0} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.60)$$

It is interesting to note that, both, the transmissibility of motion from the foundation to the structure, eq.(3.45), and the transmissibility of the force from the structure to the foundation, eq.(3.60), are given by exactly the same function. Hence the curves of transmissibility in Fig.3.13 represent either type of transmissibility. An expression for the total phase angle ϕ in eq.(3.56) may be determined by taking the tangent function to both members of eq.(3.58), so that

$$\tan \phi = \frac{\tan \theta - \tan \beta}{1 + \tan \theta \tan \beta}$$

Then, the substitution of $\tan \theta$ and $\tan \beta$, respectively, from eqs.(3.21) and (3.57) results in

$$\tan \phi = \frac{2\xi r^3}{1 - r^2 + 4\xi^2 r^2} \quad (3.61)$$

Illustrative Example 3.8

A machine of weight $W = 3860$ lb is mounted on a simple supported steel beam as shown in Fig. 3.15(a). A piston that moves up and down in the machine produces a harmonic force of magnitude $F_0 = 7,000$ lb at a frequency of $\bar{\omega} = 60$ rad/sec. Neglecting the weight of the beam and assuming 10% of the critical damping,

determine: (a) the amplitude of the motion of the machine, (b) the force transmitted to the beam supports, and (c) the corresponding phase angle.

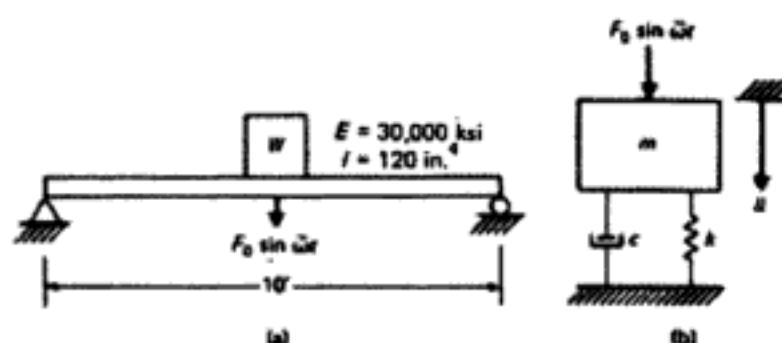


Fig. 3.14 (a) Beam-machine system for Illustrative Example 3.8.
(b) Analytical Model.

Solution:

The damped oscillator in Fig.3.14(b) is used to model the system. The following parameters are calculated:

$$k = \frac{48EI}{L^3} = 10^5 \text{ lb/in}$$

$$\omega = \sqrt{\frac{k}{m}} = 100 \text{ rad/sec}$$

$$\xi = 0.1$$

$$r = \frac{\bar{\omega}}{\omega} = 0.6$$

$$u_{st} = \frac{F_0}{k} = 0.07 \text{ in}$$

a) From eq.(3.20), the amplitude of motion is

$$U = \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.1075 \text{ in} \quad (\text{Ans.})$$

with a phase angle from eq.(3.21)

$$\theta = \tan^{-1} \frac{2r\xi}{1-r^2} = 10.6^\circ$$

b) From eq.(3.60), the transmissibility is

$$T_r = \frac{A_T}{F_0} = \sqrt{\frac{1 + (2r\xi)^2}{(1 - r^2)^2 + (2r\xi)^2}} = 1.547$$

Hence the amplitude of the force transmitted to the foundation is

$$A_T = F_0 T_r = 10,827 \text{ lb} \quad (\text{Ans.})$$

c) The corresponding phase angle from eq.(3.61) is

$$\phi = \tan^{-1} \frac{2\xi r^3}{1 - r^2 + (2r\xi)^2} = 3.78^\circ \quad (\text{Ans.})$$

3.9 Seismic Instruments

When a system of the type shown in Fig. 3.15 is used for the purpose of vibration measurement, the relative displacement between the mass and the base is recorded. Such an instrument is called a seismograph and it can be designed to measure either the displacement or the acceleration of the base. The peak relative response U/u_0 of the seismograph depicted in Fig. 3.15, for harmonic motion of the base, is given from eq.(3.52) by

$$\frac{U_r}{u_0} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad (3.62)$$

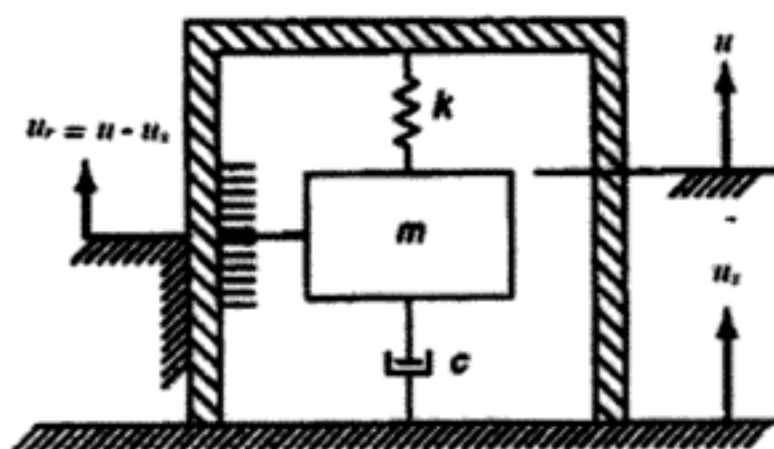


Fig. 3.15 Model of a seismograph.

A plot of this equation as a function of the frequency ratio and damping ratio is shown in Fig. 3.16. It may be seen from this figure that the response is essentially constant for frequency ratios $r > 1$ and damping ratio $\xi = 0.5$. Consequently, the response of a properly damped instrument of this type is essentially proportional to the base-displacement amplitude for high frequencies of motion of the base. The

instrument will thus serve as a displacement meter for measuring such motions. The range of applicability of the instrument is increased by reducing the natural frequency, i.e., by reducing the spring stiffness or increasing the mass.

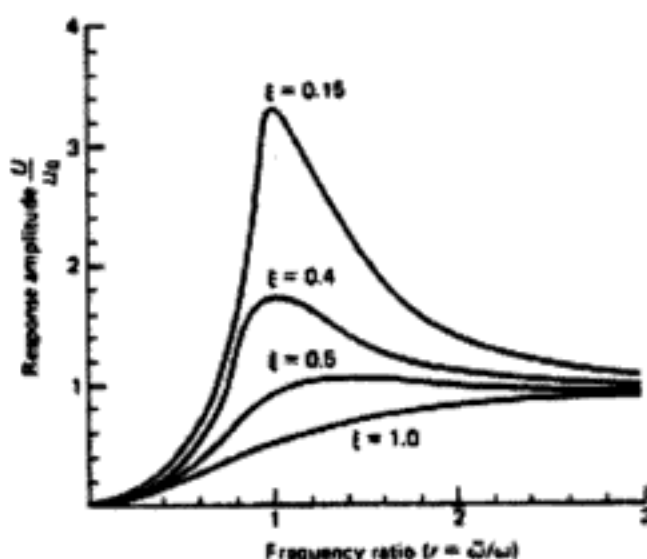


Fig. 3.16 Response of seismograph to harmonic motion of the base.

Now consider the response of the same instrument to a harmonic acceleration of the base $\ddot{u}_s = \ddot{u}_0 \sin \bar{\omega} t$. The equation of motion of this system is obtained from eq.(3.50) as

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_0 \sin \bar{\omega} t \quad (3.63)$$

The steady-state response of this system expressed as the dynamic magnification factor is then given from eq.(3.23) by

$$D = \frac{U}{m\ddot{u}_0 / k} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \quad (3.64)$$

This equation is represented graphically in Fig. 3.3. In this case, it can be seen from this figure that for a damping ratio $\xi = 0.7$, the value of the response is nearly constant in the frequency range $0 < r < 0.6$. Thus, it is clear from eq.(3.64) that the response indicated by this instrument will be directly proportional to the base-acceleration amplitude for frequencies up to about six-tenths of the natural frequency. Its range of applicability will be increased by increasing the natural frequency, that is, by increasing the stiffness of the spring or by decreasing the mass of the oscillator. Such an instrument is an accelerometer.

3.10 Response of One-Degree-of-Freedom System to Harmonic Loading Using SAP2000

Illustrative Example 3.9

The steel frame shown in Fig. 3.17 supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 5.3t$ (lb). Assume 5% of critical damping and determine: (a) The steady-state amplitude of vibration, and (b) The maximum dynamic stress in the columns. Assume the girder is rigid. (This is the same structure in Illustrative Example 3.2.)

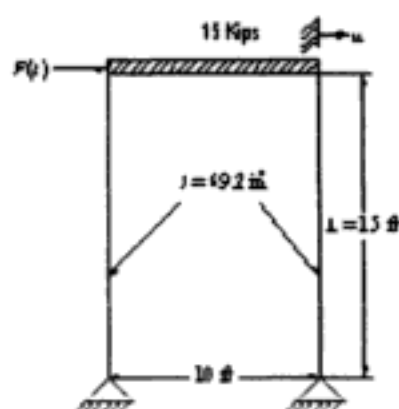


Fig. 3.17 (a) Diagram of the frame of Illustrative Example 3.9. (b) Analytical Model.

Solution:

The parameters for this structure were previously calculated in Illustrative Example 3.2 as:

Mass: $m = 38.86 \text{ (lb}\cdot\text{sec}^2/\text{in)}$

Damping: $\xi = 0.05$

Amplitude Harmonic Force: $F_0 = 200 \text{ (lb)}$

Force time function: $F(t) = \sin 5.3t$

Period (of the force) $5.3T = 2\pi$. Therefore, $T = 1.1855 \text{ sec}$

Time Step (Select 20 steps): $\Delta t = \frac{T}{20} = 0.059275 \text{ sec}$

Table 3.1 Harmonic Function

t (sec)	$F(t) = \sin 5.3t$
0	0
0.0593	0.3090
0.1186	0.5878
0.1778	0.8090
0.2371	0.9511
0.2964	1
0.3557	0.9511
0.4149	0.8090
0.4742	0.5878
0.5335	0.3090
0.5928	0
0.6520	-0.3090
0.7113	-0.5878
0.7706	-0.8090
0.8299	-0.9511
0.8891	-1
0.9484	-0.9511
1.0077	-0.8090
1.0670	-0.5878
1.1262	-0.3090
1.1855	0

Enter the following commands in SAP2000:

Begin: Install SAP2000 from the CD-ROM enclosed with this volume. Follow the installation instructions and the prompts in the software. After successful installation of the program, begin this example by opening the program.

Enter: "OK" to disable the "Tip of the Day" screen.
(Check OPTIONS>SHOW TIPS AT STARTUP to permanently disable this option).

Hint: Maximize both screens for full views of all windows.

Select: In the lower right-hand corner of the screen, use the drop-down menu to select "lb-in".

Model: From the Main Menu bar enter:
FILE>NEW MODEL FROM TEMPLATE
Select Frame icon button
Enter: Number of Stories = 1

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Number of Bays = 1
Story Height = 180
Bay Width = 120
Then OK.

Select: Click on X-Z icon to select this plane.
Maximize this view and use the PAN icon to center the figure on the screen.
To decrease size of graphic click on the scaling icon with the minus sign in the center.

Label: VIEW>SET ELEMENTS
Check Joint Labels and Frame Labels. Then OK. (Accept remaining default settings)

Boundary: Select joints 1 and 3 by clicking on them.
ASSIGN>JOINT>RESTRAINTS
Select: Restraints only in X and Z directions. Then OK.

Mass: Select joint 2 by clicking on the joint.
ASSIGN>JOINT>MASSES
Enter in Direction 1: 38.86. Then OK.

Material: DEFINE>MATERIAL
Select STEEL
Click on Modify/Show Material
Enter: E = 30E6
Mass per Unit Volume = 0
Weight per Unit Volume = 0
Then OK, OK.

Sections: DEFINE>FRAME SECTIONS
Click on "Add I/Wide Flange".
Select "Add General" (section)
Accept default frame section name FSEC2
In Property Data window enter:
Cross-Sectional Area = 10000
Moment of Inertia about 3 Axis = 69.2
Shear area in 2 direction = 0
Accept all other default entries. Then OK.
In the "General Section" window select Material = STEEL
Accept all other default entries. Then OK.

Click on "Add I/Wide Flange".
Select "Add General" (section)
Accept default frame section name FSEC3
In Property Data window enter:

Cross-Sectional Area = 1.0E6

Moment of Inertia about 3 Axis = 1.0E6

Shear in 2 Direction = 0

Accept all other default entries

In the "General Section" window select Material = STEEL

Accept all other default entries. Then OK, OK.

Load: DEFINE>STATIC LOAD CASES

Select: Click down arrow and select Load = LOAD1

Click down arrow and select Type = LIVE

Enter: Self-weight-multiplier = 0

Click on "Add New Load"

Then OK.

Assign: Click on joint 2

ASSIGN>JOINT STATIC LOADS>FORCES

Force Global X = 200.

Click "Add to existing Loads". Then OK.

Assign: Select the two columns by clicking on them

ASSIGN>FRAME SECTIONS

Select FSEC2. Then OK

Select the horizontal member by clicking on it

ASSIGN>FRAME SECTIONS

Select FSEC3. Then OK.

Load function: DEFINE>TIME HISTORY FUNCTIONS

Click on "Add New Function"

Accept the default name FUNC1

Under "Define Function" enter the Time History values for the function

$F_1 = \sin 5.3t$ calculated in Table 3.1.

Enter each line of data, then press the ADD button. Then OK, OK.

History Cases: DEFINE>TIME HISTORY CASES

Click on "Add New History"

Accept the default name HIST1

Select analysis type "Periodic"

In Modal Damping select "Modify/Show"

Enter 0.05 in Damping for all Modes.

Time steps = 20

Output time step size = 0.0593

Check mark on "Envelopes"

Under "Load Assignments"

Load – Click down arrow and select "load 1"

Function – Select FUNC1

Scale Factor – select 1

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Accept zero default values for last two columns.
Click the ADD button. Then OK, OK, OK.

Analyze: ANALYZE>SET OPTIONS

Check only UX and RY for Degree-of-Freedom (DOF's)
Check Dynamic Analysis
Set Dynamic Parameters
Accept Number of Modes = 1. Then OK, OK.

Analyze: ANALYZE>RUN

Enter filename EXD 3.9 then Save
Enter OK when the analysis is completed.

Input Tables:

FILE>PRINT INPUT TABLES

Click "Print to File" and accept the default name for the file:
"C:\SAP2000\EXD 3.9.txt". Enter OK.

Table 3.2 Edited Input Tables for Illustrative Example 3.9. (Units: lb-in)

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	LIVE	0.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF STEPS	TIME STEP INCREMENT
HIST1	PERIODIC	20	0.05928

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-60.00000	0.00000	0.00000	1 1 1 0 0 0
2	-60.00000	0.00000	180.00000	0 0 0 0 0 0
3	60.00000	0.00000	0.00000	1 1 1 0 0 0
4	60.00000	0.00000	180.00000	0 0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
2	38.860	0.000	0.000	0.000	0.000	0.000

FRAME ELEMENT DATA

FRAME	JNT1	JNT2	SECT	ANGLE	RLSES	SGMNTS	R1	R2	FACTOR	LENGTH
1	1	2	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
2	3	4	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
3	2	4	FSEC3	0.000	000000	4	0.000	0.000	1.000	120.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
2	200.000	0.000	0.000	0.000	0.000	0.000

Output Tables:

FILE>PRINT OUTPUT TABLES

Click on Print to File and on "Append". Then OK.

Note: Use Notepad or another editing program to edit and print the tables in C:/SAP2000/EXD 3.9.txt

Tables 3.2 and 3.3 contain, respectively, the edited Input and the Output Tables for Illustrative Example 3.9.

Table 3.3 Edited Output Table for Illustrative Example 3.9. (Units: lb-in)

JOINT DISPLACEMENTS

JOINT	LOAD	UX	UY	UZ	RX	RY	RZ
1	LOAD1	0.0000	0.0000	0.0000	0.0000	7.804E-04	0.0000
2	LOAD1	0.0936	0.0000	0.0000	0.0000	0.0000	0.0000
3	LOAD1	0.0000	0.0000	0.0000	0.0000	7.804E-04	0.0000
4	LOAD1	0.0936	0.0000	0.0000	0.0000	0.0000	0.0000

FRAME ELEMENT FORCES

FRAME	LOAD	LOC	P	V2	V3	T	M2	M3
1	LOAD1	0.00	0.00	100.00	0.00	0.00	0.00	0.00
		90.00	0.00	100.00	0.00	0.00	0.00	- 9000.00
		180.00	0.00	100.00	0.00	0.00	0.00	- 18000.00
2	LOAD1	0.00	0.00	100.00	0.00	0.00	0.00	0.00
		90.00	0.00	100.00	0.00	0.00	0.00	- 9000.00
		180.00	0.00	100.00	0.00	0.00	0.00	- 18000.00
3	LOAD1	0.00 - 100.00		300.00	0.00	0.00	0.00	18000.00
		30.00 - 100.00		300.00	0.00	0.00	0.00	9000.00
		60.00 - 100.00		300.00	0.00	0.00	0.00	3.844E-05
		90.00 - 100.00		300.00	0.00	0.00	0.00	- 9000.00
		120.00 - 100.00		300.00	0.00	0.00	0.00	- 18000.00

Display Max and Min Displacements:

DISPLAY>SET OUTPUT TABLE MODE

Select: HIST1 History. Then OK.

Right click on joint 2 to display Maximum and Minimum values for the displacements on joint 2:

Joint 2 HIST1 MAX: U1 = 0.1858

Joint 2 HIST1 MAX U1 = -0.1858

Close this screen by clicking the "X"

Display Time History Displacements:

Select joint 2 by clicking on it

DISPLAY>SHOW TIME HISTORY TRACES

Accept default HIST1

Click on Joint 2 and on ADD button to move joint 2 to Plot Functions

Accept time range from 0 to 1.1855.

Axis Labels:

Horizontal: Time (sec)

Vertical: Displacement Joint 2 (in)

Click on Display

(The screen will show the displacement-time function for joint 2)

Plot Displacement:

FILE>PRINT GRAPHICS

(Fig 3.18 shows the Displ.-Time plot for joint 2 of Illustrative Example 3.9)

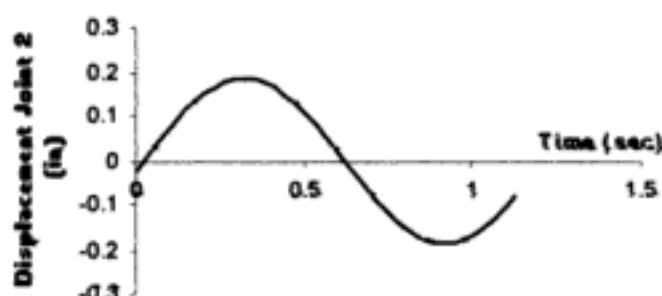


Fig. 3.18 Displacement Function at joint 2 of Illustrative Example 3.9 (in)

Illustrative Example 3.10

The steel frame In Illustrative Example 3.9 is now subjected to an acceleration at its base given by the function $F(t) = 0.3 \sin 5.3t$ in/sec². Use SAP2000 to determine the response.

Illustrative Example 3.10

The steel frame in Illustrative Example 3.9 is now subjected to an acceleration at its base given by the function $F(t) = 0.3 \sin 5.3t$ in/sec². Use SAP2000 to determine the response.

Solution:

This structure is modeled for dynamic analysis by the damped oscillator already shown in Fig. 3.2 (a). The parameters previously calculated in Illustrative Example 3.2 are the following:

Spring Constant	$k = 2136$ (lb/in)
Mass:	$m = 38.86$ (lb.sec ² /in)
Damping:	$\xi = 0.05$
Amplitude Harmonic Acceleration:	$a_0 = 0.3$ (in/sec ²)
Harmonic Function:	$A_t = \sin 5.3t$
Period:	$5.3T = 2\pi$, therefore $T = 1.1855$ sec
Time Step (20 steps):	$\Delta t = \frac{T}{20} = 0.059275$ sec
Harmonic Function Table:	Given in Table 3.1 of Illustrative Example 3.9

Enter the following commands in SAP2000:

Begin: Open SAP2000.

Enter: "OK" to disable the Tip of the Day screen.
(Check OPTIONS>SHOW TIPS AT STARTUP to permanently disable this option).

Hint: Maximize both screens for full view of all windows.

Select: In the lower right-hand corner of the screen, use the drop-down menu to select "lb-in".

Model: From the Main Menu bar enter:
FILE>NEW MODEL
Enter: Number of Grid Spaces:
X Direction = 2
Y Direction = 0

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Z Direction = 0

Grid Spaces:

X Direction = 1

Y Direction = 1

Z Direction = 1. Then OK.

Select: Click on X-Z icon to select this plane.
Maximize this view and use the PAN icon to center the figure on the screen.

Draw: From the Main menu enter:
DRAW>ADD SPECIAL JOINT
Click on the origin of coordinates and on the point X = 1. (The grid intersection to the right of the origin)
Press the "Enter" key or ESC to return to normal mode.

Draw: DRAW>DRAW NLINK ELEMENT
Click on joint 1 (at origin) and drag to joint 2. Click on joint 2 to draw the NLINK element. Press the "Enter" key or ESC to return to normal mode.

Properties: DEFINE>NLINK PROPERTIES
Click on "Modify/Show Property"
Check box U1 in Directional properties box.
Click on "Modify/Show" for U1
Enter: Effective Stiffness = 2136
Effective Damping = 0. Then OK, OK, OK.

Label: VIEW>SET ELEMENT
Check Joint Labels. Then OK. (Accept remaining default settings)

Boundary: Select joint 1 by clicking on the joint.
ASSIGN>JOINT>RESTRAINTS
Select: Restraints in all directions. Then OK.

Mass: Select joint 2 by clicking on the joint.
ASSIGN>JOINT>MASSES
Enter in Direction 1: 38.86. Then OK.

Define: DEFINE>TIME HISTORY FUNCTIONS
Click on "Add New Function"
Accept the default name FUNC1

Under **Define Function** enter the Time History values for the function
 $F_1 = \sin 5.3t$ calculated in Table 3.1 of Illustrative Example 3.9.
Enter each line of data. Then press the ADD button. Then OK, OK.

History Cases: DEFINE>TIME HISTORY CASES

Click on "Add New History"

Accept the default name HIST1

Select analysis type "Periodic"

In Modal Damping select "Modify/Show"

Enter 0.05 in Damping for all Modes.

Time steps = 20

Output time step size = 0.0593

Check mark on "Envelopes"

Under "Load Assignments"

Load – Click down arrow and select "acc dir 1"

Function – Select FUNC1

Scale Factor – select 1

Accept zero default values for last two columns.

Click the ADD button. Then OK, OK, OK.

Assign: Click on joint 1

ASSIGN>JOINT STATIC LOADS>FORCES

Force Global X = 0.3

Click "Add to existing Loads". Then OK.

Analyze: ANALYZE>SET OPTIONS

Check only UX for Degree-of-Freedom (DOF's)

Check Dynamic Analysis

Set Dynamic Parameters

Accept Number of Modes = 1. Then OK, OK.

Analyze: ANALYZE>RUN

Enter filename EXD 3.10 then Save

Enter OK when the analysis is complete.

Display Max and Min Displacements:

DISPLAY>SET OUTPUT TABLE MODE

Select: HIST1 History. Then OK.

Right click on joint 2 to display the maximum and minimum displacement values

Joint 2 HIST1 Max: U1 = 0.0369

Joint 2 HIST1 Min: U1 = -0.0369

Input Tables: FILE>PRINT INPUT TABLES

Click "Print to File" and accept the default name for the file.

C:\SAP2000e\EXD 3.10.txt. Then OK.

Output Tables: FILE>PRINT OUTPUT TABLES

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Click on "Print to File" and on "Append". Then OK.

Note: Use Notepad or another editing tool to edit and print the tables in C:/SAP2000/EXD 3.10.txt

Tables 3.4 and 3.5 contain, respectively, the edited Input and Output Tables for Illustrative Example 3.10.

Table 3.4 Edited Input Table for Illustrative Example 3.10 (Units: lb-in)

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	DEAD	1.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	TIME STEP INCREMENT
HIST1	PERIODIC	20	0.05930

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	0.00000	0.00000	0.00000	1 1 1 1 1
2	1.00000	0.00000	0.00000	0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
2	38.860	0.000	0.000	0.000	0.000	0.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
1	0.300	0.000	0.000	0.000	0.000	0.000

NONLINEAR LINK ELEMENT DATA

NLLINK	JNT-1	JNT-2	SECTION	ANGLE	AXIAL-DIR	LENGTH
1	1	2	NLPR1	0.000		1.000

Table 3.5 Edited Output Table for Illustrative Example 3.10 (Units: lb-in)**JOINT DISPLACEMENTS**

JOINT	LOAD	UX	UY	UZ	RX	RY	RZ
1	HIST1 MAX	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	HIST1 MIN	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	HIST1 MAX	0.0361	0.0000	0.0000	0.0000	0.0000	0.0000
2	HIST1 MIN	-0.0361	0.0000	0.0000	0.0000	0.0000	0.0000

JOINT REACTIONS

JOINT	LOAD	F1	F2	F3	M1	M2	M3
1	HIST1 MAX	62.2459	0.0000	0.0000	0.0000	0.0000	0.0000
1	HIST1 MIN	-100.8512	0.0000	0.0000	0.0000	0.0000	0.0000

NLLINK ELEMENT FORCES

NLLINK	LOAD	LOC	P	V2	V3	T	M2	M3
1	HIST1 MAX							
		0.00	100.85	0.00	0.00	0.00	0.00	0.00
		0.00	100.85	0.00	0.00	0.00	0.00	0.00
1	HIST1 MIN							
		0.00	-62.25	0.00	0.00	0.00	0.00	0.00
		0.00	-62.25	0.00	0.00	0.00	0.00	0.00

Display Time History Displacements:

Select joint 2 by clicking on it.

DISPLAY>SHOW TIME HISTORY TRACES

Select HIST1

Click on Joint 2 and then on the ADD button to move to the Plot Functions column. Accept the default time range.

Axis Labels:

HORIZONTAL Time (sec)

VERTICAL Displacement Joint 2 (in)

Click on "Display"

(The screen will show the Displacement-Time Plot for Joint 2 of Illustrative Example 3.10)

Plot Displacement: FILE>PRINT GRAPHICS

(Fig.3.19 shows the Displacement-Time function of Illustrative Example 3.10)

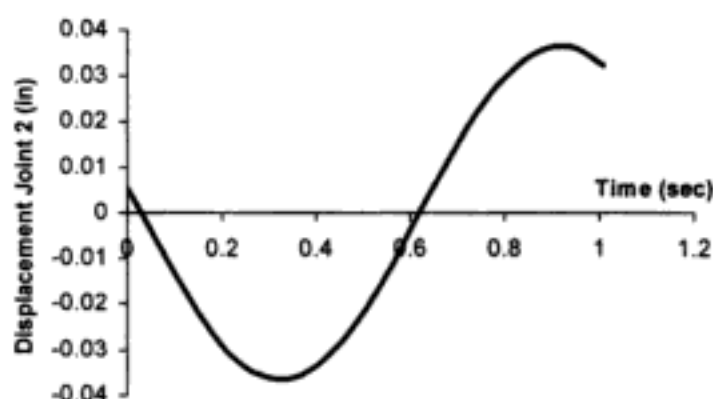


Fig. 3.19 Displacement-Time Function for Joint 2 of Illustrative Example 3.10

3.11 Summary

In this chapter, we have determined the response of a single-degree-of-freedom system subjected to harmonic loading. This type of loading is expressed as a sine, cosine, or as an exponential function and can be easily handled mathematically for the undamped or damped structure. The differential equation of motion for a linear single-degree-of-freedom system is the second-order differential equation

$$m\ddot{y} + c\dot{y} + ky = F_0 \sin \bar{\omega} t \quad (3.10) \text{ repeated}$$

or

$$\ddot{u} + 2\xi\omega \dot{u} + \omega^2 u = \frac{F_0}{m} \sin \bar{\omega} t$$

in which $\bar{\omega}$ is the forced frequency,

$$\xi = \frac{c}{c_{cr}} \quad \text{is the damping ratio}$$

and

$$\omega = \sqrt{\frac{k}{m}} \quad \text{is the natural frequency}$$

The general solution of eq.(3.10) is obtained as the combination of the complementary (transient) and the particular (steady-state) solutions, namely

$$u = \underbrace{e^{-\xi\omega t} (A \cos \omega_D t + B \sin \omega_D t)}_{\text{transient solution}} + \underbrace{\frac{F_0 / k \sin(\bar{\omega} t - \theta)}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}}_{\text{steady state solution}}$$

in which

$$r = \frac{\bar{\omega}}{\omega} \quad \text{is the frequency ratio,}$$

$$\omega_D = \omega \sqrt{1 - r^2} \quad \text{is the damped natural frequency,}$$

$$\theta = \tan^{-1} \left(\frac{2r\xi}{1 - r^2} \right) \quad \text{is the phase angle}$$

and

A and B are constants of integration which can be determined from the initial conditions.

The transient part of the solution vanishes rapidly to zero because of the negative exponential factor, thus leaving only the steady-state solution. Of particular significance is the condition of resonance ($r = \bar{\omega} / \omega = 1$) for which the amplitude of motion become very large for the damped system and tend to become infinity for the undamped system.

The response of the structure to support or foundation motion can be obtained in terms of the absolute motion of the mass or of its relative motion with respect to the support. In this latter case, the equation assumes a simpler and more convenient form, namely

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff}(t) \quad (3.50) \text{ repeated}$$

in which

$$u_r = u - u_s \quad \text{is the relative displacement}$$

and

$$F_{eff}(t) = -m\ddot{u}_s(t) \quad \text{is the effective force}$$

For harmonic excitation of the foundation, the solution of eq.(3.50) in terms of the relative motion is of the same form as the solution if eq.(3.10) in which the force is acting on the mass.

In this chapter, we have also shown that the equivalent damping in the system may be evaluated experimentally either from the peak amplitude or from the bandwidth obtained from a plot of the amplitude-frequency curve when the system is forced to harmonic vibration. Most commonly, equivalent viscous damping is evaluated by equating the experimentally measured energy dissipated in the system during a vibratory cycle at the resonant frequency to the theoretically calculated energy that the system, assumed viscously damped, would dissipate in a cycle. This approach leads to the following expression for the equivalent viscous damping:

$$\xi_{eq} = \frac{1}{4\pi} \frac{E^*}{E_s} \quad (3.35) \text{ repeated}$$

in which

E^* = energy dissipated in the system during a cycle of harmonic vibration at resonance

E_s = strain energy stored at maximum displacement if the system were elastic

r = ratio of forced vibration frequency to the natural frequency of the system

Two related problems of vibrating isolation were discussed in this chapter: (1) the motion transmissibility, that is, the relative motion transmitted from the foundation to the structure; and (2) the force transmissibility which is the relative magnitude of the force transmitted from the structure to the foundation. For both of these problems, the transmissibility is given by

$$T_r = \sqrt{\frac{1 + (2r\xi)^2}{(1 - r^2)^2 + (2r\xi)^2}}$$

3.12 Analytical Problem

Problem 3.1

Demonstrate that during one cycle in harmonic vibration, the work W_F of the external force $F = F_0 \sin \bar{\omega} t$ is equal to the energy E_D dissipated by the damping force as expressed by eq.(3.30).

$$E_D = 2\pi \xi r k U^2 \quad (3.30) \text{ repeated}$$

Solution:

During one cycle, the work of the external force $F = F_0 \sin \bar{\omega} t$ is

$$\begin{aligned} W_F &= \int_0^{2\pi/\bar{\omega}} F_0 \sin \bar{\omega} t \, dy = \int_0^{2\pi/\bar{\omega}} F_0 \sin \bar{\omega} t \, \frac{du}{dt} dt \\ &= \int_0^{2\pi/\bar{\omega}} F_0 \sin \bar{\omega} t \, \dot{u}(t) dt \end{aligned}$$

in which $\dot{u}(t)$ is given by eq.(3.29). Hence,

$$\begin{aligned} W_F &= \int [F_0 \sin \bar{\omega} t][\bar{\omega} U \cos \bar{\omega}(t - \theta)] dt \\ &= \pi F_0 U \sin \theta \end{aligned} \quad (a)$$

To demonstrate that work, W_F , of the exciting force given by eq.(a) is equal to the energy dissipated, E_D , by the viscous force in eq.(3.30), we need to substitute the sine of the phase angle θ into eq.(a).

Thus from eq.(3.21), we have:

$$\begin{aligned}\tan \theta &= \frac{2\xi r}{1-r^2} & (3.21) \text{ repeated} \\ \frac{\sin \theta}{\cos \theta} &= \frac{2\xi r}{1-r^2} \\ \frac{\sin^2 \theta}{\sin^2 \theta + \cos^2 \theta} &= \frac{(2\xi r)^2}{(1-r^2)^2 + (2r\xi)^2}\end{aligned}$$

Therefore

$$\sin \theta = \frac{2\xi r}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$$

Then using eq.(3.20),

$$\begin{aligned}U &= \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} & (3.20) \text{ repeated} \\ \sin \theta &= \frac{2\xi U}{u_{st}}\end{aligned}$$

which substituted in eq.(a) yields

$$\begin{aligned}W_F &= \pi F_0 U^2 \frac{2\xi r}{F_0 / k} \\ W_F &= 2\pi\xi r k U^2 & (b)\end{aligned}$$

Thus, the work of external force, W_F , expressed by eq.(b), is equal to the energy, E_D , dissipated per cycle by the viscous damping force as given by eq.(3.30).

3.13 Problems

Problem 3.2

An electric motor of total weight $W = 1000$ lb is mounted at the center of a simply supported beam as shown in Fig. P3.2). The unbalance in the rotor is $W'e = 1$ lb.in. Determine the steady-state amplitude of vertical motion of the motor for a speed of 900 rpm. Assume that the damping in the system is 10% of the critical damping. Neglect the mass of the supporting beam.

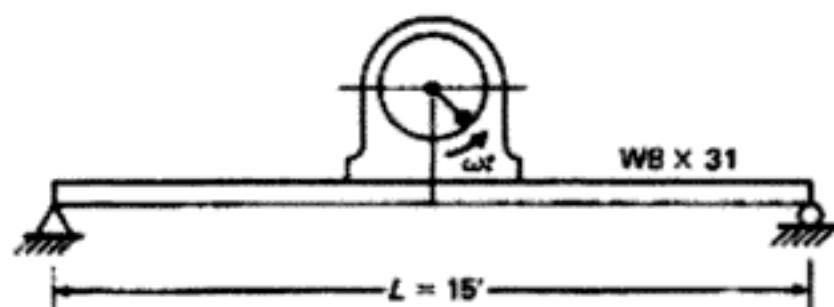


Fig. P3.2

Problem 3.3

Determine the maximum force transmitted to the supports of the beam in Problem 3.2.

Problem 3.4

Determine the steady-state amplitude for the horizontal motion of the steel frame in Fig. P3.4. Assume the horizontal girder to be infinitely rigid and neglect both the mass of the columns and damping.

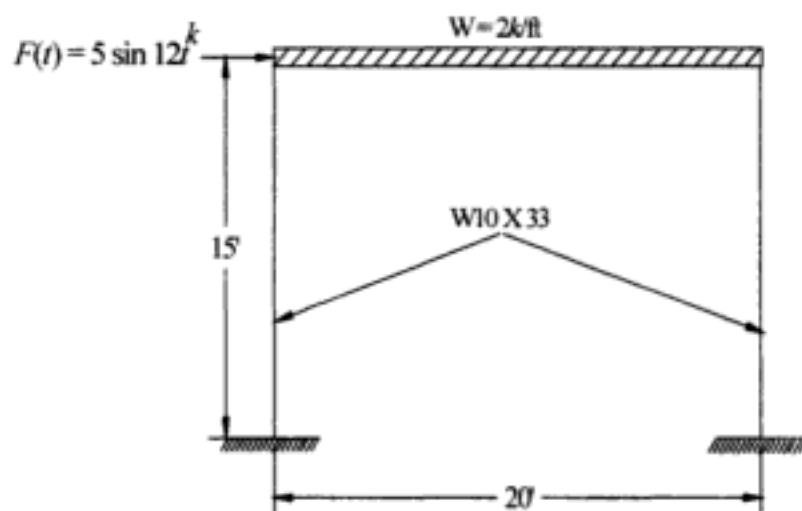


Fig. P3.4

Problem 3.5

Solve for Problem 3.4 assuming that the damping in the system is 8% of the critical damping.

Problem 3.6

For Problem 3.5 determine: (a) the maximum force transmitted to the foundation and (b) the transmissibility.

Problem 3.7

A delicate instrument is to be spring mounted to the floor of a test laboratory where it has been determined that the floor vibrates vertically with harmonic motion of amplitude 0.1 at 10 cps. If the instrument weighs 100 lb, determine the stiffness of the isolation springs required to reduce the vertical motion amplitude of the instrument to 0.01 in. Neglect damping.

Problem 3.8

Consider the water tower shown in Fig. P3.8 which is subjected to ground motion produced by a passing train in the vicinity of the tower. The ground motion is idealized as a harmonic acceleration of the foundation of the tower with an amplitude of $0.1g$ at a frequency of 10cps. Determine the motion of the tower relative to the motion of its foundation. Assume an effective damping coefficient of 10% of the critical damping in the system.

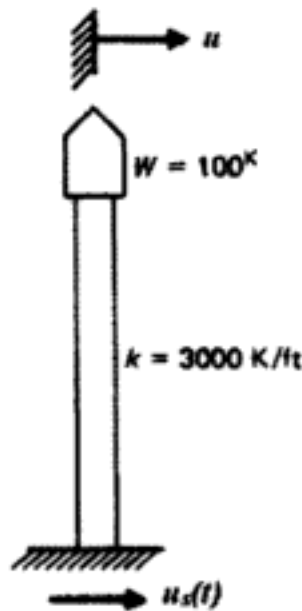


Fig. P3.8

Problem 3.9

Determine the transmissibility in Problem 3.8.

Problem 3.10

An electric motor of total weight $W = 3330$ lb is mounted on a simple supported beam with overhang as shown in Fig. P3.10. The unbalance of the rotor is $W'e = 50$ lb·in. (a) Find the amplitudes of forced vertical vibration of the motor for speeds 800, 1000, and 1200 rpm. (b) Draw a rough plot of the amplitude versus rpm. Assume damping equal to 10% of critical damping.

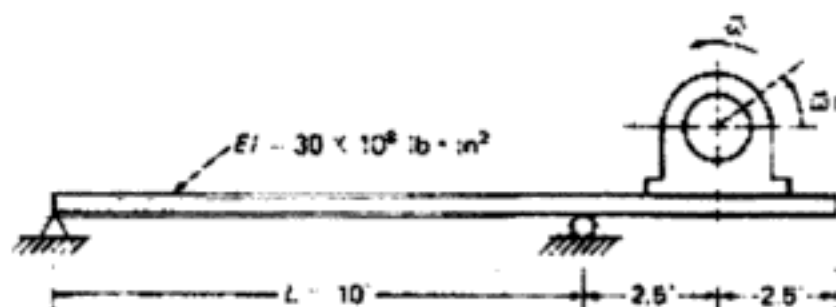


Fig. P3.10

Problem 3.11

Estimate the damping in a single-degree-of-freedom system that is excited by a harmonic force. The peak displacement amplitude at resonance was measured equal to 3 in and equal to 0.2 in at one-tenth of the natural frequency of the system.

Problem 3.12

Determine the damping in a system in which during a vibration test under a harmonic force it was observed that at a frequency 10% higher than the resonant frequency, the displacement amplitude was exactly one-half of the resonant amplitude.

Problem 3.13

Determine the natural frequency, amplitude of vibration, and maximum normal stress in the simple supported beam carrying an engine of weight $W = 30$ kN (Figure P3.13). The engine rotates at 400 rpm and induces a vertical force $F(t) = 8 \sin \bar{\omega} t$. ($E = 210 \times 10^9$ N/m², $I = 8950 \times 10^{-8}$ m⁴, $S = 597 \times 10^6$ m³)

(Problem contributed by Vladimir N. Alekhin and Aleksey A. Antipin of the Urals State University, Russia)

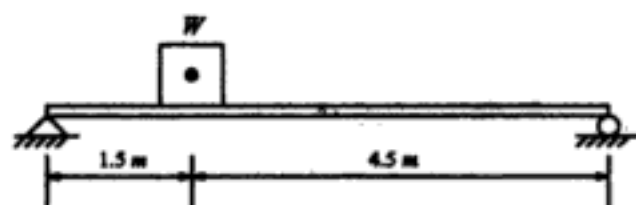


Fig. P3.13

Problem 3.14

A machine of mass m rests on an elastic floor as shown in Fig. P3.14. In order to find the natural frequency of the vertical motion, a mechanical shaker of mass m_s is bolted to the machine and run at various speeds until the resonant frequency f_r is found. Determine the natural frequency f_n of the floor-machine system in terms of f_r and the given data.

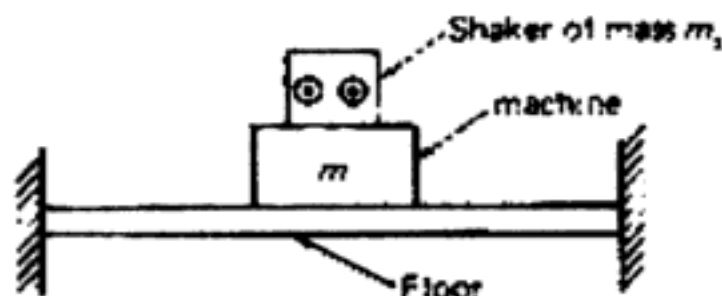


Fig. P3.14

Problem 3.15

Determine the frequency at which the peak amplitude of a damped oscillator will occur. Also, determine the peak amplitude and corresponding phase angle.

Problem 3.16

A structure modeled as a damped spring-mass system (Fig. P3.16) with $mg = 2520$ lb, $k = 89,000$ lb/in, and $c = 112$ lb.in/sec is subjected to a harmonic exciting force. Determine: (a) the natural frequency, (b) the damping ratio, (c) the amplitude of the exciting force when the peak amplitude of the vibrating mass is measured to be 0.37 in, and (d) the amplitude of the exciting force when the amplitude measured is at the peak frequency assumed to be the resonant frequency.

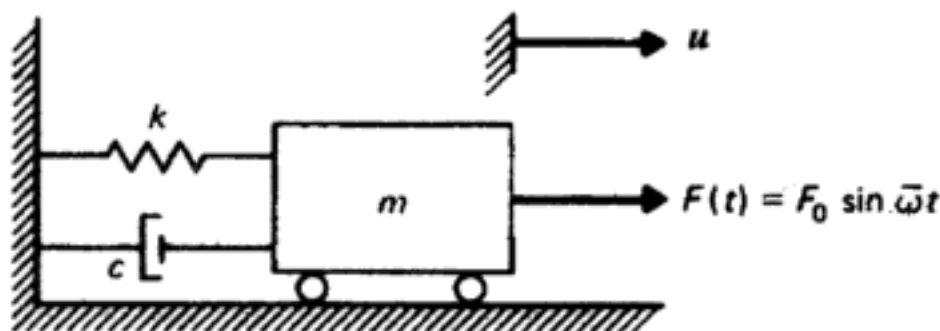


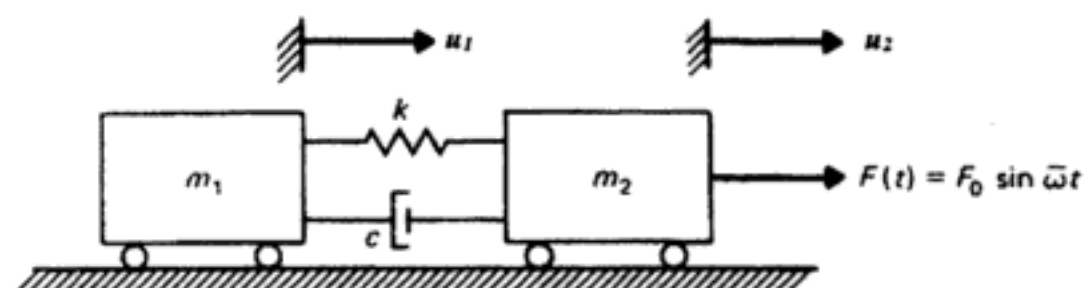
Fig. P3.16

Problem 3.17

A structural system modeled as a damped oscillator is subjected to the harmonic excitation produced by an eccentric rotor. The spring constant k and the mass m are known but not the damping and the amount of unbalance in the rotor. From measured amplitudes U_r at resonance and U_1 at a frequency ratio $r_1 \neq 1$, determine expressions to calculate the damping ratio ξ and the amplitude of the exciting force F_r at resonance.

Problem 3.18

A system is modeled by two vibrating masses m_1 and m_2 interconnected by a spring k and damper element c (Fig. P3.18). For harmonic force $F = F_0 \sin \bar{\omega} t$ acting on mass m_2 determine: (a) equation of motion in terms of the relative motion of the two masses, $u_r = u_2 - u_1$; (b) the steady-state solution of the relative motion.

**Fig. P3.18**

4 Response to General Dynamic Loading

In the preceding chapter we studied the response of a single degree-of-freedom system with harmonic loading. Through this type of loading is important, real structures are often subjected to loads that are not harmonic. In this chapter we shall study the response of the single degree-of-freedom system to a general type of force. We shall see that the response can be obtained in terms of an integral that for some simple load functions can be evaluated analytically. For the general case, however, it will be necessary to resort to a numerical integration procedure.

4.1 Duhamel's Integral – Undamped System

An impulsive loading is a load which is applied during a short duration of time. The corresponding impulse of this type of load is defined as the product of the force and the time of its duration.

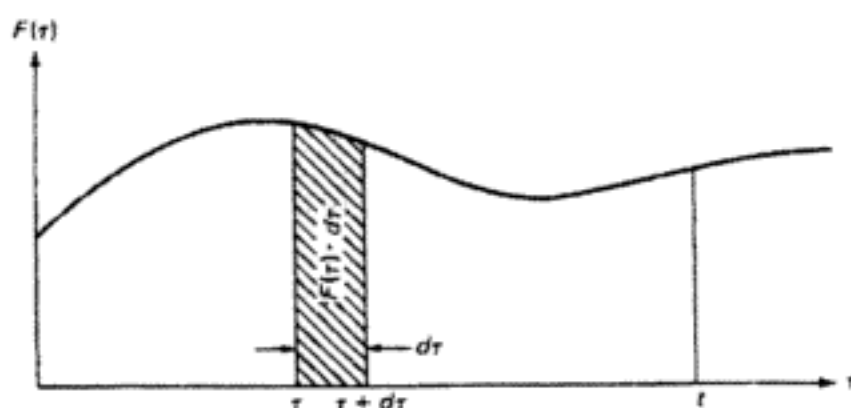


Fig. 4.1 General load function as impulsive loading.

For example, the impulse of the force $F(\tau)$ depicted in Fig. 4.1 at time τ during the time interval $d\tau$ is represented by the shaded area and it is equal to $F(\tau)d\tau$. This impulse acting on a body of mass m produces a change in velocity that can be determined from Newton's Law of Motion, namely

$$m \frac{dv}{d\tau} = F(\tau)$$

Rearrangement yields

$$dv = \frac{F(\tau)d\tau}{m} \quad (4.1)$$

where $F(\tau)d\tau$ is the impulse and dv is the incremental velocity. This incremental velocity may be considered to be an initial velocity of the mass at time τ . Let us now consider this impulse $F(\tau)d\tau$ acting on the structure represented by the undamped oscillator. At time τ the oscillator will experience a change in velocity given by eq.(4.1). This change in velocity is then introduced in eq.(1.20) as the initial velocity v_0 together with the initial displacement $u_0 = 0$ at time τ producing a displacement $du(t)$ at a later time t given by

$$du(t) = \frac{F(\tau)d\tau}{m\omega} \sin \omega(t - \tau) \quad (4.2)$$

The loading function may then be regarded as a series of short impulses at successive incremental times $d\tau$, each producing its own differential response at time t of the form given by eq.(4.2). Therefore, we conclude that the total displacement at time t due to the continuous action of the force $F(\tau)$ is given by the summation or integral of the differential displacements $du(t)$ from time $t = 0$ to time t , that is,

$$u(t) = \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau) d\tau \quad (4.3)$$

The integral in this equation is known as Duhamel's integral. Equation (4.3) represents the total displacement produced by the exciting force $F(\tau)$ acting on the undamped oscillator; it includes both the steady-state and the transient components of the motion corresponding to zero initial conditions, $u_0 = 0$ and $v_0 = 0$. If the function $F(\tau)$ cannot be expressed analytically, the integral of eq. (4.3) can always be evaluated approximately by suitable numerical methods. To include the effect of initial displacement u_0 and initial velocity v_0 at time $t = 0$, it is only necessary to add to eq. (4.3) the solution given by eq.(1.20) for the effects due to the initial conditions. Thus the total displacement of an undamped single degree-of-freedom system with an arbitrary load is given by

$$u(t) = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t + \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau) d\tau \quad (4.4)$$

Applications of eq.(4.4) for some simple forcing functions for which it is simple to obtain the explicit integration of eq.(4.4) are presented below.

4.1.1 Constant Force

Consider the case of a constant force of magnitude F_0 applied suddenly to the undamped oscillator at time $t = 0$ as shown in Fig. 4.20.

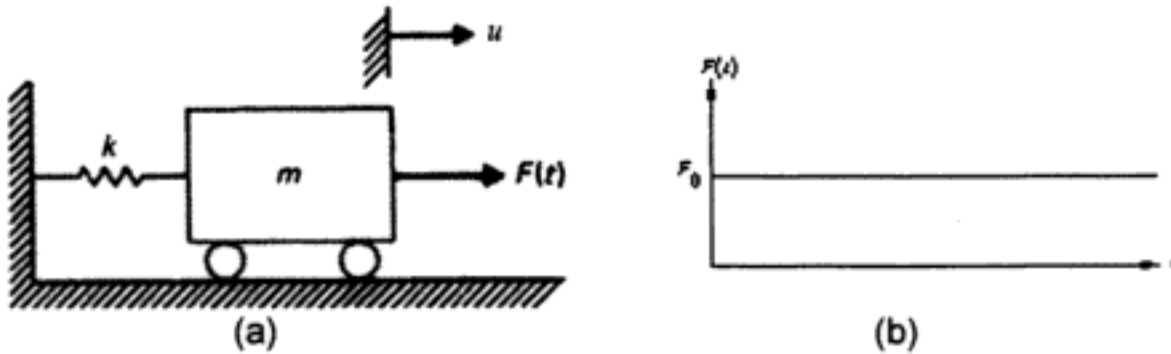


Fig. 4.2 Undamped oscillator acted upon by constant force.

For both initial displacement and initial velocity equal to zero, the application of eq.(4.4) to this case gives

$$u(t) = \frac{1}{m\omega} \int_0^t F_0 \sin \omega(t - \tau) d\tau$$

and the integration yields

$$\begin{aligned} u(t) &= \frac{F_0}{m\omega^2} \left[\cos \omega(t - \tau) \right]_0^t \\ u(t) &= \frac{F_0}{k} (1 - \cos \omega t) = u_{st} (1 - \cos \omega t) \end{aligned} \quad (4.5)$$

where $u_{st} = \frac{F_0}{k}$ is the static displacement due to a force of magnitude F_0 . The response for such a suddenly applied constant load is shown in Fig. 4.3. It will be observed that this solution is very similar to the solution for the free vibration of the undamped oscillator. The major difference is that the coordinate axis t has been shifted by an amount equal to $u_{st} = \frac{F_0}{k}$. Also, it should be noted that the maximum displacement $2u_{st}$ is exactly twice the displacement that the force F_0 would produce if it were applied statically. We have found an elementary but important result; the

maximum displacement of a linear elastic system for a constant force applied suddenly is twice the displacement caused by the same force applied statically (slowly). This result for displacement is also true for the internal forces and stresses in the structure.

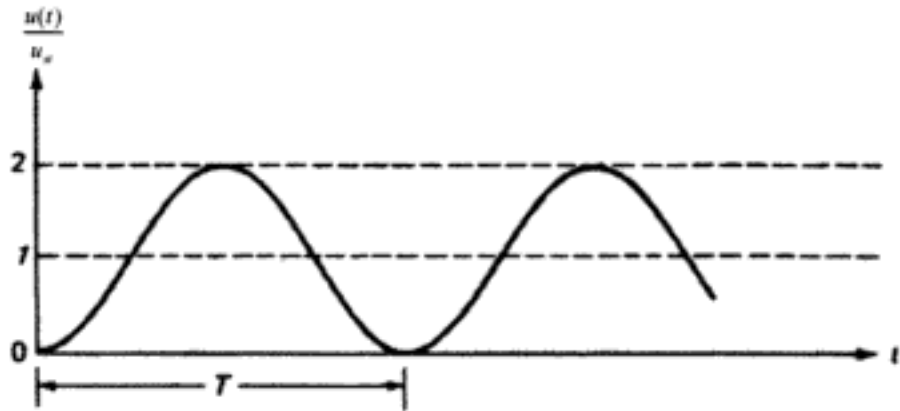


Fig. 4.3 Response of an undamped single degree-of-freedom system to a suddenly applied constant force.

4.1.2 Rectangular Load

Let us consider a second case, that of a constant force F_0 suddenly applied but only during a limited time duration, t_d as shown in Fig. 4.4:

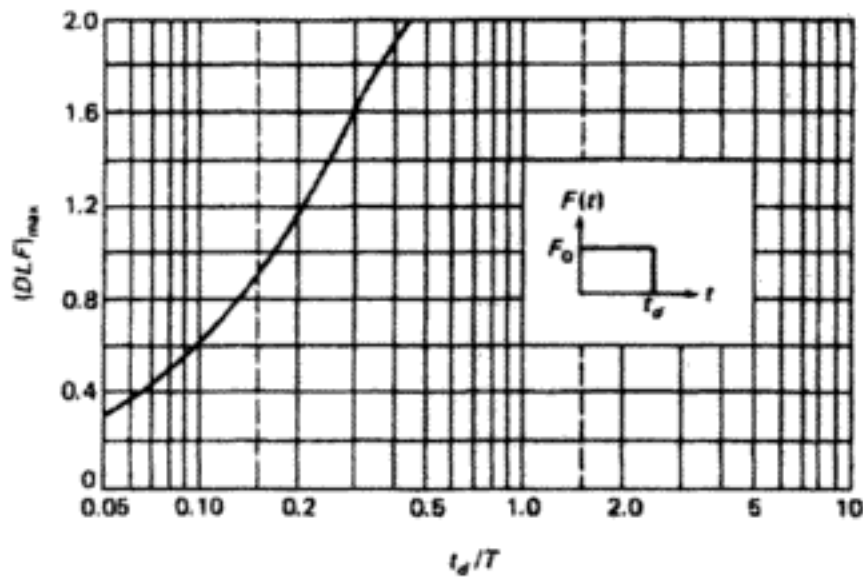


Fig. 4.4 Maximum dynamic load factor for the undamped oscillator acted on by a rectangular force.

Up to the time t_d , the solution given by eq.(4.5) applies and at that time the displacement and velocity are

$$u_d = \frac{F_0}{k} (1 - \cos \omega t_d)$$

and

$$v_d = \frac{F_0}{k} \omega \sin \omega t_d$$

For the response after time t_d we apply eq.(1.20) for free vibration, taking as the initial conditions the displacement and velocity at t_d . After replacing t by $t - t_d$ and u_0 and v_0 for u_d and v_d , respectively, we obtain

$$u(t) = \frac{F_0}{k} (1 - \cos \omega t_d) \cos \omega(t - t_d) + \frac{F_0}{k} \sin \omega t_d \sin \omega(t - t_d)$$

which can be reduced to

$$u(t) = \frac{F_0}{k} \{ \cos \omega(t - t_d) - \cos \omega t \} \quad (4.6)$$

If the dynamic load factor (DLF) is defined as the displacement at any time t divided by the static displacement $u_{st} = \frac{F_0}{k}$, we may write eqs.(4.5) and (4.6) as

$$DLF = 1 - \cos \omega t, \quad t \leq t_d$$

and

$$DLF = \cos \omega(t - t_d) - \cos \omega t, \quad t \geq t_d \quad (4.7)$$

It is often convenient to express time as a dimensionless parameter by simply using the natural period instead of the natural frequency $\left(\omega = \frac{2\pi}{T} \right)$. Hence, eqs. (4.7) may be written as

$$DLF = 1 - \cos 2\pi \frac{t}{T}, \quad t \leq t_d$$

and

$$DLF = \cos 2\pi \left(\frac{t}{T} - \frac{t_d}{T} \right) - \cos 2\pi \frac{t}{T}, \quad t \geq t_d \quad (4.8a)$$

or using the trigonometric identity

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

as

$$DLF = \left(2 \sin \frac{\pi t_d}{T} \right) \sin \left[2\pi \left(\frac{t}{T} - \frac{t_d}{2T} \right) \right], \quad t \geq t_d \quad (4.8b)$$

The use of dimensionless parameters in eqs.(4.8) serves to emphasize the fact that the ratio of duration of the time, that the constant force is applied, to the natural period rather than the actual value of either quantity is the important parameter. The maximum dynamic load factor $(DLF)_{max}$ obtained by maximizing eq.(4.8), is plotted in Fig. 4.4. It is observed from this figure that the maximum dynamic load factor for loads of duration $\left(\frac{t_d}{T} > 0.5 \right)$ is the same as if the load duration had been infinite.

In general, the maximum response occurs during the application of the load, except for loadings of very short duration $\left(\frac{t_d}{T} < 0.4 \right)$. In such cases, the maximum response may occur during the free vibration after the cessation of the load; it is then necessary to extend the loading time for a duration of about one period, in which the magnitude of the load is set equal to zero.

Charts, as shown in Fig. 4.4, which give the maximum response of a single degree-of-freedom system for a given loading function, are called response spectral charts. Response spectral charts for impulsive loads of short duration are often presented for the undamped system. For the short duration of the load, damping does not have a significant effect on the response of the system. The maximum dynamic load factor usually corresponds to the first peak of response and the amount of damping normally found in structures is not sufficient to appreciably decrease this value.

4.1.3 Triangular Load

We consider now a system represented by the undamped oscillator, initially at rest and subjected to a force $F(t)$ that has an initial value F_0 and that decreases linearly to zero at time t_d (Fig. 4.5). The response may be computed by eq.(4.4) in two intervals. For the first interval, $\tau \leq t_d$ the force is given by

$$F(\tau) = F_0 \left(1 - \frac{\tau}{t_d} \right)$$

and the initial conditions by

$$y_0 = 0, \quad \nu_0 = 0$$

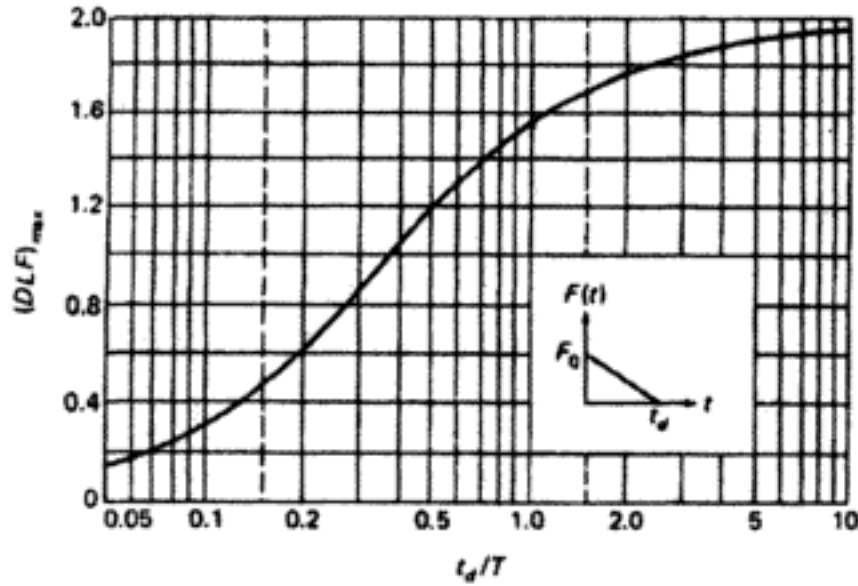


Fig 4.5 Maximum dynamic load factor for the undamped oscillator acted upon by a triangular force.

The substitution of these values in eq.(4.4) and integration gives

$$u = \frac{F_0}{k}(1 - \cos \omega t) + \frac{F_0}{k t_d} \left(\frac{\sin \omega t}{\omega} - t \right) \quad (4.9)$$

or in terms of the dynamic load factor and dimensionless parameters

$$DLF = \frac{u}{u_{st}} = 1 - \cos \left(\frac{2\pi t}{T} \right) + \frac{\sin \left(\frac{2\pi t}{T} \right)}{\left(\frac{2\pi t_d}{T} \right)} - \frac{t}{t_d} \quad (4.10)$$

$$0 \leq t \leq t_d$$

which defines the response before time t_d .

For the second interval ($t \geq t_d$), we obtain from eq.(4.9) the displacement and velocity at time t_d as

$$u_d = \frac{F_0}{k} \left(\frac{\sin \omega t_d}{\omega t_d} - \cos \omega t_d \right)$$

and

$$v_d = \frac{F_0}{k} \left(\omega \sin \omega t_d + \frac{\cos \omega t_d}{t_d} - \frac{1}{t_d} \right) \quad (4.11)$$

These values may be considered as the initial conditions at time $t = t_d$ for this second interval. Replacing in eq.(1.20) t by $t - t_d$ and u_0 and v_0 respectively, by u_d and v_d and noting that $F(\tau) = 0$ in this interval we obtain the response as

$$u = \frac{F_0}{k\omega t_d} \{ \sin \omega t - \sin \omega(t - t_d) \} - \frac{F_0}{k} \cos \omega t$$

and upon dividing by $u_{st} = \frac{F_0}{k}$ gives

$$DLF = \frac{1}{\omega t_D} \{ \sin \omega t - \sin \omega(t - t_d) \} - \cos \omega t \quad (4.12)$$

In terms of the dimensionless time parameter, this last equation may be written as

$$DLF = \frac{1}{\omega T_D} \left\{ \sin 2\pi \frac{t}{T} - \sin 2\pi \left(\frac{t}{T} - \frac{t_d}{T} \right) \right\} - \cos 2\pi \frac{t}{T} \quad (4.13)$$

$t \geq t_d$

The plot of the maximum dynamic load factor as a function of the relative time duration t_d/T for the undamped oscillator is shown in Fig. 4.5. As would be expected, the maximum value of the dynamic load factor approaches 2 as t_d/T becomes large; that is, the effect of the decay of the force is negligible for the time required for the system to reach the maximum peak.

We have studied the response of the undamped oscillator for two simple impulse loadings; the rectangular pulse and the triangular pulse. Extensive charts have been prepared by the U. S. Army Corps of Engineers¹, and are available for a variety of loading pulses. As already mentioned, the evaluation of Duhamel's Integral Method for a general forcing requires the use of numerical methods (See Problems 4.1)

Illustrative Example 4.1

A one-story building, shown in Fig. 4.6, is modeled as a 15-ft high frame with two steel columns fixed at the base and a rigid beam supporting a total weight of $W = 5000$ lb. Each column has a moment of inertia $I = 69.2$ in⁴ and a section modulus

$$S = \frac{I}{c} = 17 \text{ in}^3 \quad (E = 30 \times 10^6 \text{ psi}).$$

¹ U. S. Army Corps of Engineers, *Design of Structures to Resist the Effects of Atomic Weapons*, Manuals 415, 416, 3rd 416, March 15, 1957; Manuals 417 and 419, January 15, 1958; Manuals 418, 420, 421, January 15, 1960.

Determine the maximum response of the frame to a rectangular impulse of amplitude 3000 lb and duration $t_d = 0.1$ sec applied horizontally to the top member of the frame. The response of interest is the horizontal displacement at the top of the frame and the bending stress in the columns.

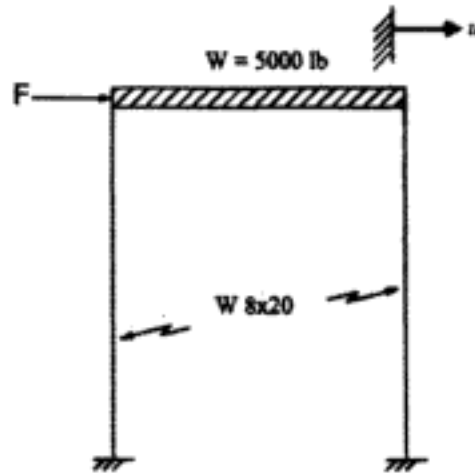


Fig. 4.6 Idealized frame for Illustrative Example 4.1.

Solution:

1. Natural period.

$$k = \frac{12E(2I)}{L^3} = \frac{12 \times 30 \times 10^6 \times 2 \times 69.2}{(15 \times 12)^3} = 8544 \text{ lb/in}$$

$$m = \frac{5000}{386} = 12.9534 \text{ lb} \cdot \text{sec}^2 / \text{in}$$

$$T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{12.9534}{8544}} = 0.2446 \text{ sec}$$

2. Maximum displacement.

$$\frac{t_d}{T} = \frac{0.1}{0.2446} = 0.408$$

$$DLF_{\max} = \frac{u_{\max}}{u_{st}} = 1.9 \quad (\text{from Fig. 4.4})$$

$$u_{st} = \frac{F_0}{k} = \frac{3000}{8544} = 0.3511 \text{ in}$$

$$u_{\max} = (1.9)(0.3511) = 0.667 \text{ in} \quad (\text{Ans.})$$

3. Maximum bending stress.

The bending moment M in the columns is given by

$$M = \frac{6EI}{L^2} u_{\max} = \frac{6 \times 30 \times 10^6 \times 69.2}{(15 \times 12)^2} 0.667 = 256,420 \text{ (lb.in)}$$

and the maximum stress σ_{\max} by

$$\sigma_{\max} = \frac{M}{S} = \frac{256,420}{17} = 15,083 \text{ psi} \quad (\text{Ans.})$$

4.2 Duhamel's Integral-Damped System

The response of a damped system expressed by the Duhamel's integral is obtained in a manner entirely equivalent to the undamped analysis except that the impulse

$F(\tau)d\tau$ producing an initial velocity $dv = F(\tau)\frac{d\tau}{m}$ is substituted into the

corresponding damped free-vibration equation (2.20). Setting $u_0 = 0$, $v_0 = F(\tau)\frac{d\tau}{m}$, and substituting t for $(t - \tau)$ in eq.(2.20), we obtain the differential displacement $du(t)$ at a time t as

$$du(t) = e^{-\xi\omega(t-\tau)} \frac{F(\tau) d\tau}{m\omega_D} \sin \omega_D(t - \tau) \quad (4.14)$$

Summing or integrating these differential response terms over the entire loading interval results in

$$u(t) = \frac{1}{m\omega_D} \int_0^t F(\tau) e^{-\xi\omega(t-\tau)} \sin \omega_D(t - \tau) d\tau \quad (4.15)$$

which is the response for a damped system in terms of the Duhamel's integral. For numerical evaluation eq.(4.15), we proceed as in the undamped case (See Problem 4.2).

4.3 Response by Direct Integration

The differential equation of motion for a one degree-of-freedom system represented by the damped simple oscillator, shown in Fig. 4.7(a), is obtained by establishing the dynamic equilibrium of the forces in the free body diagram, Fig. 4.7(b):

$$m\ddot{u} + c\dot{u} + ku = F(t) \quad (4.16)$$

in which the function $F(t)$ represents the force applied to the mass of the oscillator

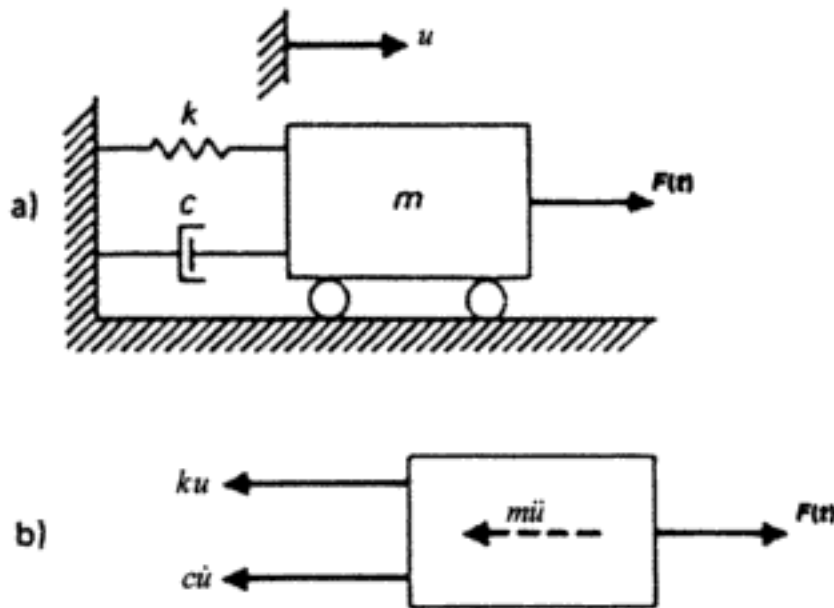


Fig. 4.7 (a) Damped simple oscillator excited by the force $F(t)$.
(b) Free body diagram.

When the structure, modeled by the simple oscillator, is excited by a motion at its support, as is shown in Fig. 4.8(a), the equation of motion obtained using the free body diagram in Fig. 4.8(b) is

$$m\ddot{u} + c(\dot{u} - \dot{u}_s) + k(u - u_s) = 0 \quad (4.17)$$

In this case, it is convenient to express the displacement u_r of the mass relative to the displacement u_s of the support, namely

$$u_r = u - u_s \quad (4.18)$$

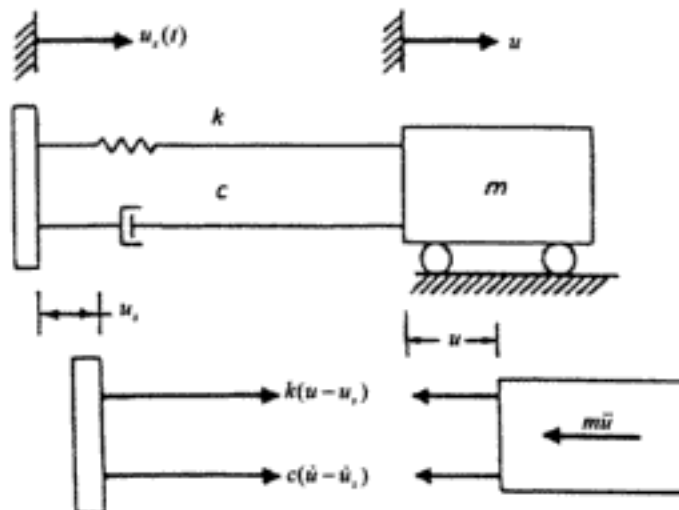


Fig. 4.8 (a) Damped Simple oscillator excited by the displacement $u_s(t)$ at its support.
(b) Free body diagram.

The substitution of u_r and its derivative from eq.(4.18) into eq.(4.17) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = -mu_s(t) \quad (4.19)$$

Comparison of eqs.(4.16) and (4.19) reveals that both equations are mathematically equivalent if the right-hand side of eq.(4.19) is interpreted as the effective force

$$F_{eff}(t) = -m\ddot{u}_s(t) \quad (4.20)$$

Equation (4.19) may then be written as

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff}(t) \quad (4.21)$$

Consequently, the solution of the second order differential equation (4.16) or equation (4.21) gives the response in terms of the absolute motion u for the case in which the mass is excited by a force, or in terms of the relative motion $u_r = u - u_s$, for the structure excited at the base.

4.4 Solution of the Equation of Motion

The method of solution for the differential equation of motion presented in this Section is exact for an excitation function described by linear segments. The process of solution requires for convenience that the excitation function be calculated at equal time intervals Δt . This result is accomplished by a linear interpolation between points defining the excitation. Thus, the time duration of the excitation, including a suitable extension of time after cessation of the excitation, is divided into N equal time intervals of duration Δt . For each interval Δt , the response is calculated by considering the initial conditions at the beginning of that time interval and the linear excitation during the interval. The initial conditions are, in this case, the displacement and velocity at the end of the preceding time interval. Assuming that the excitation function $F(t)$ is approximated by a piecewise linear function as shown in Fig. 4.9, we may express this function by

$$F(t) = \left(1 - \frac{t - t_i}{\Delta t}\right)F_i + \left(\frac{t - t_i}{\Delta t}\right)F_{i+1}, \quad t_i \leq t \leq t_{i+1} \quad (4.22)$$

in which $t_i = i \cdot \Delta t$ for equal intervals of duration Δt and $i = 1, 2, 3, \dots, N$. The differential equation of motion, eq.(4.16), is then given by

$$m\ddot{u} + c\dot{u} + ku = \left(1 - \frac{t - t_i}{\Delta t}\right)F_i + \left(\frac{t - t_i}{\Delta t}\right)F_{i+1}, \quad t_i \leq t \leq t_{i+1} \quad (4.23)$$

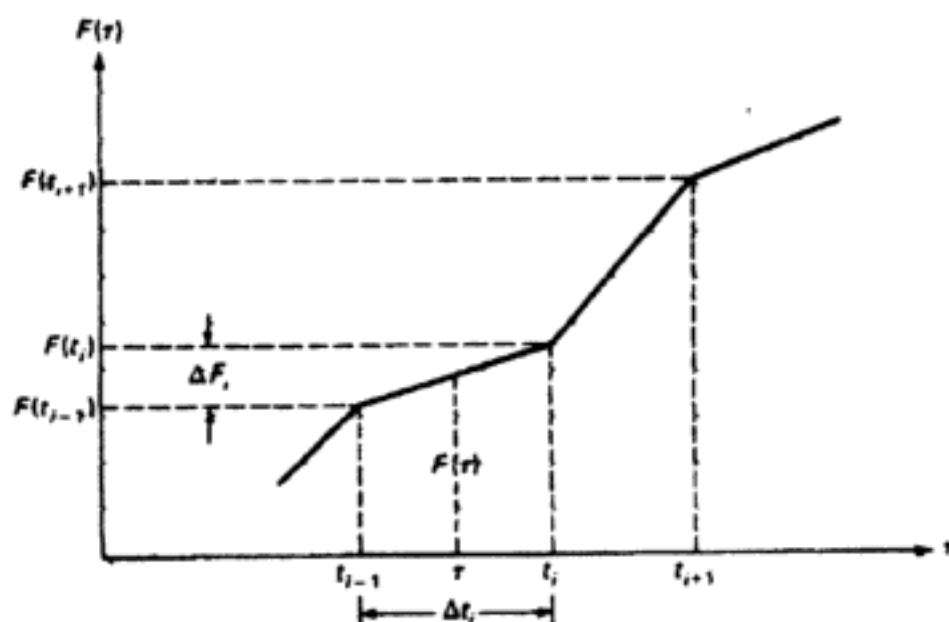


Fig. 4.9 Segmental linear loading function.

The solution of eq.(4.23) may be expressed as the sum of the complementary solution u_c for which the second member of eq.(4.23) is set equal to zero, and the particular solution u_p , that is

$$u = u_c + u_p \quad (4.24)$$

The complementary solution is given in general by eq.(2.15), which for the interval $t_i \leq t \leq t_i + \Delta t$ is

$$u_c = e^{\xi \omega_D (t-t_i)} [C_i \cos \omega_D (t-t_i) + D_i \sin \omega_D (t-t_i)] \quad (4.25)$$

On the other hand, the particular solution of eq.(4.23) takes the form

$$u_p = B_i + A_i (t-t_i) \quad (4.26)$$

which upon its substitution into eq.(4.23) gives

$$cA_i + k[B_i + A_i(t-t_i)] = \left(1 - \frac{t-t_i}{\Delta t}\right)F_i + \left(\frac{t-t_i}{\Delta t}\right)F_{i+1}$$

where A_i and B_i are constants for the interval $t_i \leq t \leq t_i + \Delta t$ and where we use the notation $F_i = F(t_i)$ and $F_{i+1} = F(t_i + \Delta t)$. Establishing the identity of terms between the left-hand and right-hand sides, that is, between the constant terms and the terms with a factor $(t-t_i)$ and then solving the resulting equations, we obtain

$$\begin{aligned} A_i &= \frac{F_{i+1} - F_i}{k \Delta t} \\ B_i &= \frac{F_i - c A_i}{k} \end{aligned} \quad (4.27)$$

The substitution into eq.(4.24) of the complementary solution u_c from eq.(4.25) and of the particular solution u_p from eq.(4.26) gives the total solution as

$$u = e^{-\xi\omega(t-t_i)} [C_i \cos \omega_D(t-t_i) + D_i \sin \omega_D(t-t_i)] + B_i + A_i(t-t_i) \quad (4.28)$$

The velocity \dot{u} is then given by the derivative of eq.(4.28) as

$$\dot{u} = e^{-\xi\omega(t-t_i)} [(\omega_D D_i - \xi\omega C_i) \cos \omega_D(t-t_i) - (\omega_D C_i + \xi\omega D_i) \sin \omega_D(t-t_i)] + A_i \quad (4.29)$$

The constants of integration C_i and D_i are obtained from eqs.(4.28) and (4.29) introducing the initial conditions for the displacement u_i and for the velocity \dot{u}_i at the beginning of the interval Δt , that is, at time t_i . Thus, introducing into eqs.(4.28) and (4.29) these initial conditions and solving for the constants C_i and D_i in the resulting relations yields

$$\begin{aligned} C_i &= u_i - B_i \\ D_i &= \frac{\dot{u}_i - A_i - \xi\omega C_i}{\omega_D} \end{aligned} \quad (4.30)$$

The evaluation of eqs.(4.28) and (4.29) at time $t_i + \Delta t$ results in the displacement u_{i+1} and the velocity \dot{u}_{i+1} at time t_{i+1} . Namely,

$$u_{i+1} = e^{-\xi\omega\Delta t} [C_i \cos \omega_D \Delta t + D_i \sin \omega_D \Delta t] + B_i + A_i \Delta t \quad (4.31)$$

and

$$\dot{u}_{i+1} = e^{-\xi\omega\Delta t} [D_i(\omega_D \cos \omega_D \Delta t - \xi\omega \sin \omega_D \Delta t) - C_i(\xi\omega \cos \omega_D \Delta t + \omega_D \sin \omega_D \Delta t)] - A_i \quad (4.32)$$

Finally, the acceleration at time $t_{i+1} = t_i + \Delta t$ is directly obtained after substituting u_{i+1} and \dot{u}_{i+1} from eqs.(4.31) and (4.32) into the differential equation (4.16) and letting $t_{i+1} = t_i + \Delta t$. Specifically,

$$\ddot{u}_{i+1} = \frac{1}{m} (F_{i+1} - c\dot{u}_{i+1} - ku_{i+1}) \quad (4.33)$$

The substitution of the coefficient A_i , B_i , C_i and D_i from eqs. (4.27) and (4.30), together with $c = \frac{2\xi k}{\omega}$ into eqs. (4.31) and (4.32), results in the following formulas to calculate the displacement, velocity and acceleration at the time step $t_{i+1} = t_i + \Delta t$:

$$u_{i+1} = A' u_i + B' \dot{u}_i + C' F_i + D' F_{i+1} \quad (4.34)$$

$$\dot{u}_{i+1} = A'' u_i + B'' \dot{u}_i + C'' F_i + D'' F_{i+1} \quad (4.35)$$

$$\ddot{u}_{i+1} = -\omega^2 u_{i+1} - 2\xi \omega \dot{u}_{i+1} + \frac{F_{i+1}}{m} \quad (4.36)$$

Equations (4.34), (4.35) and (4.36) are recurrence formulas to calculate, respectively, the displacement, velocity, and acceleration at the next time step $t_{i+1} = t_i + \Delta t$ from the previously calculated values for these quantities at the preceding time step t_i . Because these recurrence formulas are exact, the only restriction in selecting the length of the time step, Δt , is that it allows a close approximation to the excitation function and that equally spaced time intervals do not miss the peaks of this function. This numerical procedure is highly efficient because the coefficients in eqs. (4.34), (4.35) and (4.36) need to be calculated only once. The final expressions to calculate the coefficients A' , B' , ..., D'' are given in Box 4.1.

Illustrative Example 4.2

Determine the dynamic response of a tower subjected to a blast loading. The idealization of the structure and the blast loading are shown, respectively, in Figs. 4.10(a) and 4.10(b). Assume damping equal to 20% of the critical damping.

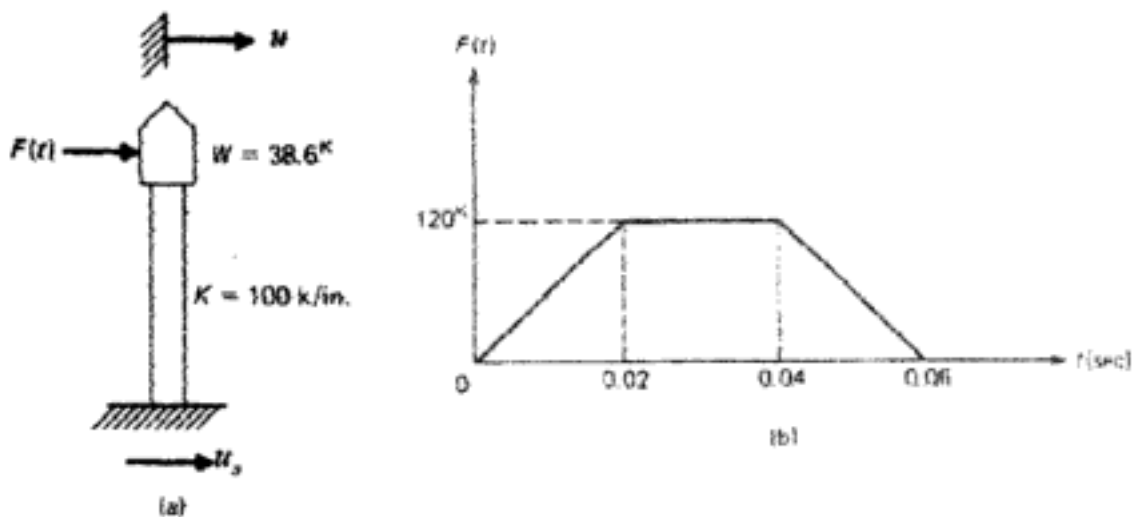


Fig. 4.10 (a) Idealized structure (b) idealized loading for Illustrative Example 4.2

$$A' = e^{-\xi\omega\Delta t} \left(\frac{\xi\omega}{\omega_D} \sin \omega_D \Delta t + \cos \omega_D \Delta t \right)$$

$$B' = e^{-\xi\omega\Delta t} \left(\frac{1}{\omega_D} \sin \omega_D \Delta t \right)$$

$$C' = \frac{1}{k} \left\{ e^{-\xi\omega\Delta t} \left[\left(\frac{1-2\xi^2}{\omega_D \Delta t} - \frac{\xi\omega}{\omega_D} \right) \sin \omega_D \Delta t - \left(1 + \frac{2\xi}{\omega \Delta t} \right) \cos \omega_D \Delta t \right] + \frac{2\xi}{\omega \Delta t} \right\}$$

$$D' = \frac{1}{k} \left\{ e^{-\xi\omega\Delta t} \left[\left(\frac{2\xi-1}{\omega_D \Delta t} \sin \omega_D \Delta t + \frac{2\xi}{\omega \Delta t} \cos \omega_D \Delta t \right) \right] + \left(1 - \frac{2\xi}{\omega \Delta t} \right) \right\}$$

$$A'' = -e^{-\xi\omega\Delta t} \left(\frac{\omega^2}{\omega_D} \sin \omega_D \Delta t \right)$$

$$B'' = e^{-\xi\omega\Delta t} (\cos \omega_D \Delta t) - \frac{\xi\omega}{\omega_D} \sin \omega_D \Delta t$$

$$C'' = \frac{1}{k} \left\{ -e^{-\xi\omega\Delta t} \left[\left(\frac{\omega^2}{\omega_D} + \frac{\omega\xi}{\Delta t \omega_D} \right) \sin \omega_D \Delta t + \frac{1}{\Delta t} \cos \omega_D \Delta t \right] - \frac{1}{\Delta t} \right\}$$

$$D'' = \frac{1}{k\Delta t} \left\{ -e^{-\xi\omega\Delta t} \left(\frac{\omega\xi}{\omega_D} \sin \omega_D \Delta t + \cos \omega_D \Delta t \right) + 1 \right\}$$

Box 4.1 Coefficients in eqs.(4.34), (4.35) and (4.36).

Solution:

Since the loading is given as a segmental linear function, the response obtained using the direct method will be exact. The necessary calculations are presented in a convenient tabular format in Table 4.1. For this system, the natural frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100,000}{100}} = 31.623 \text{ rad/sec}$$

and damped natural frequency

$$\omega_D = \omega\sqrt{1-\xi^2} = 31.623\sqrt{1-0.2^2} = 30.984 \text{ rad/sec}$$

Hence, the natural period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{31.623} = 0.20 \text{ sec}$$

Recommended practice is to select $\Delta t \leq \frac{T}{10}$. Specifically, we select $\Delta t = 0.02 \text{ sec}$.

We then calculate the coefficients of eqs.(4.34) and (4.35):

$$\begin{aligned} A' &= 0.82180 & B' &= 0.16517 & C' &= 1.16755 \times 10^{-6} & D' &= 6.1439 \times 10^{-7} \\ A'' &= -16.5170 & B'' &= 0.61286 & C'' &= 7.60730 \times 10^{-5} & D'' &= 8.9097 \times 10^{-5} \end{aligned}$$

with initial conditions $u_0 = 0$ and $v_0 = 0$, we obtain from eqs.(4.34), (4.35) and (4.36)

$$u_1 = 6.1439 \times 10^{-7} \times 1.2 \times 10^5 = 0.074 \text{ in}$$

$$\dot{u}_1 = 8.9124 \times 10^{-5} \times 1.2 \times 10^5 = 10.692 \text{ in/sec}$$

$$\ddot{u}_1 = 31.623^2 \times 0.074 - 2 \times 0.2 \times 31.623 \times 10.692 + 1.2 \times 10^5 / 100 = 990.754 \text{ in/sec}^2$$

Thus, completing the first cycle of calculations in the direct method of solution. Introducing the calculated values u_1, \dot{u}_1 , and \ddot{u}_1 into the recurrence formulas eqs.(4.34), (4.35), and (4.36), we obtain the response at time $t_2 = 0.04 \text{ sec}$. The continuation of this process results in the response of this system as shown in Table 4.1 up to 0.10 sec.

Table 4.1 Calculation of the Response for Illustrative Example 4.2

T(sec)	u_i in	\dot{u}_i in/sec	\ddot{u}_i in/sec ²	$F(\tau)$
0.000	0	0	0	0
0.020	0.074	10692	990.754	120,000
0.040	0.451	25.155	430.768	120,000
0.060	0.926	17.096	-1142.511	0
0.080	1.044	-4.821	-982.581	0
0.100	0.778	-20.191	-522.555	0

4.5 Program 2—Response by Direct Integration

The computer program described in this section calculates the response of the simple oscillator excited by a time-dependent external force acting on the mass or by an acceleration applied to the support. The excitation is assumed to be piecewise linear between defining points. The response consists of a table giving at equal increments of time the displacement, the velocity, and the acceleration of the mass for the case of the oscillator excited by a force applied to the mass. For the case of the oscillator excited at its support, the response is given in terms of the displacement and the

velocity of the mass relative respectively to the displacement and velocity of the support and of the absolute acceleration of the mass. The program implements the direct integration method to calculate the response. The program has been written in an interactive mode with the user. It starts by requesting information about the data needed in the solution of the problem, stores the data in a file, and then calculates and prints the response at equal increments of time. Maximum values for the response are also calculated and printed. A description of the programs used in this book and a form to order them is provided in Appendix II.

Illustrative Example 4.3.

Solve Example 4.2 using Program 2.

Solution:

From Fig. 4.11, we have the following data:

Mass: $m = w/g = (38.6 \times 1000)/386 = 100 \text{ (lb} \cdot \text{sec}^2/\text{in)}$

Spring constant: $k = 100 \times 1000 = 100,000 \text{ (lb} \cdot \text{/in)}$

Damping coefficient: $c = 2\xi\sqrt{km} = 1265 \text{ (lb} \cdot \text{sec/in)}$

Natural period: $T = 2\pi\sqrt{m/k} = 0.20 \text{ sec}$

Select time step for integration: $\Delta t = T/10 = 0.02 \text{ sec}$

Input Data and Output Results

PROGRAM 2: DIRECT INTEGRATION DATA FILE: D2

INPUT DATA:

NUMBER OF POINTS DEFINING THE EXCITATION	NE = 5
MASS	AM = 100
SPRING CONSTANT	AK = 100000
DAMPING COEFFICIENT	C = 1265
TIME STEP OF INTEGRATION	H = .02
INDEX (ACCELERATION OF GRAVITY OR ZERO)	G = 0

TIME	EXCITATION
0.000	0.00
0.020	12000.00
0.40	12000.00
0.060	0.00
0.20	0.00

PRINT TIME-RESPONSE TABLE Y/N? Y
OUTPUT RESULTS:

TIME	DISPL.	VELOC.	ACC.
0.000	0.000	0.000	0.000
0.020	0.074	10.692	991.023
0.040	0.451	25.155	430.768
0.060	0.926	17.096	- 1142.511
0.080	1.044	-4.821	- 982.581

Input Data and Output Results, continued

0.100	0.778	-20.191	- 522.555
0.120	0.306	-25.225	13.248
0.140	-0.165	-20.511	424.754
0.160	-0.475	-9.840	599.095
0.180	-0.553	1.809	529.696
0.200	-0.424	10.235	294.760
0.220	-0.180	13.280	11.592
0.240	0.072	11.105	-212.241
0.260	0.242	5.621	-313.497

MAX. DISPLACEMENT = 1.04
 MAX. VELOCITY = 25.22
 MAX. ACCELERATION = 1142.51

Illustrative Example 4.4.

Consider the tower shown in Fig. 4.11, but now subjected to a constant impulsive acceleration of magnitude $\ddot{u}_s = 0.5 \text{ g}$ during 0.5 sec applied at the foundation of the tower. Determine the response of the tower in terms of the displacement and velocity of the mass relative to the motion of the foundation. Also, determine the maximum acceleration of the mass.

Solution:

The following data are obtained from Fig. 4.11:

Mass: $m = 100 \text{ (lb} \cdot \text{sec}^2/\text{in)}$
 Spring constant: $k = 100,000 \text{ (lb/in)}$
 Damping coefficient: $c = 1265 \text{ (lb} \cdot \text{sec/in)}$
 Acceleration of gravity: $g = 386 \text{ (in/sec}^2\text{)}$
 Select time step: $\Delta t = 0.02 \text{ sec}$

Excitation function:	Time (sec)	Support Acceleration (g)
	0	0.5
	0.5	0.5

Input Data and Output Results

PROGRAM 2: DIRECT INTEGRATION		DATA FILE: D4.4	
INPUT DATA:			
NUMBER OF POINTS DEFINING THE EXCITATION		NE = 2	
MASS		AM = 100	
SPRING CONSTANT		AK = 100000	
DAMPING COEFFICIENT		C = 1265	
TIME STEP OF INTEGRATION		H = .02	
INDEX (ACCELERATION OF GRAVITY OR ZERO)		G = 386	
TIME	EXCITATION	TIME	EXCITATION
0.00	0.500	0.50	0.500

PRINT TIME-RESPONSE TABLE Y/N? N

Input Data and Output Results, continued
 OUTPUT RESULT:

MAX. DISPLACEMENT ²	=	0.29
MAX. VELOCITY ²	=	4.65
MAX. ACCELERATION	=	299.33

²For excitation at the support: Displacement and velocity are relative to the support whereas the acceleration is absolute value.

4.6 Program 3–Response to Impulsive Excitation

Program 3, the same as Program 2, gives the response of the simple oscillator to an excitation specified as linear segments between points defining the loading function. Program 2 may be used for any excitation function, while Program 3 is restricted to certain specific excitation functions predefined in the program. A total of eight possible excitation functions are implemented in the program. The excitation functions in the program are shown graphically in Fig. 4.11 and given analytically by the following expressions:

$$E_1(T) = P(1) * \sin [P(2) * T - P(3)] \qquad T \geq 0 \qquad (4.56)$$

$$E_2(T) = P(1) \qquad T \geq 0 \qquad (4.56)$$

$$\begin{aligned} E_3(T) &= P(1) * T & 0 \leq T \leq P(2) & \qquad (4.58) \\ &= 0 & T > P(2) & \end{aligned}$$

$$\begin{aligned} E_4(T) &= P(1)*T & 0 \leq T \leq P(2) & \qquad (4.59) \\ &= P(1)*P(2) & T > P(2) & \end{aligned}$$

$$\begin{aligned} E_5(T) &= P(1)*[1 - T/P(2)] & 0 \leq T \leq P(2) & \qquad (4.60) \\ &= 0 & T > P(2) & \end{aligned}$$

$$\begin{aligned} E_6(T) &= P(1)*[2*T/P(2)] & 0 \leq T \leq P(2)/2 & \qquad (4.61) \\ &= 2*P(1)*[1 - T/P(2)] & P(2)/2 \leq T \leq P(2) & \\ &= 0 & T > 2 & \end{aligned}$$

$$\begin{aligned} E_7 &= P(1)*\sin[P(3)*T] & 0 \leq T \leq P(2) & \qquad (4.62) \\ &= 0 & T > P(2) & \end{aligned}$$

$$E_8(T) = P(1)/EXP[P(2)*T] \qquad T \geq 0 \qquad (4.63)$$

In eqs. (4.56) through (4.63) T is the time variable and $P(1)$, $P(2)$ and $P(3)$ are constants or parameters to be input as data in order to completely define the specific excitation function selected.

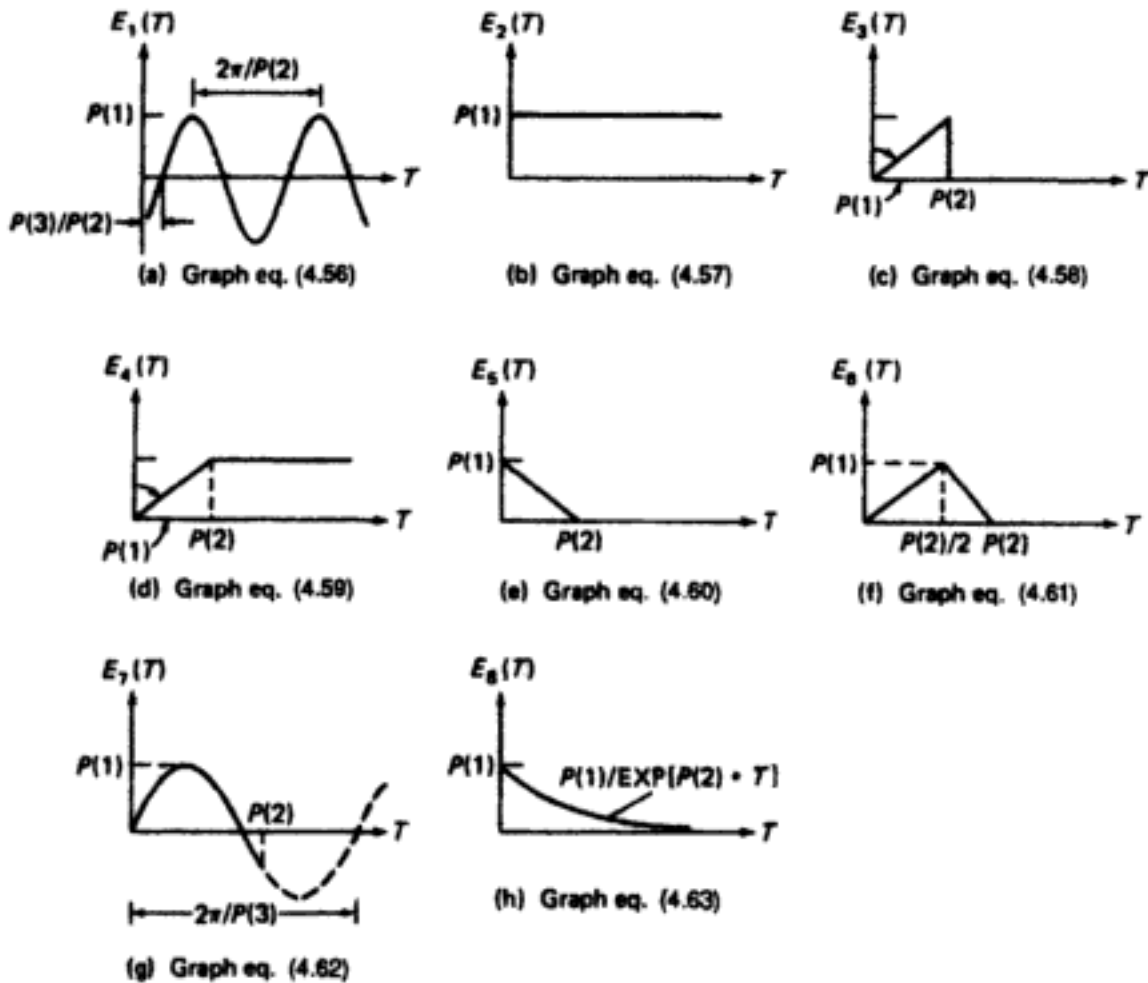


Fig. 4.11 Graphs for excitation functions given by eqs. (4.56) through (4.63).

Illustrative Example 4.5.

(a) Determine the dynamic response of the tower shown in Fig. 4.12 subjected to the sinusoidal force $F(t) = F_0 \sin \omega \cdot t$ applied at its top for 0.30 sec. (b) Check results using the exact solution which in this case is available in closed form. Neglect damping.

Solution:

(a) The following data is obtained from Fig. 4.12:

$$\text{Mass: } m = w/g = (38.6 \times 1000)/386 = 100 \text{ (lb} \cdot \text{sec}^2/\text{in)}$$

$$\text{Spring constant: } k = 100 \times 1000 = 100,000 \text{ (lb/in)}$$

$$\text{Natural frequency: } \omega = \sqrt{k/m} = 31.623 \text{ rad/sec}$$

$$\text{Natural period: } T = 2\pi/\omega = 0.20 \text{ sec}$$

$$\text{Select time step: } \Delta T = 0.01 \text{ sec}$$

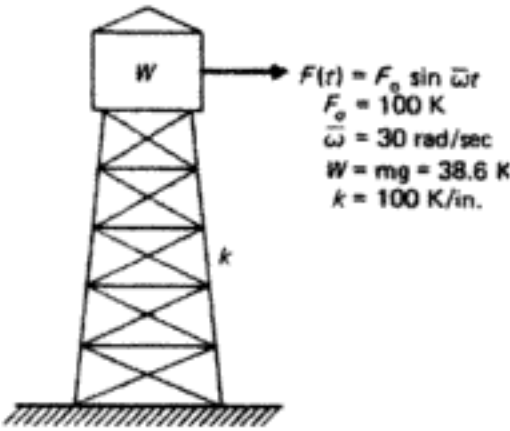


Fig. 4.12 Tower for Example 4.5.

Input Data and Output Results

PROGRAM 3--RESPONSE TO IMPULSIVE EXCITATION-- DATA FILE: D3

INPUT DATA:

MASS	M = 100
SPRING CONSTANT	K = 100000
DAMPING COEFFICIENT	C = 0
TIME STEP OF INTEGRATION	H = .01
RESPONSE MAX. TIME	TMAX = .3
INDEX (GRAVITY OR ZERO)	G = 0

PARAMETERS OF THE EXCITATION:

$$E(T) = P(1) * \sin[P(2) * T - P(3)] \quad T > 0$$

PARAMETER P (1) = 100000
PARAMETER P (2) = 30
PARAMETER P (3) = 0

PRINT TIME--RESPONSE TABLE Y/N? Y

OUTPUT RESULTS:

TIME	DISPL.	VELOC.	ACC.
0.000	0.000	0.000	0.000
0.010	0.005	1.485	290.619
0.020	0.038	5.585	526.458
0.030	0.123	11.570	660.487
0.040	0.272	18.229	660.204
0.050	0.485	24.142	512.573
0.060	0.747	27.868	226.285
0.070	1.031	28.165	-167.612
0.080	1.296	24.189	-620.916
0.090	1.499	15.651	-1071.980
0.100	1.595	2.916	-1454.280
0.110	1.547	-13.012	-1704.758
0.120	1.330	-30.544	-1772.318
0.130	0.937	-47.675	-1625.244
0.140	0.385	-62.205	-1256.580
0.150	-0.291	-72.003	-686.736
0.160	-1.033	-75.279	37.082

OUTPUT RESULTS: (continued)

TIME	DISPL	VELOC.	ACC.
0.170	-1.711	-70.833	844.732
0.180	-2.423	-58.245	1649.939
0.190	-2.910	-38.039	2359.407
0.200	-3.163	-11.605	2883.269
0.210	-3.129	18.794	3145.744
0.220	-2.783	50.259	3094.781
0.230	-2.131	79.525	2709.542
0.240	-1.211	103.295	2004.786
0.250	-0.093	118.604	1031.496
0.260	1.125	123.168	-126.486
0.270	2.330	115.674	-1359.642
0.280	3.398	95.996	-2543.164
0.290	4.213	65.276	-3549.554
0.300	4.674	25.889	-4262.176
0.310	4.713	-18.736	-4588.323

MAX. DISPLACEMENT = 4.71
 MAX. VELOCITY = 123.17
 MAX. ACCELERATION = 4588.32

(b) The exact solution for the response of a simple oscillator to the sinusoidal force $F_0 \sin \omega t$, with zero initial displacement and velocity, from eq. (3.8) is

$$u(t) = \frac{F_0}{k - m\omega^2} \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t \right)$$

where ω is the natural frequency in rad/sec, ω the forced frequency also in rad/sec, and F_0 the amplitude of the sinusoidal force.

Substituting corresponding numerical values for this example yields

$$\begin{aligned}
 u(t) &= \frac{100,000}{100,000 - 100(30)^2} \left(\sin 30t - \frac{30}{31.623} \sin 31.623t \right) \\
 &= 10(\sin 30t - 0.94868 \sin 31.623t)
 \end{aligned}$$

The velocity and acceleration functions are then given by

$$\dot{u}(t) = 300 \cos 30t - 300 \cos 31.623t$$

and

$$\ddot{u}(t) = -9000 \sin 30t + 948.7 \sin 31.623t$$

The evaluation of the response at specific values of time results in the following table:

t (sec)	$u(t)$ (in)	$\dot{u}(t)$ (in /sec)	$\ddot{u}(t)$ (in /sec ²)
0.1	1.6076	2.9379	-1466.51
0.2	-3.1865	-11.6917	2907.53
0.3	4.7420	26.0822	- 4298.04

Results shown in the above table are sufficiently close to corresponding values given by the computer in part (a) of this problem.

Illustrative Example 4.6.

Solve Example 4.5 selecting the time step Δt equal to 0.02, 0.01, and 0.005. Then compare the displacements at time $t = 0.1, 0.2$, and 0.3 sec with the response obtained in Example 4.5 using the exact solution of the differential equation.

Solution:

In solving the problem using Program 3, it is only necessary to modify existing file by selecting option 2 (Modify Existing Data File) at the computer request for file information. The only modification necessary in the data is to change the time step to the values prescribed by this problem. The following table records the results provided by the computer and their comparison with the exact solution calculated in Example 4.5:

Time (sec)	Exact Displa (in)	$\Delta t = 0.02$ sec		$\Delta t = 0.01$ sec		$\Delta t = 0.005$ sec	
		Displ. (in)	% error	Displ. (in)	% error	Displ. (in)	% error
0.1	1.6076	1.560	2.96	1.595	0.78	1.604	0.22
0.2	-3.1865	3.092	2.96	-3.163	0.73	-3.181	0.17
0.3	4.7420	4.570	3.62	4.674	1.43	4.701	0.86

Illustrative Example 4.7.

A structural system modeled in Fig. 4.13(a) by the simple oscillator with 10% ($\xi = 0.10$) of critical damping is subjected to the impulsive load as shown in Fig. 4.13(b). Determine the response.

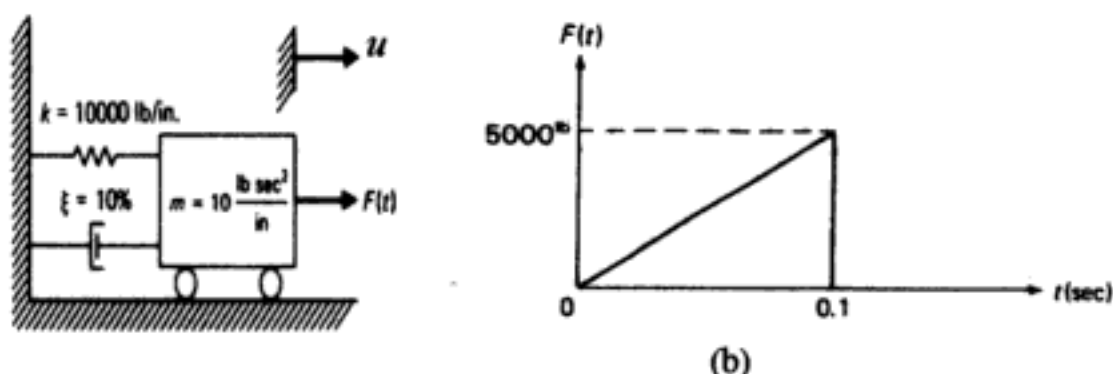


Fig. 4.13 Mathematical model and load function for Example 4.7

Solution:

The following data are obtained from Fig. 4.13:

Mass: $m = 10 \text{ lb} \cdot \text{sec}^2 / \text{in}$

Spring constant: $k = 10,000 \text{ lb} / \text{in}$

Damping coefficient: $c = \xi c_{cr} = 2\xi\sqrt{km} = 63.25 \text{ lb} \cdot \text{sec} / \text{in}$

Natural period: $T = 2\pi\sqrt{m/k} = 0.20 \text{ sec}$

Select time step for integration: $\Delta t = 0.02 \text{ sec}$

Excitation function: $F(t) = 50,000t \text{ lb}, 0 \leq t \leq 0.1 \text{ sec.}$
 $= 0 \quad t > 0.1 \text{ sec.}$

Maximum time: $t_{\max} = 0.2 \text{ sec.}$

Input Data and Output Results

PROGRAM 3—RESPONSE TO IMPULSIVE EXCITATION— DATA FILE: D4.7

INPUT DATA:

MASS	M = 10
SPRING CONSTANT	K = 10000
DAMPING COEFFICIENT	C = 63.25
TIME STEP OF INTEGRATION	H = .02
RESPONSE MAX. TIME	TMAX = .2
INDEX (GRAVITY OR ZERO)	G = 0

PARAMETERS OF THE EXCITATION:

$E(T) = P(1) \cdot T$	$0 < T < P(2)$
$= 0$	$T > P(2)$
PARAMETER P (1) = 50000	
PARAMETER P (2) = .1	

PRINT TIME—RESPONSE TABLE Y/N? Y

OUTPUT RESULTS:

TIME	DISPL.	VELOC.	ACC.
0.000	0.000	0.000	0.000
0.020	0.006	0.928	87.797
0.040	0.046	3.225	133.258
0.060	0.138	5.894	124.894
0.080	0.278	7.922	72.140
0.100	0.446	8.646	-0.560
0.120	0.576	2.393	-591.199
0.140	0.511	-8.4333	-457.841
0.160	0.268	-14.907	-173.960
0.180	-0.043	-15.195	139.402
0.200	-0.302	-9.928	364.872
0.220	-0.420	-1.679	430.965

MAX. DISPLACEMENT 0.58
MAX. VELOCITY 15.20
MAX. ACCELERATION 591.20

4.7 Response to General Dynamic Loading Using SAP2000

Illustrative Example 4.8

Use computer program SAP2000 to solve the structure of Illustrative Example 4.2.

Begin: Open SAP2000 (already installed in your computer).

Hint: Maximize both windows for a full view of all windows.

Select: In the lower right-hand corner of the screen use the drop-down menu to select “kip-in.”

Select: From the Main Menu bar:

FILE>NEW MODEL

Number of Grid Spaces:

x – direction = 2

y – direction = 0

z – direction = 0

Grid Spacing:

x – direction = 1

y – direction = 1

z – direction = 1

Then OK.

Edit: Click on XZ icon in the menu bar

Maximize the XZ window and use the PAN icon in the tool bar to drag the plot into the center of the screen.

Draw: From the main menu enter:

DRAW>ADD SPECIAL JOINT

Click at the origin of coordinates (0,0,0) and also at coordinates (1,0,0)

Then OK.

Label: From the Main Menu enter:
 VIEW>SET ELEMENTS
 Click on Joint Labels. Then OK.
 Use the PAN icon on the tool Bar to drag the figure to the center of the screen.

Boundary: Select joint 1 by clicking on it and enter:
 ASSIGN>JOINT>RESTRAINTS
 Select restraints in all directions. Then OK.

Draw: DRAW>NLINK ELEMENT
 Click on joint 1 and drag to joint 2. Then press the enter key.

Select: Select the link element by clicking on it.

Assign: ASSIGN>NLINK>PROPERTIES
 Click on Modify/Show Property
 Check on direction U1 and click on Modify/Show for U1.
 Then enter:
 Effective Stiffness = 100
 Effective Damping = 0.0
 Then OK, OK, OK.

Mass: Select Joint 2 by clicking on it.
 ASSIGN>JOINT>MASSES
 Enter Mass in direction 1 = 0.1. Then OK.

Load DEFINE > STATIC LOAD CASES
 Change Type of Load to LIVE
 Set Self Weigh multiplier to 0
 Click "Change Load." Then OK.

Load Function: DEFINE>TIME HISTORY FUNCTION
 Enter: Add New Function
 Click the Add button (to add the point 0, 0).
 Type 0.02 in the time box and 120 in the Value edit box.
 Then click the Add button.
 Enter: 0.04 in the Time box and 120 in the Value edit box and
 Click the Add button.
 Enter: 0.06 in the Time box and 0 in the Value edit box.
 Click the Add button.
 Enter: 0.10 in the Time box and 0 in the Value edit box
 Click the Add button. Then OK, OK.

History Cases:
 DEFINE>TIME HISTORY CASES
 Click Add New History button.

Accept the default HIST1.
Select: Analysis Type: Linear
Click the Modify/Show button
Enter 0.0 in the damping for mode 1
Then OK.

Check the "Envelopes" button
Type 10 in the Number of Output Time Steps
And enter 0.01 for Output Time Step Size.

In the Load Assignment: Select LOAD1
In the Function select FUNC1

Click the Add button
Click the OK button to return to the Define History Case dialog box.
Then OK.

Load: Select joint 2 by clicking on it.
ASSIGN>JOINT STATIC LOADS>FORCES
Enter: Force Global X = 1. Then OK.

Options: ANALYZE>SET OPTIONS
Click only UX for available DOF's
Click Dynamic Analysis and on Set Dynamic Parameters.
Accept Number of Modes = 1. Then OK, OK.

Analyze: ANALYZE>RUN
Enter: Filename "EXD 4.8"
Then SAVE.
When the calculations are concluded enter OK.

Display Max and Min Displacements:
DISPLAY>SET OUTPUT TABLE MODE
Select: HIST1 History. Then OK.
Right click on joint 2 to display Maximum and Minimum values for the displacements on joint 2:
Joint 2 HIST1 MAX: UX = 1.3951
Joint 2 HIST1 MIN: UX = 0.0000
Close this screen by clicking the "X"

Display Time History Displacements::
Select joint 2 by clicking on it
DISPLAY>SHOW TIME HISTORY TRACES
Click on Define Functions and on Add Input Functions
Select: Add Joint Displs/Forces from the drop down menu. Then OK.
In Time History Joint Function enter "2" for Joint ID
In Vector Type select Displ and in Component select UX. Then OK, OK.

Highlight Joint 2 in the "list of Functions" column and click on ADD to move joint 2 to the "Plot Functions" column.

Enter time range from 0 to 0.10.

Axis Labels:

Horizontal: TIME (sec)

Vertical: DISPLACEMENT JOINT 2 (in)

Click on Display

The screen will show the displacement-time function for joint 2.

Plot Displacement:

FILE>PRINT GRAPHICS

Fig 4.14 shows the Displacement-Time plot for joint 2

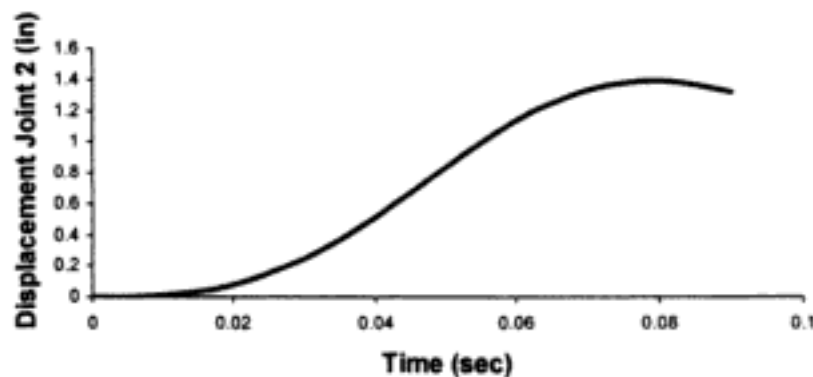


Fig. 4.14 Displacement-Time Plot for Joint 2 of Illustrative Example 4.8.

Input Tables:

FILE>PRINT INPUT TABLES

Check: Coordinates, Masses, Nllinks, Joints, Properties,

Click "Print to File" and accept the default name for the file:

"C:\SAP2000E\EXD 4.8.txt". Enter OK.

Output Tables:

FILE>PRINT OUTPUT TABLES

Click: Displacements, Reactions, Nllinks.

"Select Loads"— select both HIST1 History and LOAD1 (hold down CTRL key to select both) Then OK.

Click on "Print to File" and on "Append". Then OK.

Note: Use Notepad or another text editor to edit and print the file:

C:/SAP2000/EXD 4.8.txt

Tables 4.2 and 4.3 contain, respectively, the edited Input and Output Tables for Illustrative Example 4.8.

Table 4.2 Edited Input Tables for Illustrative Example 4.8.**JOINT DATA**

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	0.00000	0.00000	0.00000	1 1 1 1 1
2	1.00000	0.00000	0.00000	0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
2	0.100	0.000	0.000	0.000	0.000	0.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
2	1.000	0.000	0.000	0.000	0.000	0.000

Table 4.3 Edited Output Tables for Illustrative Example 4.8**JOINT DISPLACEMENTS**

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
2 (HIST1 MAX)						
:	1.3951	0.0000	0.0000	0.0000	0.0000	0.0000
2 (HIST1 MIN)						
:	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

JOINT REACTIONS

JOINT LOAD	F1	F2	F3	M1	M2	M3
1 HIST1 MAX:	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1 HIST1 MIN:	-139.5103	0.0000	0.0000	0.0000	0.0000	0.0000

NLLINK ELEMENT FORCES

NLLINK LOAD	LOC	P	V2	V3	T	M2	M3
1 HIST1 MAX							
	0.00	139.51	0.00	0.00	0.00	0.00	0.00
1 HIST1 MIN							
	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Illustrative Example 4.9

The steel frame of the Illustrative Example 3.10 reproduced, for convenience in Fig. 4.15, is subjected to acceleration at the base represented by the first 12 seconds of the North-South component of the El Centro earthquake of 1940. The acceleration record of this earthquake is reproduced in Fig. 4.16, and its digitized values have been stored in the file SAP2000\EXAMPLES\ELCENTRO.

Use SAP2000 to determine the response. (Assume 5% of critical damping)

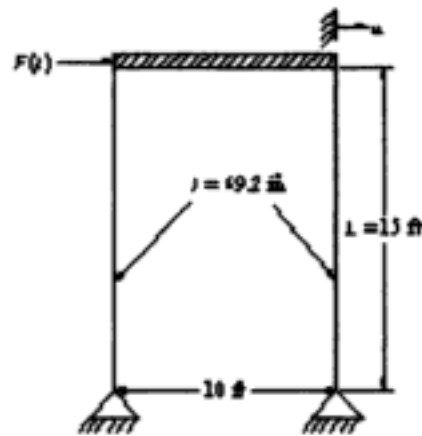


Fig. 4.15 Diagram of the frame for Illustrative Example 4.9.

Solution:

This structure may be modeled for dynamic analysis as the damped oscillator of Fig. 4.15. The parameters previously calculated for this structure in Illustrative Example 3.2 are:

Spring Constant:	$k = 2136 \text{ (lb/in)}$
Mass:	$m = 38.86 \text{ (lb.sec}^2\text{/in)}$
Damping:	$\xi = 0.05$

Prior to the start the solution of this example in SAP2000, place a copy of the file "Elcentro" (found in C:\SAP2000\EXAMPLES subdirectory) directly into the main SAP2000 directory.

Open SAP2000 and enter the following commands:

Enter: "OK" to disable the Tip of the Day screen.
(If desired, check OPTIONS>SHOW TIPS AT STARTUP to permanently disable this option).

Hint: Maximize both screens for full view of all windows.

Select: In the lower right-hand corner of the screen, use the drop-down menu to select "lb-in".

Model: From the Main Menu bar enter:

FILE>NEW MODEL

Enter: Number of Grid Spaces:

X Direction = 2

Y Direction = 0

Z Direction = 0

Grid Spaces:

X Direction = 1

Y Direction = 1

Z Direction = 1. Then OK.

Select: Click on X-Z icon to select this plane.

Maximize this view and use the PAN icon to center the figure on the screen.

Draw: From the Main menu enter:

DRAW>ADD SPECIAL JOINT

Click on the origin of coordinates and on the point X = 1. (The grid intersection to the right of the origin)

Press the "Enter" key or ESC to return to normal mode.

Label: From the Main Menu enter:

VIEW>SET ELEMENTS

Click on Joint Labels. Then Ok

Use the PAD icon on the tool bar to drag the

Figure to the center of the screen

Draw: DRAW>NLINK ELEMENT

Click on joint ① (at origin) and drag to joint ②. Then click on joint ② to draw the NLINK element. Press the "Enter" key or ESC to return to normal mode.

Properties: DEFINE>NLINK PROPERTIES

Click on "Modify/Show Property"

Check U1 in Directional properties box.

Click on "Modify/Show" for U1

Enter: Effective Stiffness = 2136

Effective Damping = 0.05 Then OK, OK, OK.

Boundary: Select joint ① by clicking on the joint.

ASSIGN>JOINT>RESTRAINTS

Select: Restraints in all directions. Then OK.

Mass: Select joint ② by clicking on the joint.

ASSIGN>JOINT>MASSES

For Direction 1 enter: 38.86. Then OK.

Load: DEFINE>STATIC LOAD CASES

Rename the load case "ELCENTRO"
Change Type of Load to QUAKE
Set Self-Weight Multiplier to 0
Click "Change Load". Then OK.

Define: DEFINE>TIME HISTORY FUNCTIONS

Click to "Add function from file" and on "Open File"
Select "Elcentro" from C:\SAP2000\Elcentro
Rename function under "function Name" as ELCENTRO.
Number of points per line = 3
(The ELCENTRO file contains 3 data points per line)
Select "Time and Function Values", then OK, OK.

Define: Select joint ② by clicking on it

DEFINE>TIME HISTORY CASES

Click on "Add New History"
Rename Time History Case ELCENTRO.

Under Options

Analysis type = linear
Number of output time steps = 215
Output time step size = 0.02
Modal Damping: Click on "Modify/Show"
Enter Damping for all Modes = 0.05
Then OK.

Load Assignments

Load = acc dir 1
Function = ELCENTRO (use the drop down menu to select)
Scale factor = 386.4 (convert g's units to in/sec²)
Arrival time = 0
Angle = 0
Then click ADD. Then OK, OK.

Analyze: ANALYZE>SET OPTIONS

Available Degrees-of-Freedom = UX only.
Click on Dynamic analysis parameters
Number of Modes = 1
Accept the rest of the default values.
Then OK, OK.

Analyze: ANALYZE>RUN

Enter filename: EXD 4.9, and SAVE
At the completion of the analysis check OK.

Plot El Centro Earthquake:

DISPLAY>SHOW TIME HISTORY TRACES

Click on “Define Functions”

Select “Add input Functions” using the drop down menu.

Click on ELCENTRO.

Then OK, OK to return to Time History Definition screen

Then, in “Choose Functions” section highlight ELCENTRO

Click ADD to move ELCENTRO to the “Plot Functions” column.

In the Axis Labels section enter:

Horizontal = Time (sec)

Vertical = N-S Acc. Component -- El Centro Earthquake 1940 (g's)

Click “Display” to view the plot on the screen

If a printed version is desired enter:

FILE>PRINT GRAPHICS in the Time History Definition Screen

Then OK, DONE.

(The Time-history of the North-South Acceleration Component of the El Centro Earthquake (1940) is reproduced in Fig. 4.16)

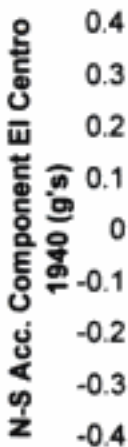


Fig. 4.16 N-S acceleration component of the El Centro Earthquake, 1940 (g's).

Plot Displacement at Joint 2:

DISPLAY>SHOW TIME HISTORY TRACES

In “List of Functions” click “Define Functions”

In the “Time History Functions” window click on

Add Joint Displs/Forces:

Joint ID: 2

Vector type = DISPL

Component = UX

Mode Number = Include ALL

Then OK, OK.

Under "Choose Function" select Joint 2-1, then click on ADD to move Joint 2-1 to the Plot Functions column. Highlight ELCENTRO to move it back to the List of Functions column.

Under Axis labels enter:

Horizontal = Time (sec)

Vertical = Displacement of Joint 2 (in)

Click "Display"

The Time-displacement plot of joint 2 of Illustrative Example 4.9 is reproduced in Fig. 4.17)

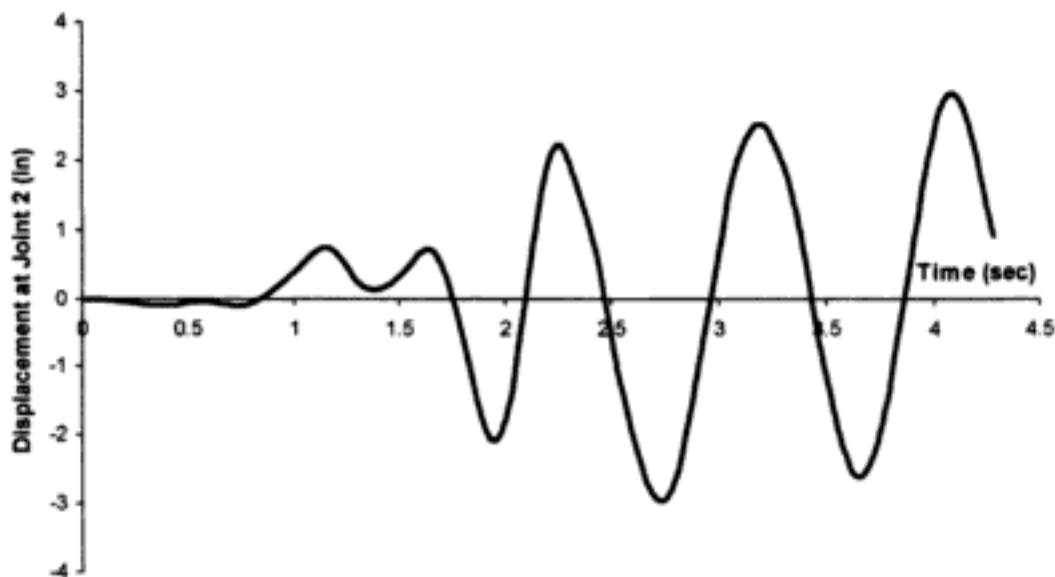


Fig. 4.17 Time-displacement plot for joint 2 of Illustrative Example 4.9.

Print Input Tables:

FILE>PRINT INPUT TABLES

Click: Coordinates, Masses, Nllinks and Properties

Click "Print to File" accept filename C:\SAP2000E\EXD 4.9.txt.

Then OK.

Output Tables:

FILE>PRINT OUTPUT TABLES

Click: Displacements, Reactions, Nllinks.

"Select Loads"– select both HIST1 History and LOAD1 (hold down CTRL key to select both)

Click on "Print to File" and on "Append". Then OK.

Note: Use Notepad or another text editor to edit and print the file:

C:/SAP2000/EXD 4.9.txt

(The edited input and output tables of Illustrative Example 4.9 are reproduced, respectively, as Table 4.4 and 4.5)

Table 4.4 Edited input tables of Illustrative Example 4.9 (Units: lb-in)**STATIC LOAD CASES**

STATIC CASE	CASE TYPE	SELF WT FACTOR
-------------	-----------	----------------

ELCENTRO	QUAKE	0.0000
----------	-------	--------

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	TIME STEP INCREMENT
--------------	--------------	----------------------	---------------------

HIST1	LINEAR	215	0.02000
-------	--------	-----	---------

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
-------	----------	----------	----------	------------

1	0.00000	0.00000	0.00000	1 1 1 1 1 1
2	1.00000	0.00000	0.00000	0 0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
-------	------	------	------	------	------	------

2	38.860	0.000	0.000	0.000	0.000	0.000
---	--------	-------	-------	-------	-------	-------

Table 4.5 Edited Output Tables of Illustrative Example 4.9 (Units: lb-in)**JOINT DISPLACEMENTS**

JOINT	LOAD	U1	U2	U3	R1	R2	R3
1	HIST1 MAX	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	HIST1 MIN	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	HIST1 MAX	2.9555	0.0000	0.0000	0.0000	0.0000	0.0000
2	HIST1 MIN	-2.9592	0.0000	0.0000	0.0000	0.0000	0.0000

JOINT REACTIONS

JOINT	LOAD	F1	F2	F3	M1	M2	M3
1	HIST1 MAX	6320.8447	0.0000	0.0000	0.0000	0.0000	0.0000
1	HIST1 MIN	-6312.9395	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.5 (Continued.)**NLLINK ELEMENT FORCES**

NLLINK	LOAD	LOC	P	V2	V3	T	M2	M3
1	HIST1 MAX	0.00	6312.94	0.00	0.00	0.00	0.00	0.00
		0.00	6312.94	0.00	0.00	0.00	0.00	0.00
1	HIST1 MIN	0.00	-6320.84	0.00	0.00	0.00	0.00	0.00
		0.00	-6320.84	0.00	0.00	0.00	0.00	0.00

4.8 Summary

In this chapter, we have shown that the differential equation of motion for a single-degree-of-freedom linear system can be solved for any forcing function $F(\tau)$ in terms of Duhamel's integral which for the undamped system is given by

$$u(t) = \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau) d\tau \quad \text{eq.(4.3) repeated}$$

and for a damped system by

$$u(t) = \frac{1}{m\omega_D} \int_0^t F(\tau) e^{-\xi\omega(t-\tau)} \sin \omega_D(t - \tau) d\tau \quad (4.15) \text{ repeated}$$

where

$$\omega = \sqrt{\frac{k}{m}} \quad \text{is the (undamped) natural frequency}$$

$$\omega_D = \omega \sqrt{1 - \xi^2} \quad \text{is the damped frequency}$$

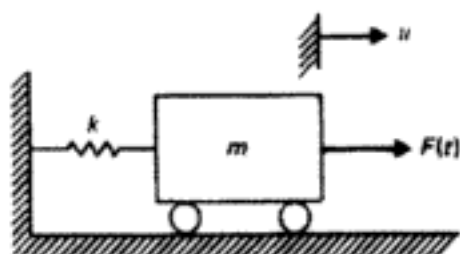
$$\xi = \frac{c}{c_{cl}} \quad \text{is the damping ratio}$$

The solution of the differential equation of motion may also be obtained by application of the Direct Method. In this method, it is assumed that the forcing function is given by a segmental linear function between defining points. Based on this assumption, the solution obtained is exact. The response is calculated at each time increment for the conditions existent at the end of the preceding time interval (initial conditions for the new time interval) and the action of the excitation applied during the time interval, which is assumed to be linear.

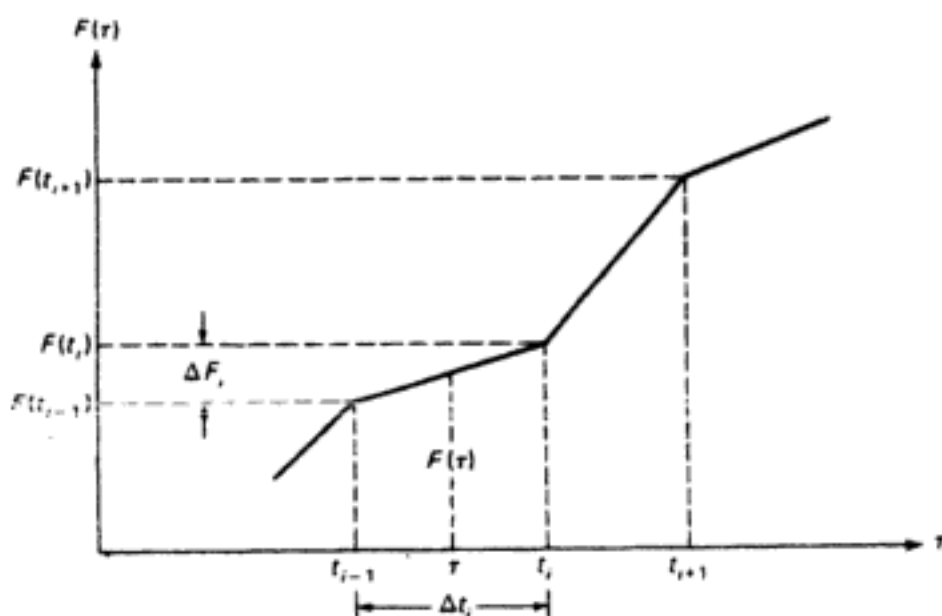
4.9 Analytical Problems**Problem 4.1**

Develop a numerical method to evaluate Duhamel's Integral which gives the response of an undamped elastic one-degree-of-freedom structure modeled by the simple

oscillation shown in Fig. P4.1(a). Assume that the forcing function $F(t)$ may be represented by a segmental linear function as shown in Fig. P4.1(b).



(a)



(b)

Fig. P4.1 (a) Structure modeled by the undamped simple oscillator.
(b) Segmental linear forcing function.

Solution:

Assuming zero initial conditions, we obtain Duhamel's integral from eq.(4.4) as

$$u(t) = \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau) d\tau$$

Then, using the trigonometric identity

$$\sin \omega(t - \tau) = \sin \omega t \cos \omega \tau - \cos \omega t \sin \omega \tau$$

we obtain

$$u(t) = \sin \omega t \frac{1}{m\omega} \int_0^t F(\tau) \cos \omega \tau \, d\tau - \cos \omega t \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega \tau \, d\tau$$

or

$$u(t) = \{A(t) \sin \omega t - B(t) \cos \omega t\} / m\omega \quad (a)$$

where

$$A(t) = \int_0^t F(\tau) \cos \omega \tau \, d\tau \quad (b)$$

$$B(t) = \int_0^t F(\tau) \sin \omega \tau \, d\tau \quad (c)$$

The calculation of Duhamel's integral thus requires the numerical evaluation of the integrals $A(t)$ and $B(t)$. It is more convenient to express the integrations in eqs. (b) and (c) in incremental form, namely

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) \cos \omega \tau \, d\tau \quad (d)$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) \sin \omega \tau \, d\tau \quad (e)$$

where $A(t_i)$ and $B(t_i)$ represent the values of the integrals in eq.(a) at time t_i . Assuming that the forcing function $F(\tau)$ is approximated by a piecewise linear function as shown in Fig. P4.1(b), we may write

$$F(\tau) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} (\tau - t_{i-1}), \quad t_{i-1} \leq \tau \leq t_i \quad (f)$$

where

$$\Delta F_i = F(t_i) - F(t_{i-1})$$

and

$$\Delta t_i = t_i - t_{i-1}$$

The substitution of eq.(4.20) into eq.(4.18) and integration yield

$$\begin{aligned} A(t_i) = A(t_{i-1}) + & \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) (\sin \omega t_i - \sin \omega t_{i-1}) / \omega \\ & + \frac{\Delta F_i}{\omega^2 \Delta t_i} \{ \sin \omega t_i - \sin \omega t_{i-1} + \omega(t_i \cos \omega t_i - \omega t_{i-1} \cos \omega t_{i-1}) \} \end{aligned} \quad (4.21)$$

Analogously from eq.(4.19),

$$B(t_i) = B(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) (\cos \omega t_{i-1} - \cos \omega t_i) / \omega$$

$$+ \frac{\Delta F_i}{\omega^2 \Delta t_i} \{ \sin \omega t_i - \sin \omega t_{i-1} - \omega(t_i \cos \omega t_i - t_{i-1} \cos \omega t_{i-1}) \}$$
(4.22)

Equations (4.21) and (4.22) are recurrent formulas for the evaluation of the integrals in eq.(4.15) at any time $t = t_i$.

Problem 4.2

Develop a numerical method to evaluate Duhamel's integral including damping in the system.

Solution:

The response of a damped single-degree-of-freedom system in terms of Duhamel's integral is given by eq.(4.15) as

$$u(t) = \frac{1}{m\omega_D} \int_0^t F(\tau) e^{-\xi\omega(t-\tau)} \sin \omega_D(t-\tau) d\tau \quad (4.15) \text{ repeated}$$

For numerical evaluation, we proceed as in the undamped case and obtain from eq.(4.15)

$$u(t) = \{A_D(t) \sin \omega_D t - B_D(t) \cos \omega_D t\} \frac{e^{-\xi\omega t}}{m\omega_D} \quad (g)$$

where

$$A_D(t_i) = A_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) e^{\xi\omega\tau} \cos \omega_D \tau d\tau \quad (h)$$

$$B_D(t_i) = B_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau) e^{-\xi\omega\tau} \sin \omega_D \tau d\tau \quad (i)$$

For a linear piecewise loading function, $F(\tau)$ given by eq.(f) of Problem 4.1, is substituted into eqs.(h) and (i) which require the evaluation of the following integrals:

$$I_1 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau} \cos \omega_D \tau d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \omega_D^2} (\xi\omega \cos \omega_D \tau + \omega_D \sin \omega_D \tau) \Big|_{t_{i-1}}^{t_i} \quad (j)$$

$$I_2 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau} \sin \omega_D \tau d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \omega_D^2} (\xi\omega \sin \omega_D \tau - \omega_D \cos \omega_D \tau) \Big|_{t_{i-1}}^{t_i} \quad (k)$$

$$I_3 = \int_{t_{i-1}}^{t_i} \tau e^{\xi\omega\tau} \sin \omega_D \tau d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \omega_D^2} \right) I_2 + \frac{\omega_D}{(\xi\omega)^2 + \omega_D^2} I_1 \Big|_{t_{i-1}}^{t_i} \quad (l)$$

$$I_4 = \int_{t_{i-1}}^{t_i} \tau e^{-\xi\omega\tau} \cos \omega_D \tau d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \omega_D^2} \right) I_1 - \frac{\omega_D}{(\xi\omega)^2 + \omega_D^2} I_2 \Big|_{t_{i-1}}^{t_i} \quad (m)$$

where I_1' and I_2' are the integrals indicated in eqs. (j) and (k) before their evaluation at the limits. In terms of these integrals, $A_D(t_i)$ and $B_D(t_i)$ may be evaluated after substituting eq.(f) of Problem 4.1 into eqs.(h) and (i) as

$$A_D(t_i) = A_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) I_1 + \frac{\Delta F_i}{\Delta t_i} I_4 \quad (n)$$

$$B_D(t_i) = B_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) I_2 + \frac{\Delta F_i}{\Delta t_i} I_3 \quad (o)$$

Finally, the substitution of eqs.(n) and (o) into eq.(g) gives the displacement at time t_i as

$$u_i(t_i) = \frac{e^{-\xi \omega_D t_i}}{m \omega_D} \{ A_D(t_i) \sin \omega_D t_i - B_D(t_i) \cos \omega_D t_i \} \quad (p)$$

4.10 Problems

Problem 4.3

The steel frame shown in Fig. P4.3 is subjected to a horizontal force $F(t)$ applied at the girder level. The force decreases linearly from 5 kip at time $t = 0$ to zero at $t = 0.6$ sec. Determine: (a) the horizontal deflection at $t = 0.5$ sec and (b) the maximum horizontal deflection. Assume the columns massless and the girder rigid. Neglect damping.

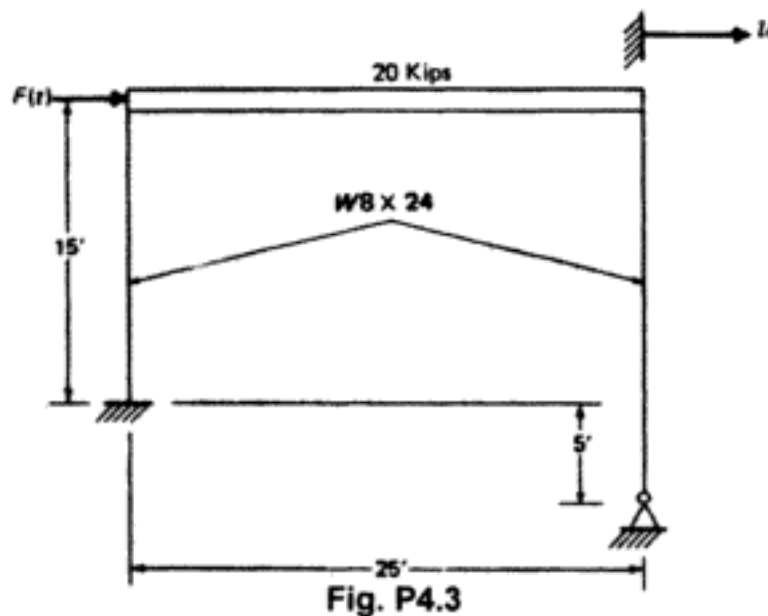


Fig. P4.3

Problem 4.4

Repeat Problem 4.3 for 10% of critical damping.

Problem 4.5

For the load-time function in Fig. P4.5, derive the expression for the dynamic load factor for the undamped simple oscillator as a function of t , ω , and t_d .

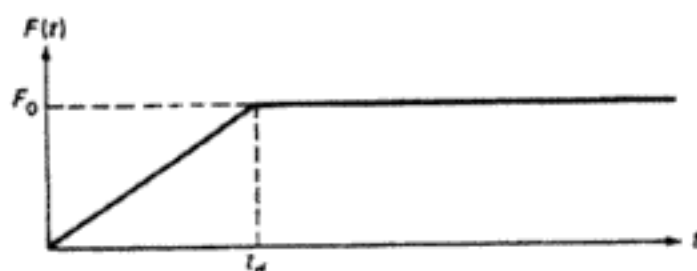


Fig P4.5

Problem 4.6

The frame shown in Fig. P4.3 is subjected to a sudden acceleration of 0.5 g applied to its foundation. Determine the maximum shear force in the columns. Neglect damping.

Problem 4.7

Repeat Problem 4.6 for 10% of critical damping.

Problem 4.8

Use Duhamel's integral to obtain the response of a damped simple oscillator of stiffness k , mass m , and damping ratio ξ , subjected to a suddenly applied force of magnitude F_0 . Assume initial displacement and initial velocity equal zero.

Problem 4.9

Establish the equation of motion for the system in Problem 4.8 and solve it by superposition of complementary and particular solutions with initial conditions for displacement and velocity equal to zero.

Problem 4.10

A trailer being pulled by a truck moving at constant speed v is idealized as a mass m connected to the truck by a spring of stiffness k . Determine the governing equation and its solution if the truck starts from rest.

Problem 4.11

Determine the response of an undamped system to a ramp force (Fig. P4.11) of maximum magnitude F_0 and duration t_d starting with zero initial conditions of displacement and velocity.

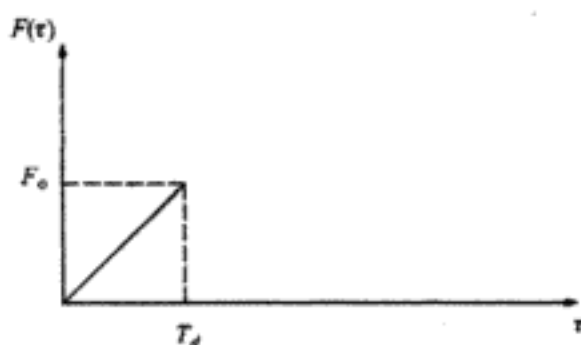


Fig. P4.11

Problem 4.12

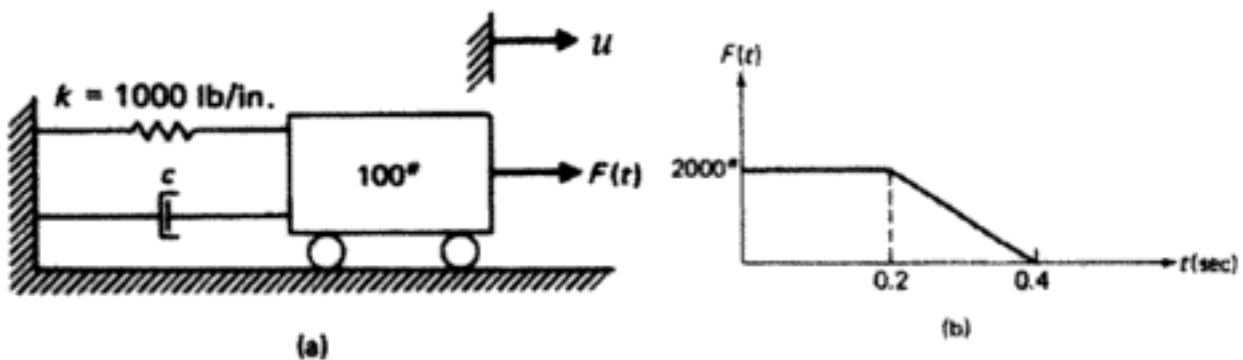
Determine the maximum displacement at the top of the columns and maximum bending stress in the frame of Fig. P4.3 assuming that the columns are pinned at the base. Discuss the effect of base fixity.

Problem 4.13

Determine the maximum response (displacement and bending stress) for the frame of Illustrative Example 4.1 subjected to a triangular load of initial force $F_0 = 6000$ lb linearly decreasing to zero at time $t_d = 0.1$ sec.

Problem 4.14.

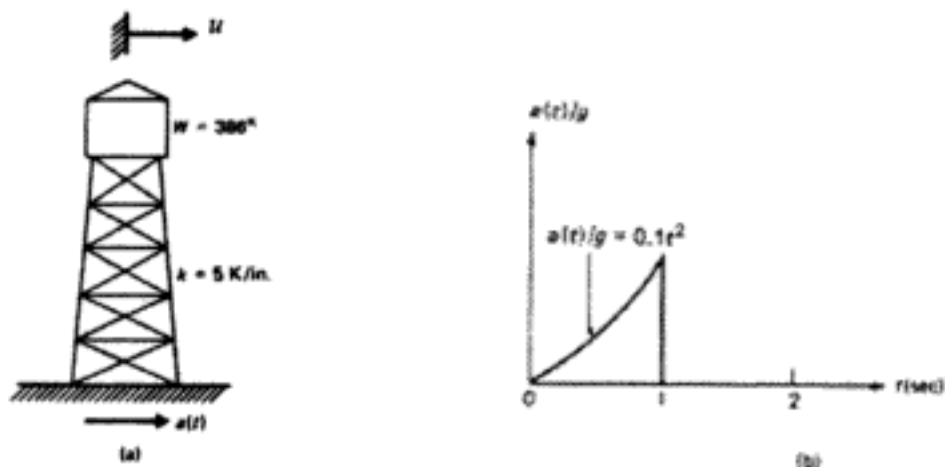
For the dynamic system shown in Fig. P4.14, determine and plot the displacement as a function of time for the interval $0 \leq t \leq 0.5$ sec. Neglect damping.

**Fig. P4.14****Problem 4.15**

Repeat problem 4.14 for 10% critical damping.

Problem 4.16

The tower of Fig. P4.16(a) is subjected to horizontal ground acceleration $a(t)$ shown in Fig. P4.16(b). Determine the relative displacement at the top of the tower at time $t = 1.0$ sec. Neglect damping.

**Fig. P4.16**

Problem 4.17

Repeat problem 4.16 for 20% of critical damping.

Problem 4.18

Determine for the tower of Problem 4.17, the maximum displacement at the top of the tower relative to the ground displacement.

Problem 4.19

The frame of Fig. P4.19(a) is subjected to horizontal support motion shown in Fig. P4.19(b). Determine the maximum absolute deflection at the top of the frame. Assume no damping.

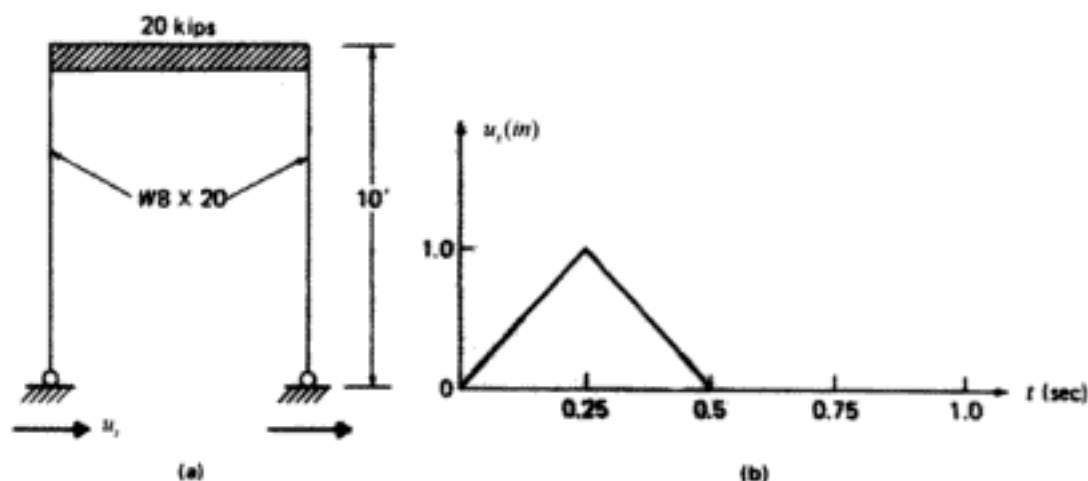


Fig. P4.19

Problem 4.20

Repeat Problem 4.19 for 10% of critical damping.

Problem 4.21

A structural system modeled by the simple oscillator with 10% ($\xi = 0.10$) of critical damping is subjected to the impulsive load as shown in Fig. P4.21. Determine the response.

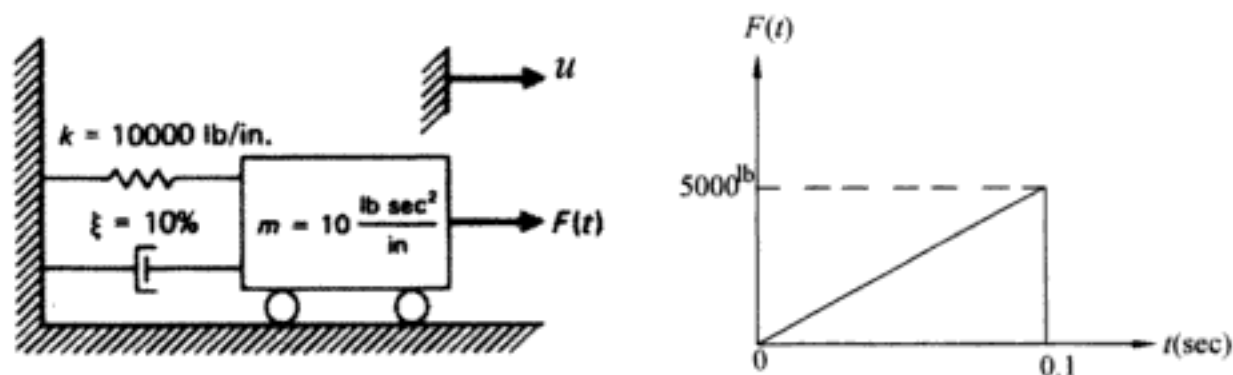


Fig. P4.21

Problem 4.22

A water tower modeled as shown in Fig. P4.22(a) is subjected to ground shock given by the function depicted in Fig. P4.22(b). Determine: (a) the maximum displacement at the top of the tower and (b) the maximum shear force at the base of the tower. Neglect damping. Use time step for integration $\Delta t = 0.005$ sec.

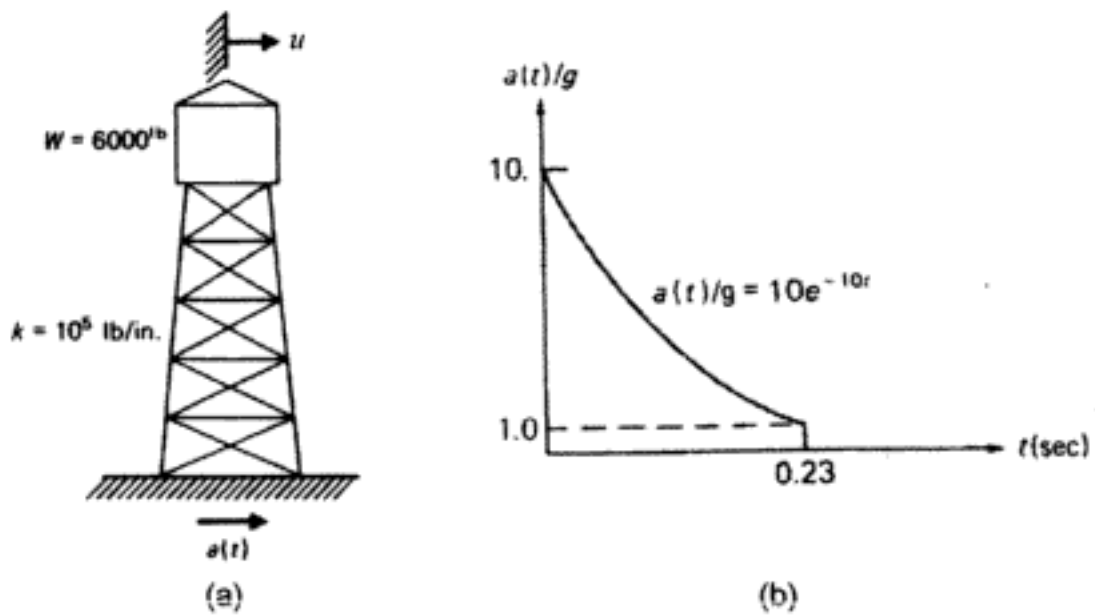


Fig P4.22

Problem 4.23

Repeat Problem 4.22 for 20% of critical damping.

Problem 4.24

Determine the maximum response of the tower of Problem 4.22 when subjected to the impulsive ground acceleration depicted in Fig. P4.24.

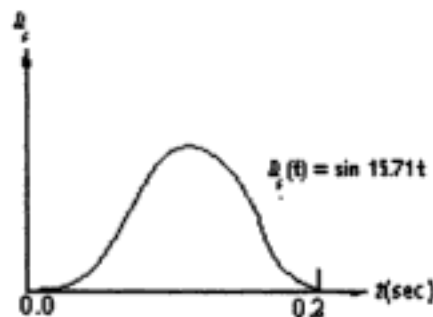
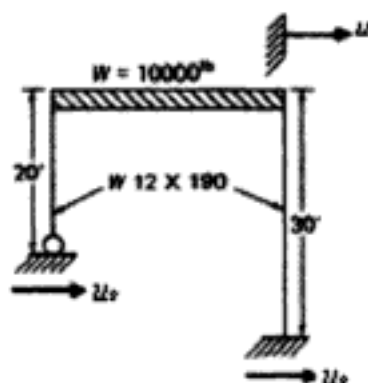


Fig. P4.24

Problem 4.25

The steel frame in Fig. P4.25 is subjected to the ground motion produced by a passing train in its vicinity. The ground motion is idealized as a harmonic acceleration of the foundation of the frame with amplitude 0.1 g at frequency 10 cps . Determine the maximum response in terms of the relative displacement of the girder of the frame and the motion of the foundation. Assume 10% of the critical damping.

**Fig. P4.25****Problem 4.26**

Determine the maximum stresses in the columns of the frame in Problem 4.25 using the maximum response in terms of relative motion. Also check that the same results may be obtained using the response in terms of the maximum absolute acceleration.

Problem 4.27

A machine having a weight $W = 3,000\text{ lb}$ is mounted through coil springs to a steel beam of rectangular cross-section as shown in Fig. P4.27(a). Due to malfunctioning, the machine produces a shock force represented in Fig. 4.27(b). Neglecting the mass of the beam and damping in the system, determine the maximum displacement of the machine.

Problem 4.28

For Problem 4.27 determine: (a) the maximum tensile and compressive stresses in the beam and (b) the maximum force experienced by the coil springs during the shock.

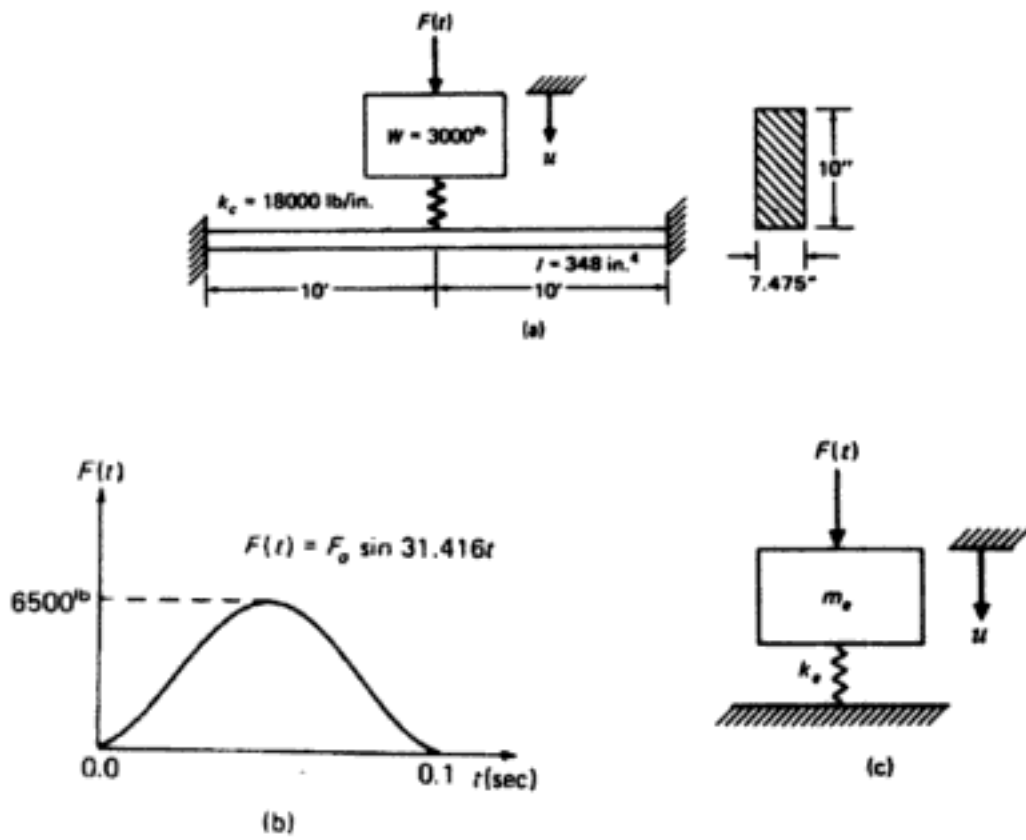


Fig.P4.27

5 Response Spectra

In this chapter, we introduce the concept of response spectrum, which in recent years has gained wide acceptance in structural dynamic practice, particularly in earthquake engineering design. Stated brief, the response spectrum is a plot of the maximum response (maximum displacement, velocity, acceleration, or any other quantity of interest) to a specified load function for all possible single-degree-of-freedom systems. The abscissa of the spectrum is the natural frequency (or period) of the system, and the ordinate the maximum response. A plot of this type is shown in Fig. 5.1, in which a one-story building is subject to a ground displacement indicated by the function $u_g(t)$. The response spectral curve shown in Fig. 5.1 (a) gives, for any single-degree-of-freedom system, the maximum displacement of the response from an available spectral chart, for a specified excitation, we need only to know the natural frequency of the system.

5.1 Construction of Response Spectrum

To illustrate the construction of a response spectral chart, consider in Fig. 5.2(a) the undamped oscillator subject to one-half period of the sinusoidal exciting force shown in Fig. 5.2(b).

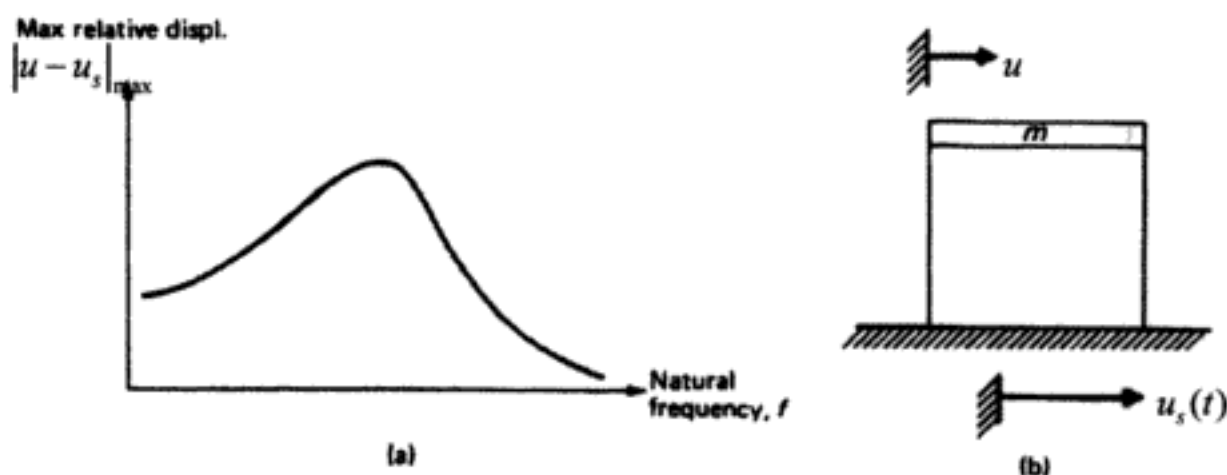


Fig. 5.1 (a) Typical response spectrum. (b) Single-degree-of-freedom system subject to ground excitation.

The system is assumed to be initially at rest. The duration of the sinusoidal impulse is denoted by t_d . The differential equation of motion is obtained by equating to zero the sum of the forces in the corresponding free body diagram shown in Fig. 5.2 (c), that is,

$$m\ddot{u} + ku = F(t) \quad (5.1)$$

in which

$$F(t) = \begin{cases} F_0 \sin \omega t & \text{for } 0 \leq t \leq t_d \\ 0 & \text{for } t > t_d \end{cases} \quad (5.2)$$

and

$$\omega = \frac{\pi}{t_d} \quad (5.3)$$

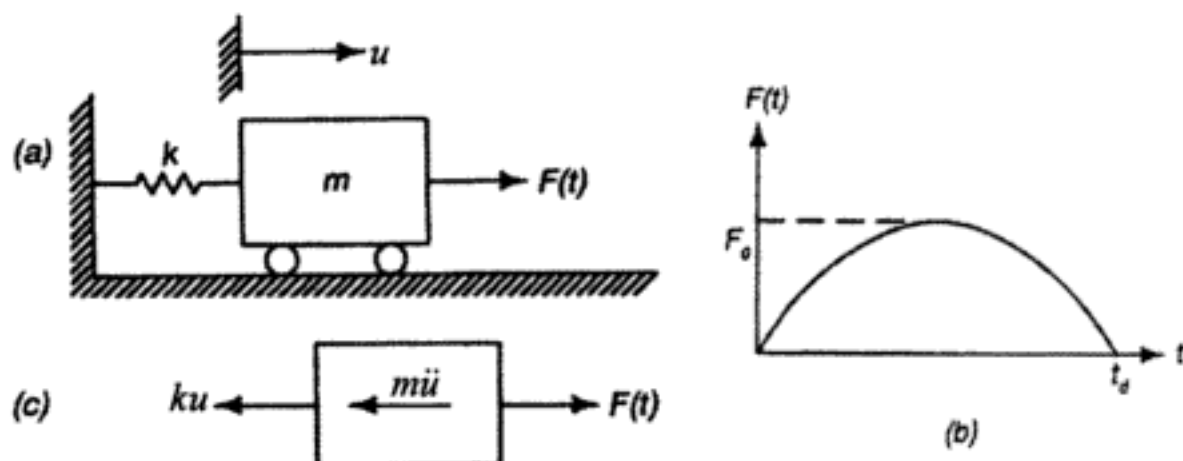


Fig. 5.2 (a) Undamped simple oscillator subjected to load $F(t)$. (b) Loading function $F(t) = F_0 \sin \omega t$ ($0 \leq t \leq t_d$). (c) Free body diagram.

The solution of eq. (5.1) may be found by any of the methods studied in the preceding chapter 4 such as the use of Duhamel's integral or the direct method. However, for this example, owing to the simplicity of the exciting force, we can obtain the solution of eq. (5.1) by the general method of integration of a linear differential equation, that is, the superposition of the complementary solution u_c and the particular solution u_p ,

$$u = u_c + u_p \quad (5.4)$$

The complementary solution of eq. (5.1) (right-hand side equals zero) is given by eq. (1.17) as

$$u_c = A \cos \omega t + B \sin \omega t \quad (5.5)$$

in which $\omega = \sqrt{k/m}$ is the natural frequency. The particular solution for the time interval $0 \leq t \leq t_d$ is suggested by the right-hand side of eq. (5.1) to be of the form

$$u_p = C \sin \varpi t \quad (5.6)$$

The substitution of eq. (5.6) into eq. (5.1) and solution of the resulting identity gives

$$C = \frac{F_0}{k - m\varpi^2} \quad (5.7)$$

Combining (5.4) through (5.7), we obtain the response for $0 \leq t \leq t_d$ as

$$u = A \cos \omega t + B \sin \omega t + \frac{F_0 \sin \varpi t}{k - m\varpi^2} \quad (5.8)$$

Introducing the initial conditions $u(0) = 0$ and $\dot{u}(0) = 0$ into eq. (5.8) and calculating the constants of integration A and B , we obtain

$$u = \frac{F_0/k}{1 - (\varpi/\omega)^2} [\sin \varpi t - (\varpi/\omega) \sin \omega t] \quad (5.9)$$

It is convenient to introduce the following notation:

$$u_{st} = \frac{F_0}{k}, \quad \varpi = \frac{\pi}{t_d}, \quad \omega = \frac{2\pi}{T}$$

Then eq. (5.9) becomes

$$\frac{u}{u_{st}} = \frac{1}{1 - \left(\frac{T}{2t_d}\right)^2} \left[\sin \pi \frac{t}{t_d} - \frac{T}{2t_d} \sin 2\pi \frac{t}{T} \right] \quad \text{for } 0 \leq t \leq t_d \quad (5.10a)$$

After a time t_d the external force becomes zero and the system is then in free vibration. Therefore, the response for $t > t_d$ is of the form given by eq. (5.5) with the constants of integration determined from known values of displacement and velocity calculated from eq. (5.10a) at time $t = t_d$. The expression obtained for the response is then given by

$$\frac{u}{u_{st}} = \frac{T/t_d}{\left(\frac{T}{2t_d}\right)^2 - 1} \cos \pi \frac{t}{t_d} \sin 2\pi \left(\frac{t}{T} - \frac{t_d}{2T} \right) \quad \text{for } t \geq t_d \quad (5.10b)$$

It may be seen from eq. (5.10) that the response in terms of u/u_{st} is a function of the ratio of the pulse duration to the natural period of system (t_d/T) and of time expressed as t/T . Hence for any fixed value of the parameter t_d/T , we can obtain the maximum response from eq. (5.10). The plot in Fig. 5.3 of these maximum values as a function of t_d/T is the response spectrum for the half-sinusoidal force duration considered in this case. It can be seen from the response spectrum in Fig 5.3 that the maximum value of the response (amplification factor) $u/u_{st} = 1.76$ occurs for this particular pulse when $t_d/T = 0.8$.

Owing to the simplicity of the input force, it was possible in this case to obtain a closed solution and to plot the response spectrum in terms of dimensionless ratios, thus making this plot valid for any impulsive force described by one-half of the sine cycle. However, in general, for an arbitrary input load, we cannot expect to obtain such a general plot of the response spectrum and we normally have to be satisfied with the response spectrum calculated and plotted for a completely specified input excitation.

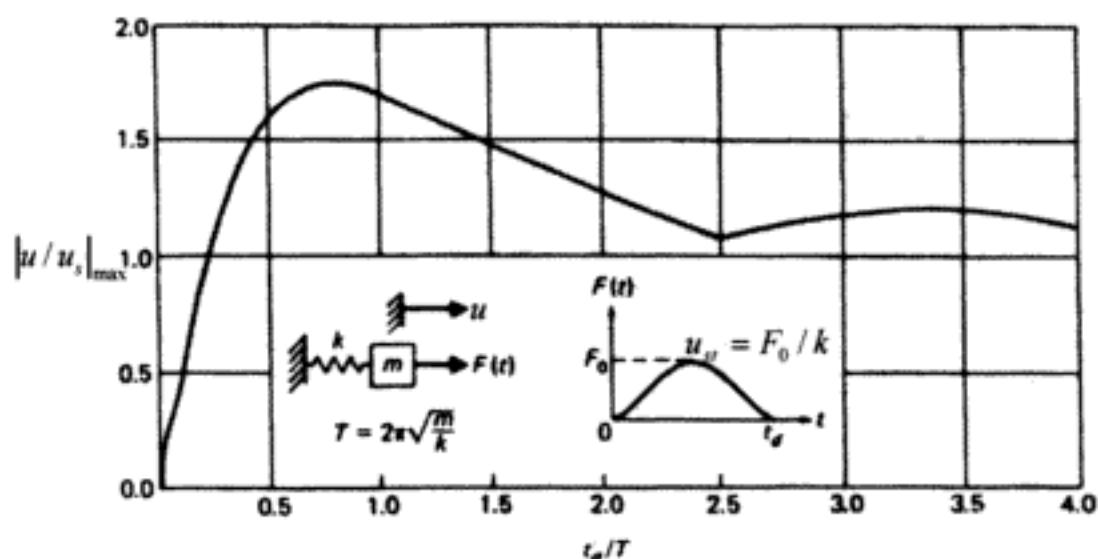


Fig. 5.3 Response spectrum for half-sinusoidal force of duration t_d

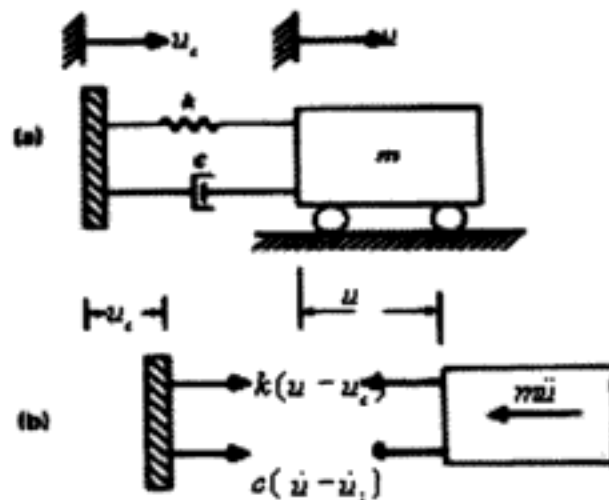


Fig. 5.4 (a) Damped simple oscillator subjected to support excitation.
(b) Free body diagram.

5.2 Response Spectrum for Support Excitation

An important problem in structural dynamic is the analysis of a system subjected to excitation applied to the base or foundation of the structure. An example of such input excitation of the base acting on a damped oscillator which serves to model certain structures is shown in Fig. 5.4. The excitation in this case is given as an acceleration function which is represented in Fig. 5.5. The equation of motion that is obtained by equation to zero the sum of the forces in the corresponding free body diagram in Fig. 5.4(b) is

$$m\ddot{u} + c(\dot{u} - \dot{u}_s) + k(u - u_s) = 0 \quad (5.11)$$

or, with the usual substitution $\omega = \sqrt{k/m}$ and $\xi = c/c_{cr}$ ($c_{cr} = 2\sqrt{km}$),

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \omega^2 u_s(t) + 2\xi\omega\dot{u}_s(t) \quad (5.12)$$



Fig. 5.5 Acceleration function exciting the support of the oscillator in Fig. 5.4.

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Equation (5.12) is the differential equation of motion for the damped oscillator in terms of its absolute motion. A more useful formulation of this problem is to express eq. (5.12) in terms of the relative motion of the mass with respect to the motion of the support, that is, in terms of the spring deformation. The relative displacement u_r is then defined as

$$u_r = u - u_s \quad (5.13)$$

Substitution into eq. (5.12) yields

$$\ddot{u}_r + 2\xi\omega\dot{u}_r + \omega^2 u_r = -\ddot{u}_s(t) \quad (5.14)$$

The formulation of the equation of motion in eq. (5.14) as a function of the relative motion between the mass and the support is particularly important since in design it is the deformation or stress in the "spring element" that is required. Besides, the input motion at the base is usually specified by means of an acceleration function (e.g., earthquake accelerograph record); thus eq. (5.14) containing in the right-hand side the acceleration of the excitation is a more convenient form than eq. (5.12) which in the right-hand side has the support displacement and the velocity.

The solution of the differential equation, eq. (5.14), may be obtained by any of the methods presented in previous chapters for the solution of one-degree-of-freedom systems. In particular, the solution is readily expressed using Duhamel's integral as

$$u(t) = -\frac{1}{\omega} \int_0^t \ddot{u}_s(\tau) e^{-\xi\omega(t-\tau)} \sin \omega(t-\tau) d\tau \quad (5.15)$$

5.3 Tripartite Response Spectra

It is possible to plot in a single chart using logarithmic scales the maximum response in terms of the acceleration, the relative displacement, and a third quantity known as the relative pseudovelocity. The pseudovelocity is not exactly the same as the actual velocity, but it is closely related and provides for a convenient substitute for the true velocity. These three quantities the maximum absolute acceleration, the maximum relative displacement, and the maximum relative pseudovelocity are known, respectively, as the spectral acceleration, spectral displacement, and spectral velocity.

It is significant that the spectral displacement S_D , that is, the maximum relative displacement, is proportional to the spectral acceleration S_a the maximum absolute acceleration. To demonstrate this fact, consider the equation of motion, eq. (5.11), which, after using eq. (5.13), becomes for the damped system

$$m\ddot{u} + c\dot{u}_r + ku_r = 0 \quad (5.16)$$

and for the undamped system

$$m\ddot{u} + ku_r = 0 \quad (5.17)$$

We observe from eq. (5.17) that the absolute acceleration is at all times proportional to the relative displacement. In particular, at maximum values, the spectral acceleration is proportional to the spectral displacement, that is, from eq. (5.17)

$$S_a = -\omega^2 S_D \quad (5.18)$$

where $\omega = \sqrt{k/m}$ is the natural frequency of the system, $S_a = \ddot{u}_{\max}$, and $S_D = u_{r,\max}$.

When damping is considered in the system, it may be rationalized that the maximum relative displacement occurs when the relative velocity is zero ($\dot{u} = 0$). Hence we again obtain eq. (5.18) relating spectral acceleration and spectral displacement. However, for a damped system, the spectral acceleration S_a is not exactly equal to the maximum acceleration, although in general, it provides a good approximation. Equation (5.18) is by mere coincidence the same as the relationship between acceleration and displacement for a simple harmonic motion. The fictitious velocity associated with the apparent harmonic motion is the pseudovelocity and, its maximum value S_v is defined as the spectral velocity, that is

$$S_v = \omega S_D = \frac{S_a}{\omega} \quad (5.19)$$

Dynamic response spectra for a single-degree-of-freedom elastic systems have been computed for a number of input motions. A typical example of displacement response spectrum for a single-degree-of freedom system subjected to support motion is shown in Fig. 5.6. This plot is the response for the input motion given by the recorded ground acceleration of the North-South component of the 1940 El Centro earthquake. The acceleration record of this earthquake has been used extensively in earthquake engineering investigations. A plot of the acceleration record for this earthquake is shown in Fig. 5.7. Until the time of the San Fernando, California earthquake of 1971, the El Centro record was one of the few records available for long and strong earthquake motions. In Fig. 5.8, the same type of data that were used to obtain the displacement response spectrum in Fig. 5.6 are plotted in terms of the spectral velocity, for several values of the damping coefficient, with the difference that the abscissa as well as the ordinate are in these cases plotted on a logarithmic scale. In this type of plot, because of eqs. (5.18) and (5.19), it is possible to draw diagonal scales for the displacement sloping 135° with the abscissa, and for the acceleration 45° , so that we can from a single plot read values of spectral acceleration, spectral velocity, and spectral displacements.

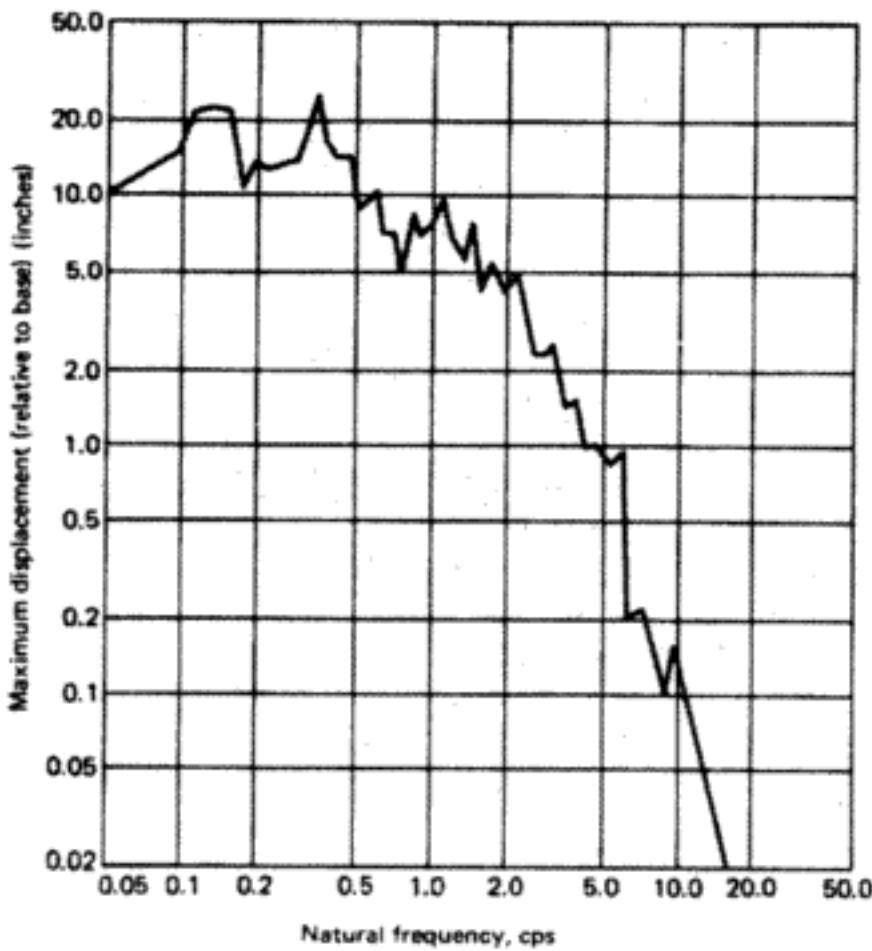


Fig. 5.6 Displacement response spectrum for elastic system, subjected to the ground motion of the N-S component of the 1940 El Centro earthquake. (from *Design of Multistory Reinforced Building for Earthquake Motions* by J. A. Blum, N. M. Newmark, and L. H. Corning, Portland Cement Association 1961.)

To demonstrate the construction of a tripartite diagram such as the one of Fig. 5.8, we write eq. (5.19) in terms of the natural frequency f in cycles per second (cps) and take the logarithm of the terms, so that

$$\begin{aligned} S_v &= \omega S_D = 2\pi f S_D \\ \log S_v &= \log f + \log (2\pi S_D) \end{aligned} \tag{5.20}$$

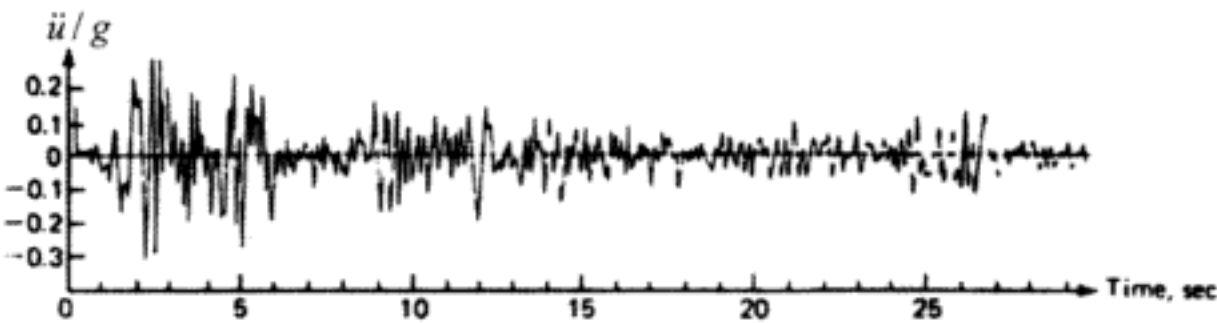


Fig. 5.7 Ground acceleration record for El Centro, California earthquake of May 18, 1940 north-south component.

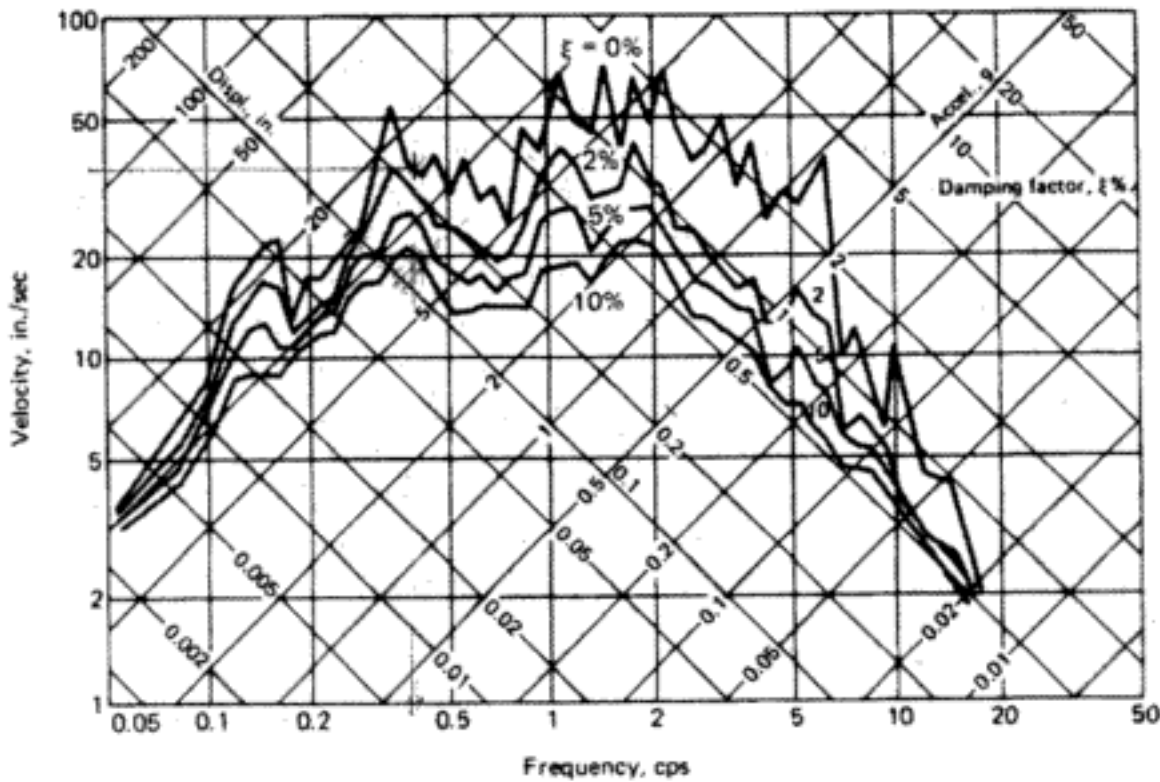


Fig. 5.8 Response spectra for elastic system for the 1940 El Centro earthquake (from Blume et al. 1961.)

For constant values of S_D , eq. (5.20) is the equation of a straight line of $\log S_v$ versus $\log f$ with a slope of 45° . Analogously, from eq. (5.19)

$$S_v = \frac{S_a}{\omega} = \frac{S_a}{2\pi f}$$

$$\log S_v = -\log f + \log \frac{S_a}{2\pi} \quad (5.21)$$

For a constant value of S_a , eq. (5.21) is the equation of a straight line of $\log S_v$ versus $\log f$ with a slope of 135° .

5.4 Response Spectra for Elastic Design

In general, response spectral charts are prepared by calculating the response to a specified excitation of single-degree-of-freedom systems with various amounts of damping. Numerical integration with short time intervals are applied to calculate the response of the system. The step-by-step process is continued until the total earthquake record has been completed. The greatest value of the function of interest is recorded and becomes the maximum response of the system to that excitation. Changing the parameters of the system to change the natural frequency, the process is continued and a new maximum response is recorded. This process is continued until all frequencies of interest have been covered and the results plotted. Since no two earthquakes are alike, this process must be repeated for all earthquakes of interest.

As already stated, until the San Fernando, California earthquake of 1971, there

were few recorded strong earthquake motions because there were few accelerometers emplaced to measure them. The El Centro, California earthquake of 1940 was the most severe earthquake recorded and was used as the basis for much analytical work. Since that year, however, many other strong earthquakes have been recorded. Maximum values of ground motion of about 0.32 g for the El Centro earthquake to values of more than 0.5 g for other earthquakes have been recorded. It can be expected that even larger values will be recorded as more instruments are placed closer to the epicenters of earthquakes.

Earthquakes consist of a series of essentially random ground motions. Usually the north-south, east-west, and vertical components of the ground acceleration are measured. Currently, no accurate method is available to predict the particular motion that a site can be expected to experience in future earthquakes. Thus it is reasonable to use a design response spectrum which incorporates the spectra for several earthquakes and which represents a kind of “average” response spectrum for design. Such a design response spectrum is shown in Fig. 5.9 normalized for a maximum ground acceleration of 1.0 g.

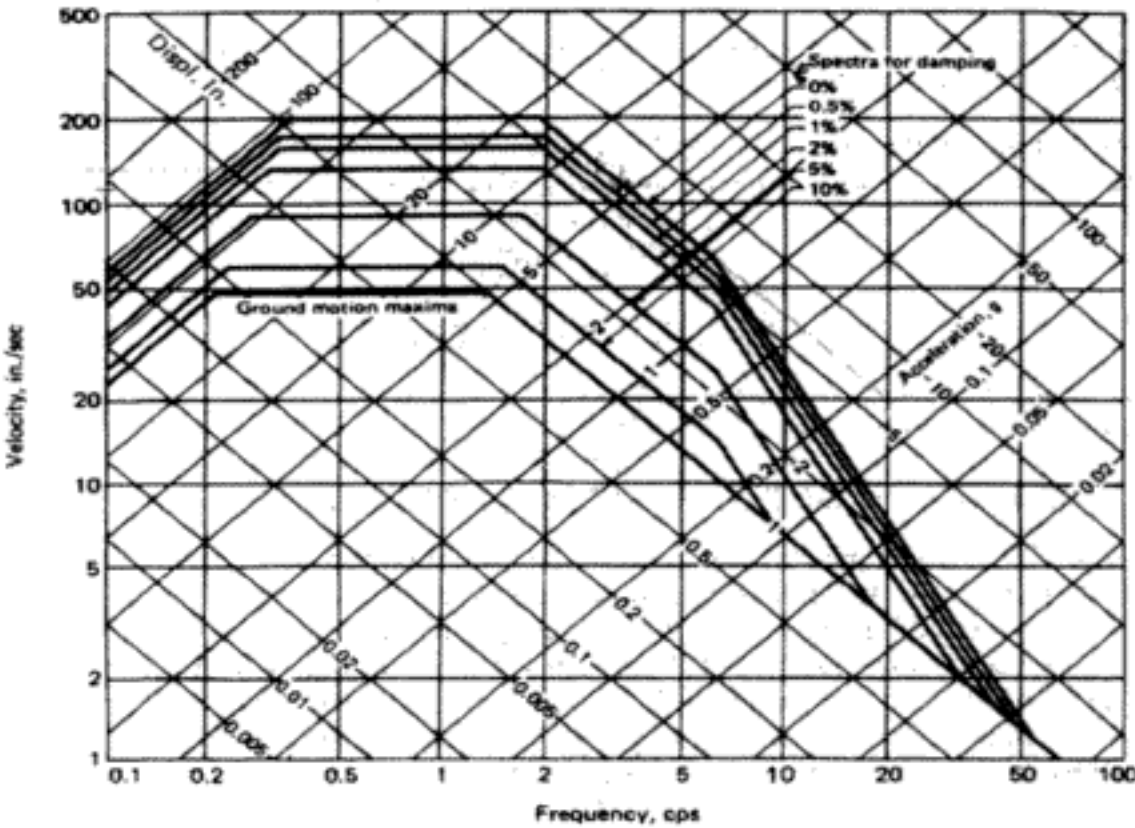


Fig. 5.9 Basic design spectra normalized to peak ground acceleration of 1.0 g. (From Newark and Hall 1973.)

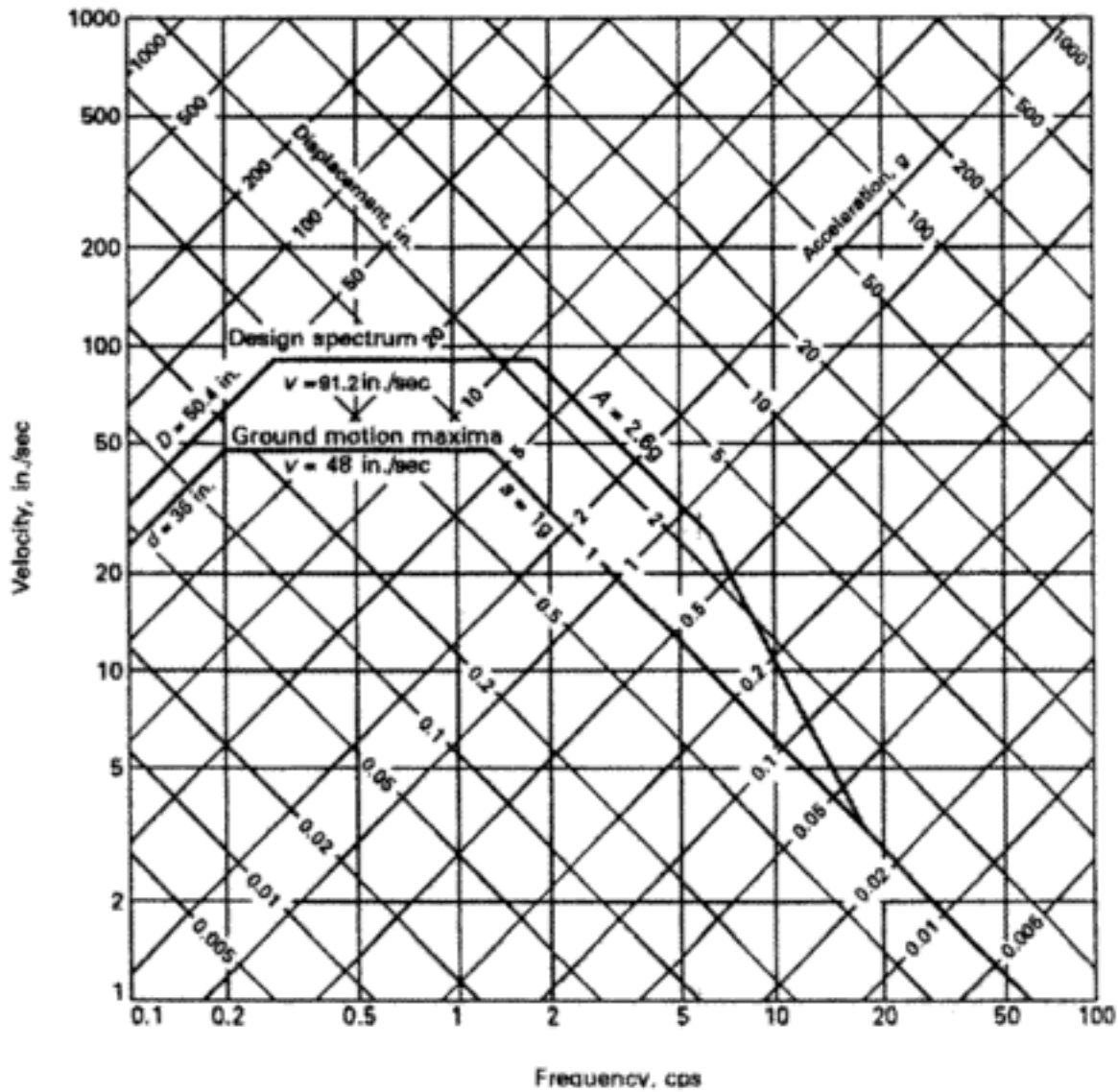


Fig. 5.10 Elastic design spectrum normalized to peak ground acceleration of 1.0 g for 5% damping. (Newmark and Hall 1973.)

This figure shows the design maximum ground motion and a series of response spectral plots corresponding to various values of the damping ratio in the system.

Details for the construction of the basic spectrum for design purposes are given by Newmark and Hall (1973), who have shown that smooth response spectra of idealized ground motion may be obtained by amplifying the ground motion by factors depending on the damping in the system. In general, for any given site, estimates might be made of the expected maximum ground acceleration, maximum ground velocity, and maximum ground displacement. The lines representing these maximum values are drawn on a tripartite logarithmic paper of which Fig. 5.10 is an example. The lines in this figure are shown for a maximum ground acceleration of 1.0 g , a velocity of 48 in./sec, and a displacement of 36 in. These values correspond to motions that are more intense than those generally expected in seismic design. They are, however, of proportional magnitudes that are generally acceptable in practice.

TABLE 5.1 Design Spectrum Amplification Factors
(Newmark and Hall, 1973)

Amplification Factors			
Percent Damping	Displacement	Velocity	Acceleration
0	2.5	4.0	6.4
0.5	2.2	3.6	5.8
1	2.0	3.2	5.2
2	1.8	2.8	4.3
5	1.4	1.9	2.6
7	1.2	1.5	1.9
10	1.1	1.3	1.5
20	1.0	1.1	1.2

These maximum values normalized for a ground acceleration of the ground of 1 g are simply scaled down for other than 1 g maximum acceleration of the ground. Recommended amplification factors to obtain the response spectra from maximum values of the ground motion are given in Table 5.1. For each value of the damping coefficient, the amplified displacement lines are drawn at the left, the amplified velocities at the top, and the amplified acceleration at the right of the chart. At a frequency of approximately 6 cps (Fig. 5.9), the amplified acceleration region line intersects a line sloping down toward the maximum ground acceleration value at a frequency of about 30 cps for a system with 2% damping. The lines corresponding to other values of damping are drawn parallel to the 2% damping line as shown in Fig. 5.9.

The amplification factors in Table 5.1 were developed on the basis of earthquake records available at the time. As new records of more recent earthquakes become available, these amplification factors have been recalculated. Table 5.2 shows the results of a statistical study based on a selection of 10 strong motion earthquakes. The table gives recommended amplification factors as well as the corresponding standard deviation values obtained in the study. The relatively large values shown in Table 5.2 for the standard deviation of the amplification factors provides further evidence on the uncertainties surrounding earthquake prediction for structural analysis. The response spectra for designs presented in Fig. 5.9 has been constructed using the amplification factors shown in Table 5.1.

TABLE 5.2 Spectral Amplification Factors and Standard Deviation Values*

Percent Damping	Displacement		Velocity		Acceleration	
	Factor	Standard Deviation	Factor	Standard Deviation	Factor	Standard Deviation
2	1.691	0.828	2.032	0.853	3.075	0.738
5	1.465	0.630	1.552	0.605	2.281	0.502
10	1.234	0.481	1.201	0.432	1.784	0.321

* Newmark, N. M., and Riddell, R., Inelastic Spectra for Seismic Design: Seventh World Earthquake Conference, Istanbul, Turkey, Vol. 4, pp. 129-136, 1980.

Illustrative Example 5.1

A structure modeled as a single-degree-of-freedom system has a natural period, $T = 1$ sec. Use the response spectral method to determine the maximum absolute acceleration, the maximum relative displacement, and the maximum relative pseudovelocity for: (a) a foundation motion equal to the N-S component of the El Centro earthquake of 1940, and (b) the design spectra earthquake with a maximum ground acceleration equal to $0.32 g$. Assume 10% of the critical damping.

Solution:

(a) From the response spectra in Fig. 5.8 with $f = 1/T = 1.0$ cps, corresponding to the curve labeled $\xi = 0.10$, we read on the three scales the following values:

$$\begin{aligned} S_D &= 3.3 \text{ in.} \\ S_v &= 18.5 \text{ in/sec} \\ S_a &= 0.30 g \end{aligned}$$

(b) From the basic design spectra in Fig. 5.9 with frequency $f = 1$ cps and 10% critical damping, we obtain after scaling for $0.32 g$ maximum ground acceleration, the following results:

$$\begin{aligned} S_D &= 9.5 \times 0.32 = 3.04 \text{ in} \\ S_v &= 60 \times 0.32 = 19.2 \text{ in/sec} \\ S_a &= 0.95 \times 0.32 = 0.304 g \end{aligned}$$

5.5 Influence of Local Soil Conditions

Before the San Fernando earthquake of 1971, earthquake accelerograms were limited in number, and the majority had been recorded on alluvium. Therefore, it is only natural that the design spectra based on those data, such as those suggested by Housner (1959) and Newmark-Hall (1973), mainly represent alluvial sites. Since 1971, the wealth of information obtained from earthquakes worldwide and from subsequent studies have shown the very significant effect that the local site conditions have on spectral shapes.

An example of a conservative design spectrum is shown in Fig. 5.11. This figure shows four spectral acceleration curves representing the average of normalized spectral values corresponding to several sets of earthquake records registered on four types of soils. The dashed line through the points A, B, C, and D defines a possible conservative design spectrum for rock and stiff soil sites. Normalized design spectral shapes, as those included in the recent edition of the Uniform Building Code (ICBO, 1994) (Fig. 5.12) are based on such simplifications. The UBC spectral shapes become trilinear, when drawn on a tripartite logarithmic chart, similar in shape to a Newmark-Hall spectrum. UBC spectral shapes (ICBO, 1994) can be expressed¹ by the following

¹ From Recommended Lateral Force Requirement and Tentative Commentary SEAOL-90, P. 36-C.

rather simple formulas:

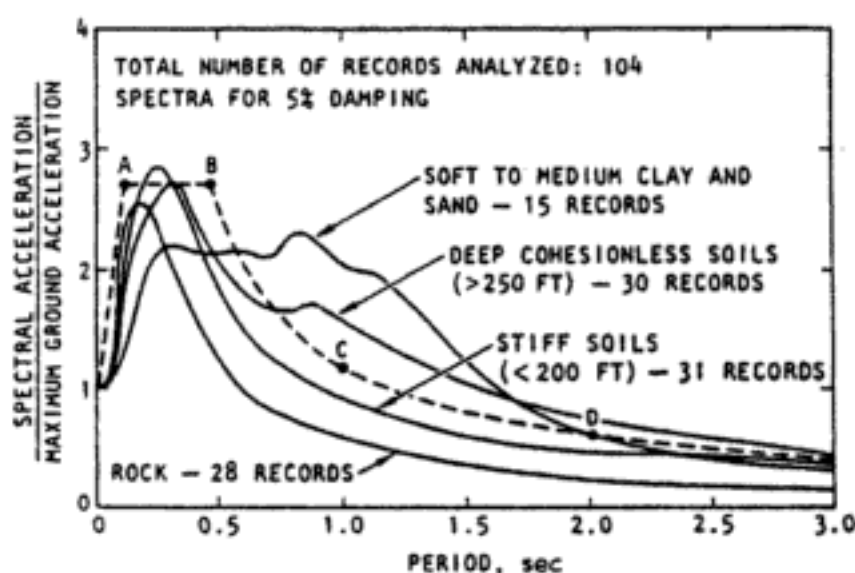


Fig. 5.11 Average acceleration spectra for different soil conditions (After Seed et al. 1976; from Seed and Idriss 1982.)

Soil Type I (Rock and Stiff Soils):

$$\begin{aligned} S_A &= 1 + 10 T & \text{for} & \quad 0 < T \leq 0.15 \text{ sec} \\ S_A &= 2.5 & \text{for} & \quad 0.15 < T \leq 0.39 \text{ sec} \\ S_A &= 0.975 / T & \text{for} & \quad T > 0.39 \text{ sec} \end{aligned} \quad (5.22)$$

Soil Type II (Deep Cohesionless or Stiff Clay Soils):

$$\begin{aligned} S_A &= 1 + 10 T & \text{for} & \quad 0 < T \leq 0.15 \text{ sec} \\ S_A &= 2.5 & \text{for} & \quad 0.15 < T \leq 0.585 \text{ sec} \\ S_A &= 1.463 / T & \text{for} & \quad T > 0.585 \text{ sec} \end{aligned} \quad (5.23)$$

Soil Type III (Soft to Medium Clays and Sands):

$$\begin{aligned} S_A &= 1 + 75 T & \text{for} & \quad 0 < T \leq 0.2 \text{ sec} \\ S_A &= 2.5 & \text{for} & \quad 0.2 < T \leq 0.915 \text{ sec} \\ S_A &= 2.288 / T & \text{for} & \quad T > 0.915 \text{ sec} \end{aligned} \quad (5.24)$$

where S_A is the spectral acceleration for 5% damping normalized to a peak ground acceleration of one g , and T is the fundamental period of the building. It should be noted that values obtained from the UBC spectral chart of Fig. 5.12, or alternatively, calculated with eqs. (5.22) to (5.24) are too conservative. In actual design practice, these values are scaled down by the structural factor R (Chapter 24) having values between 4 and 12, depending on the type of structural resisting system.

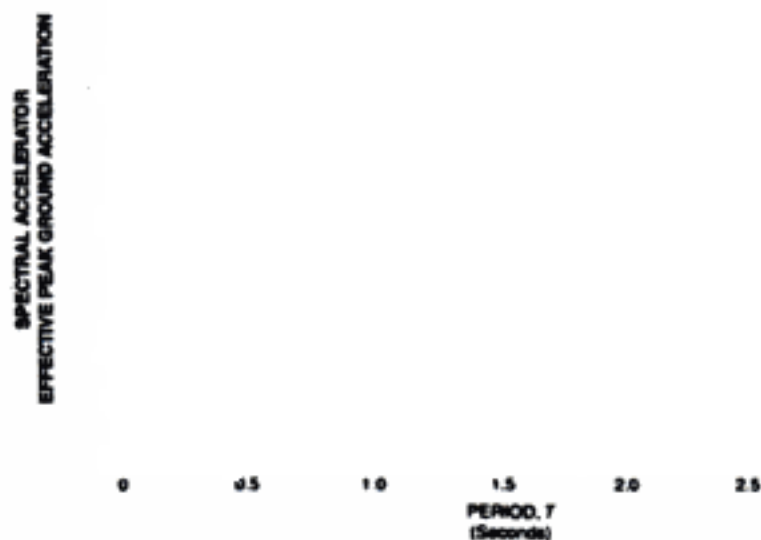


Fig. 5.12 Normalized design spectra shapes contained in Uniform Code. (ICBO 1994)

5.6 Response Spectra for Inelastic Systems

For certain types of extreme events such as nuclear blast explosions motion earthquakes, it is sometimes necessary to design structures strains beyond the elastic limit. For example, in seismic design for an very strong withstand of

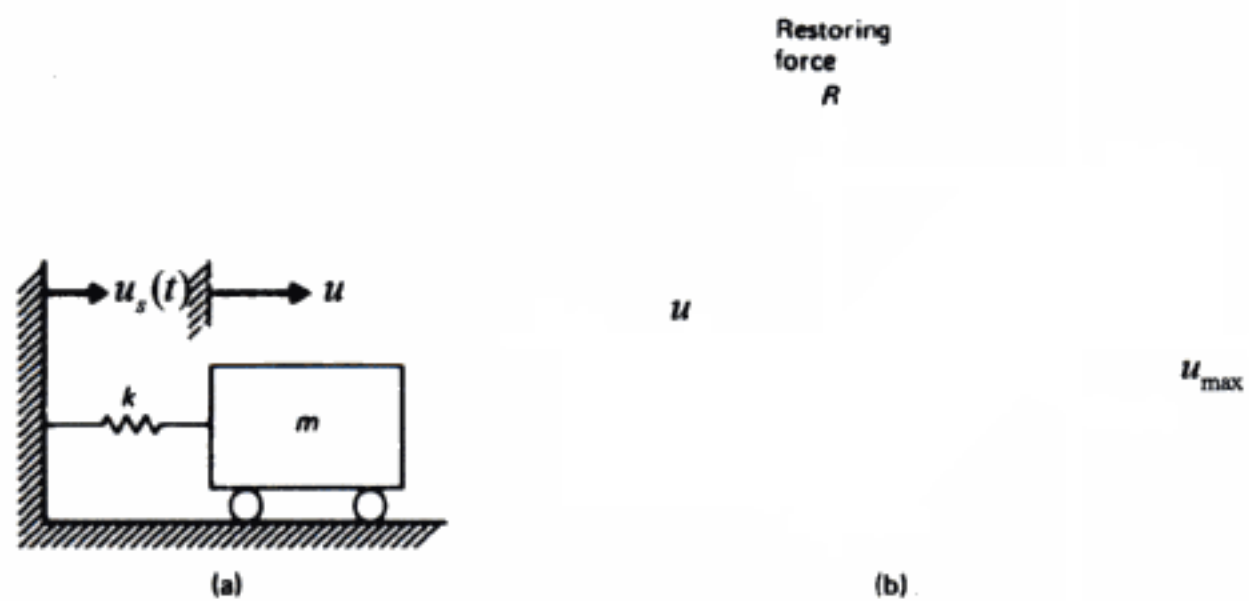


Fig. 5.13 Force-displacement relationship for an elastoplastic freedom system.

moderate intensity, it is reasonable to assume elastic behavior for a well-constructed structure. However, for very strong motions, this is and a realistic

assumption even for a well-designed structure. Although structures can be designed to resist severe earthquakes, it is not feasible economically to design buildings to elastically withstand earthquakes of the greatest foreseeable intensity. In order to design structures for strain levels beyond the linear range, the response spectrum has been extended to include the inelastic range (Newmark and Hall 1973). Generally, the elastoplastic relation between force and displacement is assumed in structural dynamics. Such a force-displacement relationship is shown in Fig 5.13(b). Because of the assumption of elastoplastic behavior, if the force is removed prior to the occurrence of yielding, the material will return along its loading line to the origin. However, when yielding occurs at displacement u_y , the restoring force remains constant at a magnitude R_y . If the displacement is not reversed, the displacement may reach a maximum value u_{max} . If, however, the displacement is reversed, the elastic recovery follows along a line parallel to the initial line and the recovery proceeds elastically until a negative yield value R_c is reached in the opposite direction.

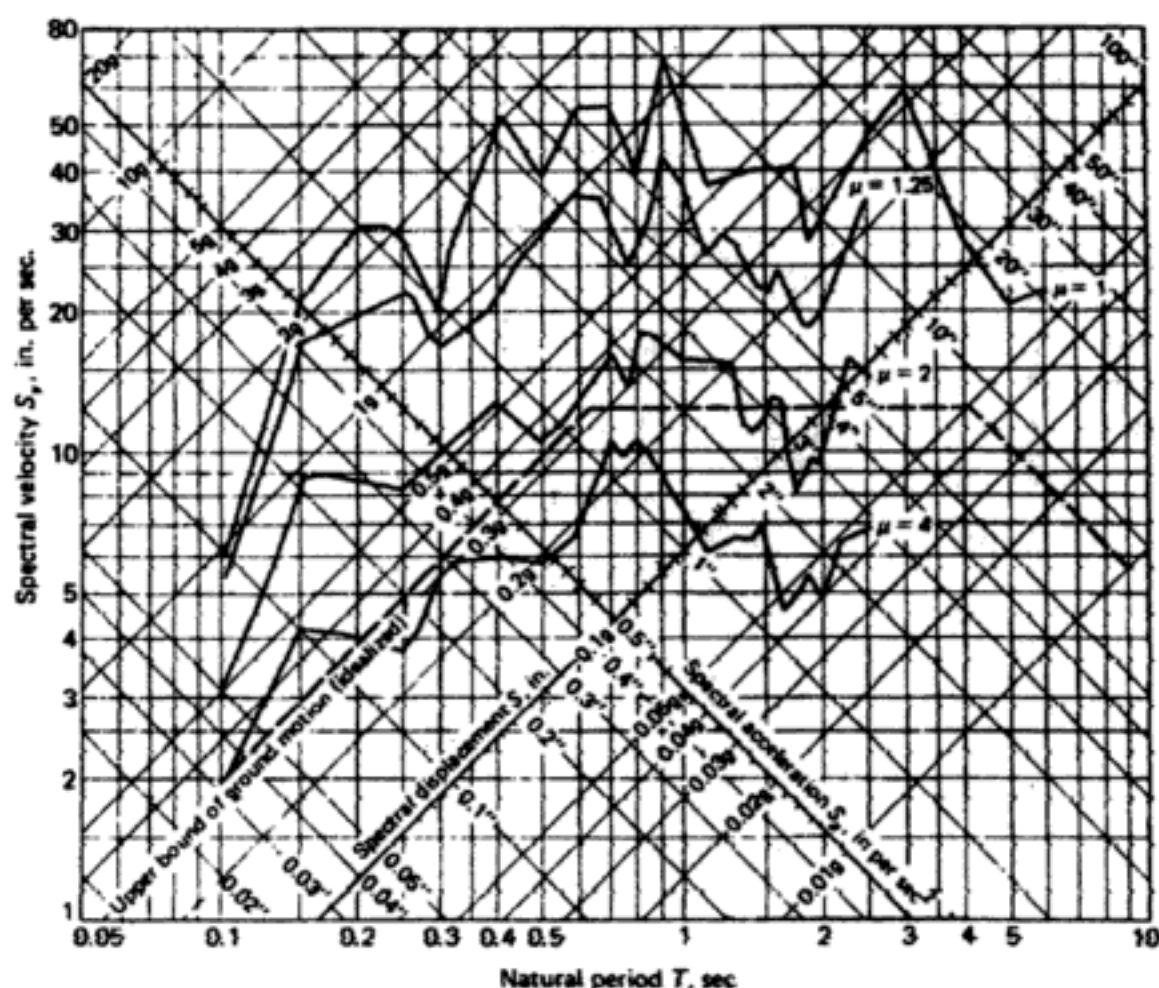


Fig. 5.14 Response spectra for undamped elastoplastic system for the 1940 El Centro earthquake. (From Blume et al. 1961.)

The preparation of response spectra for such an inelastic system is more complex than that for elastic systems. However, response spectra have been prepared for several different earthquake motions. These spectra are usually plotted as a series of curves corresponding to definite values of the ductility ratio μ . The ductility ratio μ is defined as the ratio of the maximum displacement u_{max} of the structure in the inelastic range to

the displacement corresponding to the yield point u_y , that is,

$$\mu = \frac{u_{\max}}{u_y} \quad (5.25)$$

The response spectra for an undamped single-degree-of-freedom system subjected to a support motion equal to the El Centro 1940 earthquake is shown in Fig. 5.14 for several values of the ductility ratio. The tripartite logarithmic scales used to plot these spectra give simultaneously for any single-degree-of-freedom system of natural period T and specified ductility ratio μ , the spectral values of displacement, velocity, and acceleration. Similarly, in Fig. 5.15 are shown the response spectra for an elastoplastic system with 10% of critical damping. The spectral velocity and the spectral acceleration are read directly from the plots in Figs. 5.14 and 5.15, whereas the values obtained for the spectral displacement must be multiplied by the ductility ratio in order to obtain the correct value for the spectral displacement.

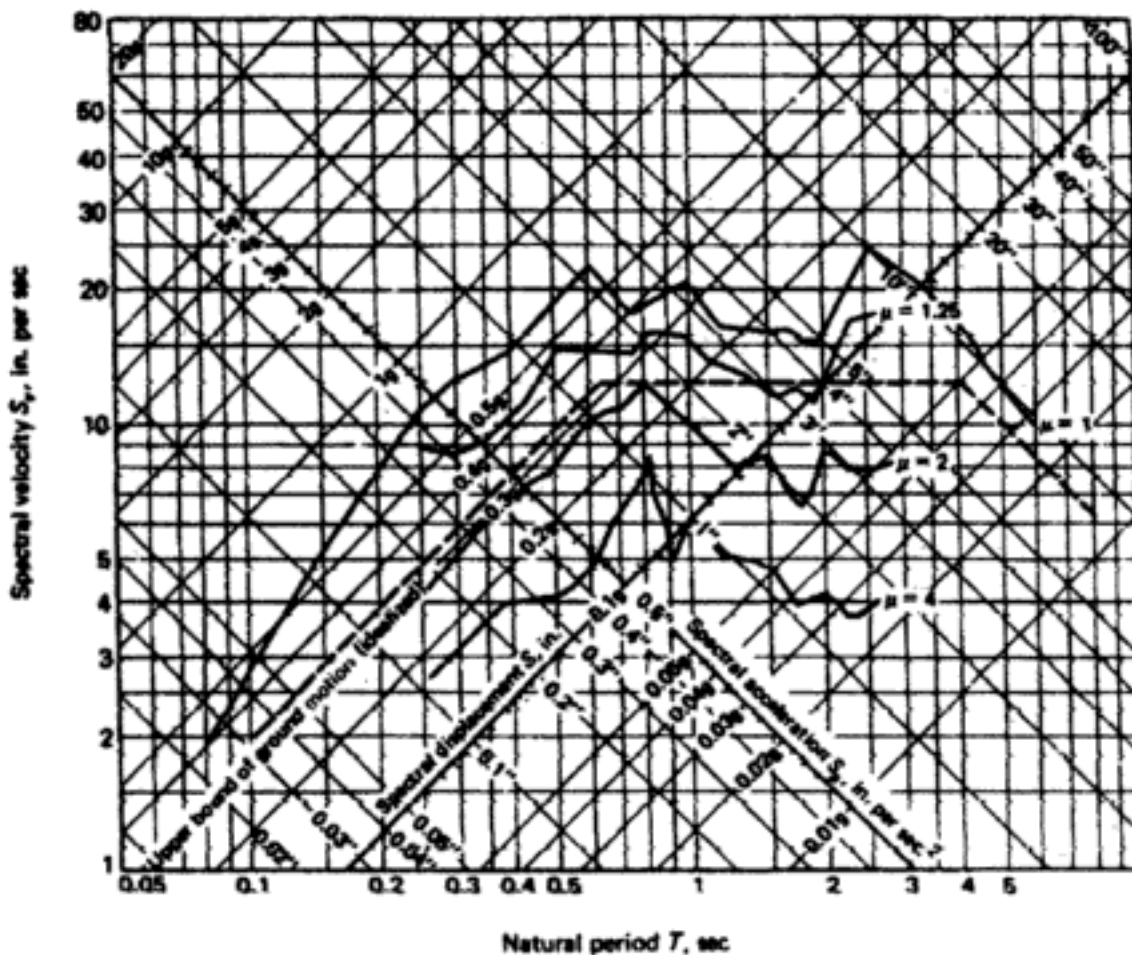


Fig. 5.15 Response spectra for elastoplastic system with 10% critical damping for the 1940 El Centro earthquake. (From Blume et al. 1961.)

The concept of ductility ratio has been associated mainly with steel structures. These structures have a load-deflection curve that is often approximated as the elastoplastic curve shown in Fig. 5.13(b). For other types of structures, such as reinforced concrete structures or masonry shear walls, still, conveniently, the load-deflection curve is modeled as the elastoplastic curve. Although for steel structures, ductility factors as high as 6 are often used in collapse-level earthquake design, lower values for the

ductility ratio are applicable to reinforced concrete or masonry shear walls. The selection of ductility values for seismic design must also be based on the design objectives and the loading criteria as well as the risk level acceptable for the structure as it relates to its use.

For reinforced concrete structures or masonry walls, a ductility factor of 1.0 to 1.5 seems appropriate for earthquake design where the objective is to limit damage. In other words, the objective of limit damage requires that structural members should be designed to undergo little if any yielding. When the design objective is to prevent collapse of the structure, ductilities of 2 to 3 are appropriate in this case.

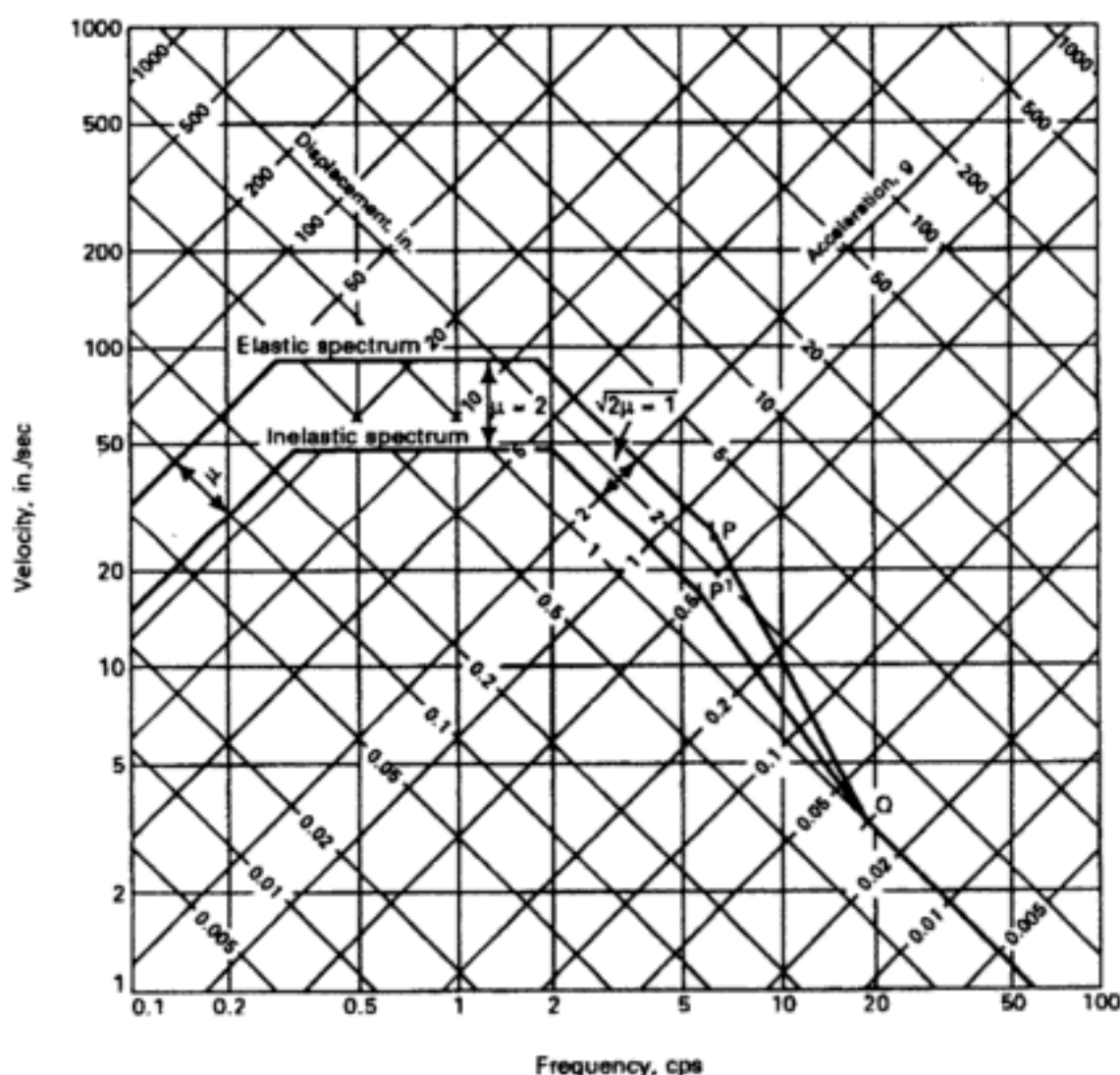


Fig. 5.16 Inelastic design spectrum normalized to peak ground acceleration of 1.0 g for 5% damping and ductility ratio $\mu = 2.0$. (NOTE: Chart gives directly S_v and S_a ; however, displacement values obtained from the chart should be multiplied by μ to obtain S_D .)

5.7 Response Spectra for Inelastic Design

In the preceding section of this chapter, we discussed the procedure for calculating the seismic response spectra for elastic design. Figure 5.9 shows the elastic design spectra for several values of the damping ratio. The same procedure of constructing of a basic response spectrum that consolidates the "average" effect of several earthquake records may also be applied to design in the inelastic range. The spectra for

elastoplastic systems have the same appearance as the spectra for elastic systems, but the curves are displaced downward by an amount that is related to the ductility ratio μ . Figure 5.16 shows the construction of a typical design spectrum currently recommended (Newmark and Hall 1973) for use when inelastic action is anticipated.

The elastic spectrum for design from Fig. 5.9 corresponding to the desired damping ratio is copied in a tripartite logarithmic paper as shown in Fig. 5.16 for the spectra corresponding to 5% damping. Then lines reduced by the specified ductility ratio are drawn parallel to elastic spectral lines in the displacement region (the left region) and in the velocity region (the central region). However, in the acceleration region (the right region) the recommended reducing factor is $\sqrt{2\mu-1}$. This last line is extended up to a frequency of about 6 cps (point P' in Fig. 5.16). Then the inelastic design spectrum is completed by drawing a line from this last point P' to point Q , where the descending line from point P of the elastic spectrum intersects the line of constant acceleration as shown in Fig. 5.16.

The development of the reduction factors in the displacement and velocity regions is explained with the aid of Fig. 5.17(a). This figure shows the force-displacement curves for elastic and for elastoplastic behavior. At equal maximum displacement u_{max} for the two curves, we obtain from Fig. 5.17(a) the following relationship:

$$\mu = \frac{u_{max}}{u_y} = \frac{F_E}{F_y}$$

or

$$F_y = \frac{F_E}{\mu} \quad (5.26)$$

where F_E is the force corresponding to the maximum displacement u_{max} in the elastic curve and F_y is the force at the yield condition. Equation (5.26) then shows that the force and consequently the acceleration in the elastoplastic system is equal to the corresponding value in the elastic system reduced by the ductility ratio. Therefore, the spectral acceleration S_a for elastoplastic behavior is related to the elastic spectral acceleration S_{aE} by

$$S_a = \frac{S_{aE}}{\mu} \quad (5.27)$$

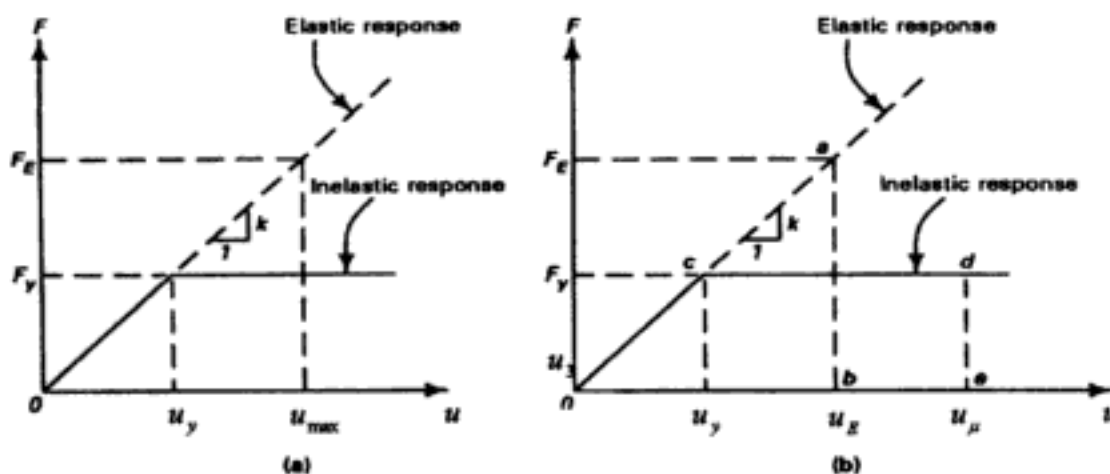


Fig. 5.17 Elastic and inelastic force-displacement curves. (a) Displacement and velocity regions. (b) Acceleration region.

However, in the acceleration region of the response spectrum, a ductility reduction factor μ does not result in close agreement with experimental data as does the recommended reduction factor $\sqrt{2\mu-1}$. This factor may be rationally obtained by establishing the equivalence of the energy between the elastic and the inelastic system. In reference to Fig. 5.17(b) this equivalence is established by equating the area under elastic curve "Oab" with the area under the inelastic curve "Ocde." Namely,

$$\frac{F_E u_E}{2} = \frac{F_y u_y}{2} + F_y (u_\mu - u_y) \quad (5.28)$$

where F_E and F_y are the elastic and inelastic forces corresponding respectively to the maximum elastic displacement u_E and to the maximum inelastic displacement u_μ and where u_y is the yield displacement.

The substitution of $u_E = F_E/k$, $u_y = F_y/k$, and $u_\mu = \mu u_y$ into eq. (5.28) gives

$$\frac{F_E^2}{2k} = \frac{F_y^2}{2k} (2\mu - 1)$$

or

$$\frac{F_E}{F_y} = \sqrt{2\mu - 1}$$

Thus,

$$F_y = \frac{F_E}{\sqrt{2\mu - 1}}$$

and, consequently

$$S_a = \frac{S_{aE}}{\sqrt{2\mu - 1}}$$

Thus, the inelastic spectral acceleration in the acceleration region is obtained by reducing the elastic spectral by the factor $\sqrt{2\mu-1}$.

The inelastic response spectrum thus constructed and shown in Fig. 5.16 gives directly the values for the spectral acceleration S_a and spectral velocity S_v . However, the values read from this chart in a displacement scale must be multiplied by the ductility ratio μ to obtain the spectral displacement S_D . Inelastic response spectral charts for design developed by the procedure explained are presented in Figs. 5.18, 5.19, and 5.20 correspondingly to damping factors $\xi = 0, 5\%$, and 10% and for ductility ratios $\mu = 1, 2, 5$, and 10 .

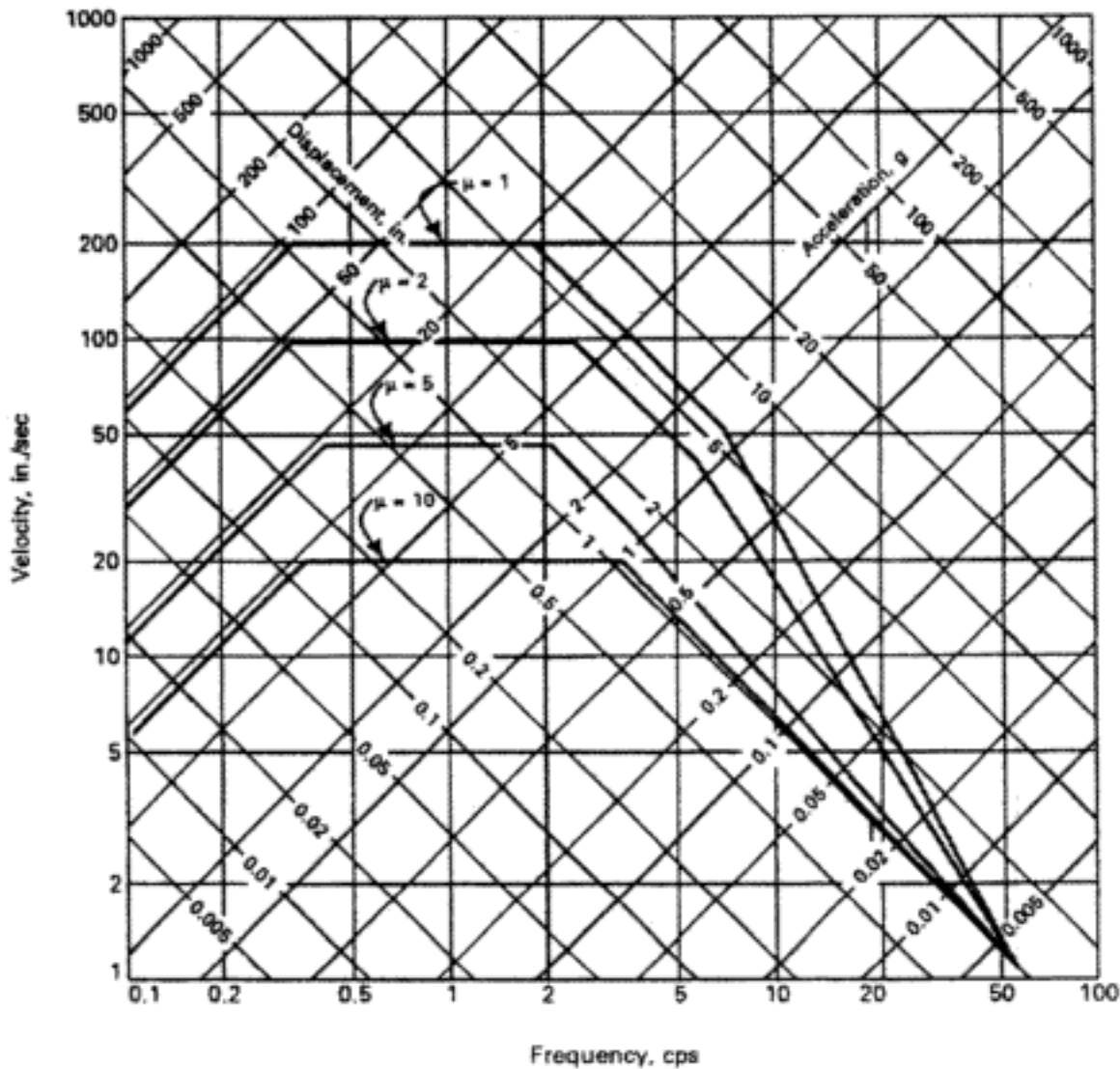


Fig. 5.18 Undamped inelastic design spectra normalized to peak ground acceleration of 1.0 g . (Spectral values S_a for acceleration and S_v for velocity are obtained directly from the graph; however, values of S_D for spectral displacement should be amplified by the ductility ratio μ .)

Illustrative Example 5.2

Calculate the response of the single-degree-of-freedom system of Example 5.1, assuming that the structure is designed to withstand seismic motions with an elastoplastic behavior having a ductility ratio $\mu = 4.0$. Assume damping equal to 10% of the critical damping. (a) Use the response spectra of the El Centro earthquake and (b) Use the design response spectra.

Solution:

(a) From the response spectrum corresponding to 10% of the critical damping (Fig. 5.15), we read for $T = 1$ sec and the curve labeled $\mu = 4.0$

$$S_D = 1.0 \times 4.0 = 4.0 \text{ in, } S_v = 6.2 \text{ in/sec and } S_a = 0.1 g.$$

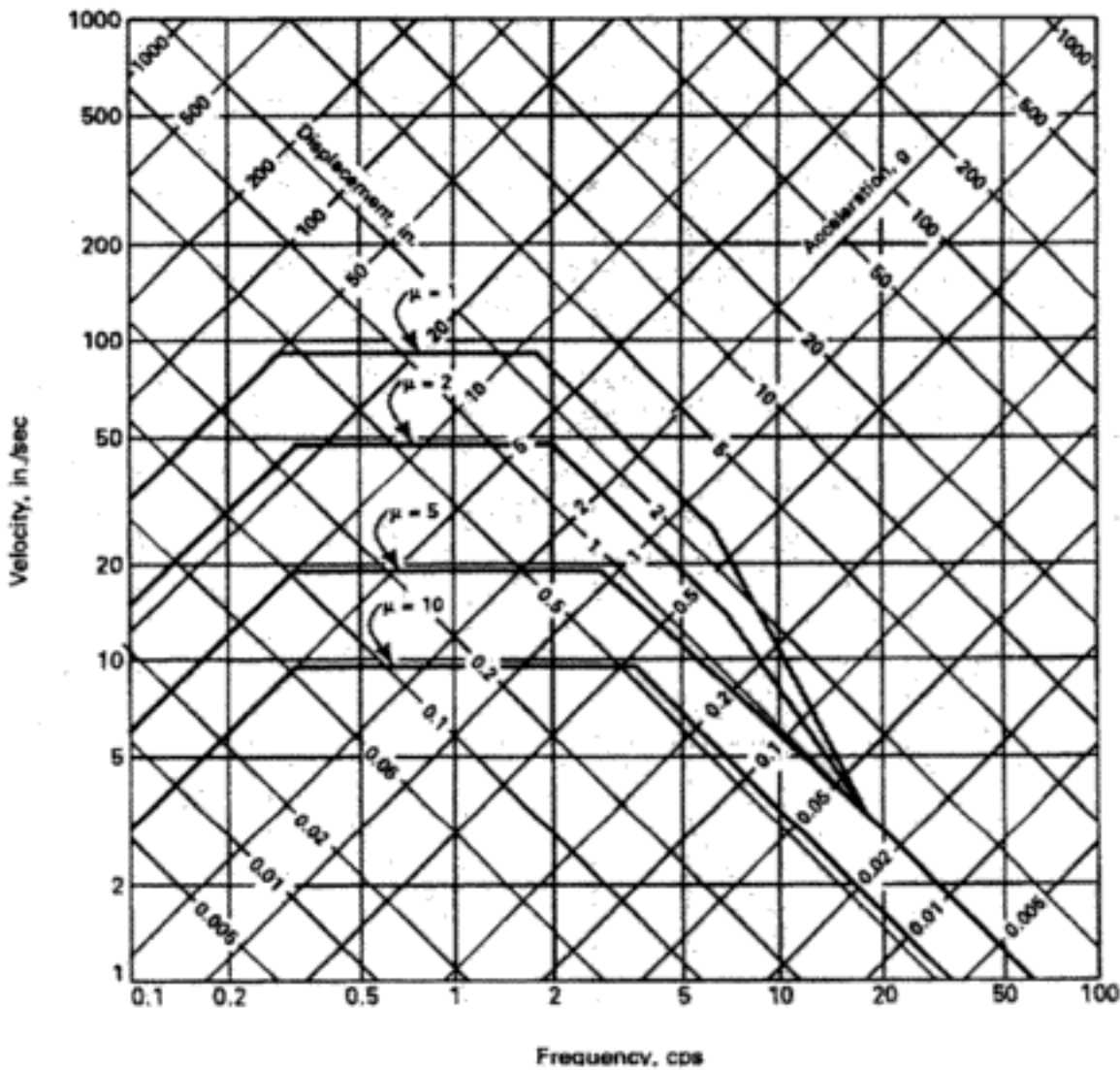


Fig. 5.19 Inelastic design spectra normalized to peak ground acceleration of 1.0 *g* for 5% damping. (Spectral values *S_a* for acceleration and *S_v* for velocity are directly obtained from the graph. However, values of *S_D* for spectral displacement should be amplified by the ductility ratio *μ*.)

The factor 4.0 is required in the calculation of *S_D* since as previously noted the spectra plotted in Fig. 5.15 are correct for acceleration and for pseudovelocity, but for displacements it is necessary to amplify the values read from the chart by the ductility ratio

(b) Using the inelastic design spectra with 10% damping for design in Fig. 5.20, corresponding to 1 cps, we obtain the following maximum values for the response:

$$\begin{aligned} S_D &= 3.0 \times 0.32 \times 4.0 = 3.84 \text{ in} \\ S_v &= 15.6 \times 0.32 = 5.00 \text{ in/sec} \\ S_a &= 0.3 \times 0.32 = 0.096 \text{ g} \end{aligned}$$

As it can be seen, these spectral values based on the design spectrum are somewhat different than those obtained from the response spectrum of the El Centro earthquake

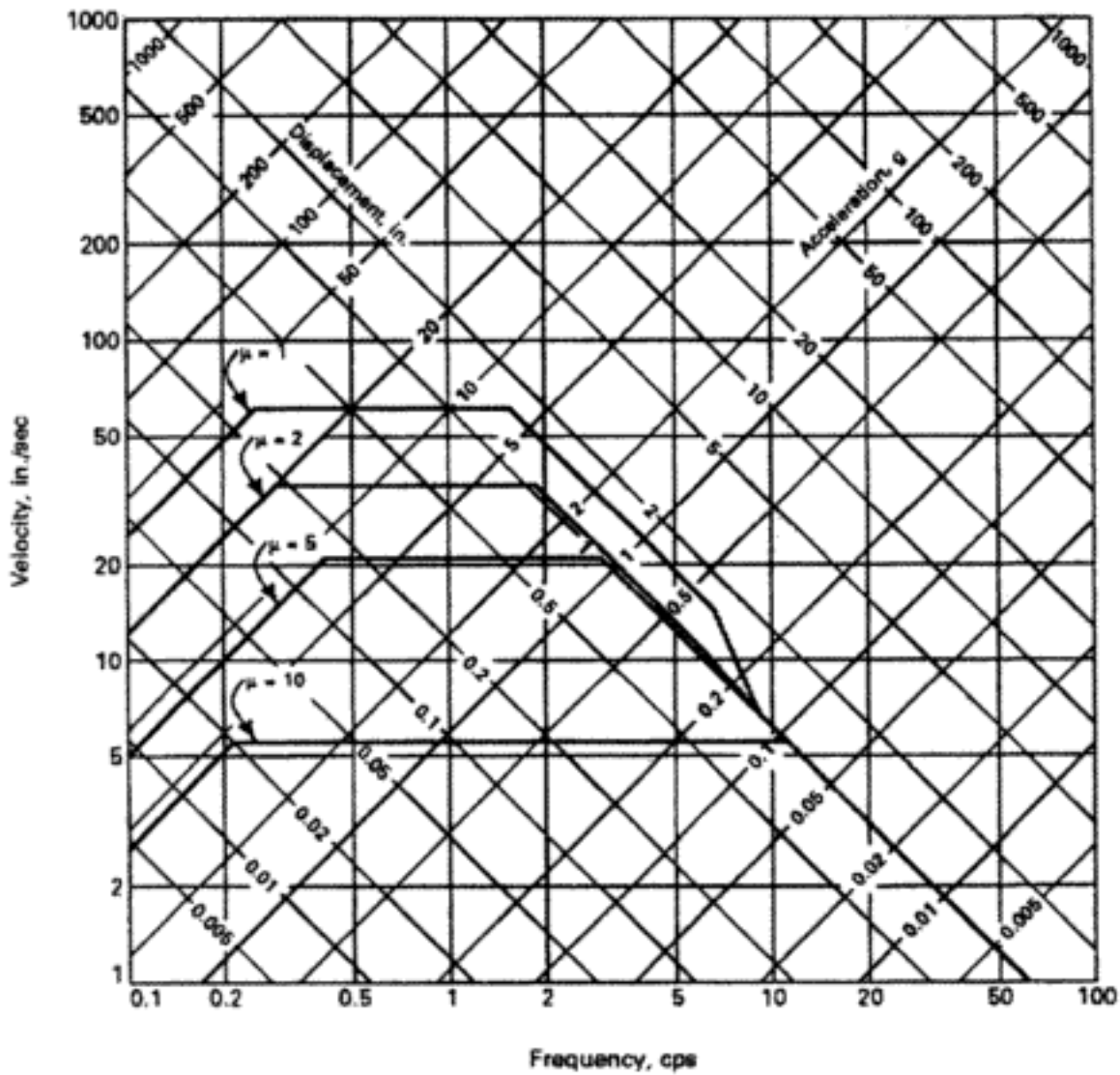


Fig. 5.20 Inelastic design spectra normalized to peak ground acceleration of $1.0 g'$ for 10% damping. (Spectral values S_a for acceleration and S_v for velocity are directly obtained from the graph; however, values of S_D for spectral displacement should be amplified by the ductility ratio μ .)

of 1940. Also, if we compare these results for the elastoplastic behavior with the results in Example 5.1 for the elastic structure, we observe that the maximum relative displacement has essentially the same magnitude whereas the acceleration and the relative pseudovelocity are appreciably less. This observation is in general true for any structure when inelastic response is compared with the response based on elastic behavior.

5.8 Program 6—Seismic Response Spectra

The computer program described in this chapter serves to calculate elastic response spectra in terms of spectral displacement (maximum relative displacement), spectral velocity (maximum relative pseudovelocity), and spectral acceleration (maximum absolute acceleration) for any prescribed time-acceleration-seismic excitation. The response is calculated in the specified range of frequencies using the direct integration method presented in Chapter 4.

Illustrative Example 5.3

Use Program 6 to develop the response spectra for elastic systems subjected to the first ten seconds of the 1940 El Centro earthquake. Assume 10% damping.

The digitized values corresponding to the accelerations recorded for the first 10 sec of the El Centro earthquake are given in Table 5.3.

Solution:

PROGRAM 6, SEISMIC RESPONSE SPECTRA

DATA FILE: D6

INPUT DATA:

NUMBER OF POINTS DEFINING THE EXCITATION	NE = 186
INITIAL FREQUENCY (C.P.S.)	FI = .05
FREQUENCY INCREMENT (C.P.S.)	DF = .05
FINAL FREQUENCY (S.P.S.)	F = 20
DAMPING RATIO	XSI = .1
TIME STEP INTEGRATION	H = .01
ACCELERATION GRAVITY	G = 386

OUTPUT RESULTS:

FREQ. (C.P.S.)	SPECT. DISPL. (IN)	SPECT. VELOC. (IN/SEC)	SPECT. ACC. (IN/SEC/SEC)
0.05	24.65	7.70	2.41
0.10	21.86	13.67	8.54
0.15	15.36	14.41	13.51
0.20	12.63	15.79	19.75
0.25	13.33	20.83	32.56
0.30	13.62	25.55	47.92
0.35	12.33	26.97	59.02
0.40	12.02	30.06	75.17
0.45	8.75	24.63	69.28
0.50	6.31	19.72	61.64
1.00	3.26	20.38	127.40
1.50	2.43	22.78	213.57
2.00	2.02	25.29	316.18
2.50	1.22	19.13	299.00
3.00	0.74	13.96	261.73
3.50	0.53	11.49	251.39
4.00	0.41	10.28	257.19
4.50	0.32	9.08	255.40
5.00	0.30	9.42	294.58
10.00	0.05	3.20	200.10
15.00	0.02	1.77	165.54
20.00	0.01	0.99	124.09

Table 5.3 Digitized Values of the Acceleration Recorded for the First Ten Second of the N-S component for the El Centro Earthquake of 1940

Time (sec)	Acc. (acc. g)	Time (sec)	Acc. (acc. g)	Time (sec)	Acc. (acc. g)	Time (sec)	Acc. (acc. g)
0.0000	0.0108	0.0420	0.0020	0.0970	0.0159	0.1610	-0.0001
0.2210	0.0189	0.2630	0.0001	0.2910	0.0059	0.3320	-0.0012
0.3740	0.0200	0.4290	-0.0237	0.4710	0.0076	0.5810	0.0425
0.6230	0.0094	0.6650	0.0138	0.7200	-0.0088	0.7400	-0.0256
0.7890	-0.0387	0.8290	-0.0568	0.8720	-0.0232	0.9020	-0.0343
0.9410	-0.0402	0.9610	-0.0603	0.9970	-0.0789	1.0660	-0.0666
1.0760	-0.0381	1.0940	-0.0429	1.1680	0.0897	1.3150	-0.1696
1.3840	-0.0828	1.4120	-0.0828	1.4400	-0.0945	1.4810	-0.0885
1.5090	-0.1080	1.5370	-0.1280	1.6280	0.1144	1.7030	0.2355
1.8550	0.1428	1.8800	0.1777	1.9240	-0.2610	2.0070	-0.3194
2.2150	0.2952	2.2700	0.2634	2.3200	-0.2984	2.3950	0.0054
2.4500	0.2865	2.5190	-0.0469	2.5750	0.1516	2.6520	0.2077
2.7080	0.1087	2.7690	-0.0325	2.8930	0.1033	2.9760	-0.0803
3.0680	0.0520	3.1290	-0.1547	3.2120	0.0065	3.2530	-0.2060
3.3860	0.1927	3.4190	-0.0937	3.5300	0.1708	3.5990	-0.0359
3.6680	0.0365	3.7380	-0.0736	3.8350	0.0311	3.9040	-0.1833
4.0140	0.0227	4.0560	-0.0435	4.1060	0.0216	4.2220	-0.1972
4.3140	-0.1762	4.4160	0.1460	4.4710	-0.0047	4.6180	0.2572
4.6650	-0.2045	4.7560	0.0608	4.8310	-0.2733	4.9700	0.1779
5.0390	0.0301	5.1080	0.2183	5.1990	0.0267	5.2330	0.1252
5.3020	0.1290	5.3300	0.1089	5.3430	-0.0239	5.4540	0.1723
5.5100	-0.1021	5.6060	0.0141	5.6900	-0.1949	5.7730	-0.2420
5.8000	-0.0050	5.8090	-0.0275	5.8690	-0.0573	5.8830	-0.0327
5.9250	0.0216	5.9800	0.0108	6.0130	0.0235	6.0850	-0.0665
6.1320	0.0014	6.1740	0.0493	6.1880	0.0149	6.1980	-0.0200
6.2290	-0.0381	6.2790	0.0207	6.3260	-0.0058	6.3680	-0.0603
6.3820	-0.0162	6.4090	0.0200	6.4590	-0.1760	6.4780	-0.0033
6.5200	0.0043	6.5340	-0.0040	6.5620	-0.0099	6.5750	-0.0017
6.6030	-0.0170	6.6450	0.0373	6.6860	0.0457	6.7140	0.0385
6.7280	0.0009	6.7490	-0.0288	6.7690	0.0016	6.8110	0.0113
6.8520	0.0022	6.9080	0.0092	6.9910	-0.0996	7.0740	0.0360
7.1210	0.0078	7.1430	-0.0277	7.1490	0.0026	7.1710	0.0272
7.2260	0.0576	7.2950	-0.0492	7.3700	0.0297	7.4060	0.0109
7.4250	0.0186	7.4610	-0.2530	7.5250	-0.0347	7.5720	0.0036
7.6000	-0.0628	7.6410	-0.0280	7.6690	-0.0196	7.6910	0.0068
7.7520	-0.0054	7.7940	-0.0603	7.8350	-0.0357	7.8770	-0.0716
7.9600	-0.0140	7.9870	-0.0056	8.0010	0.0222	8.0700	0.0468
8.1260	0.0260	8.1660	-0.0335	8.1950	-0.0128	8.2230	0.0661
8.2780	0.0305	8.3340	0.0246	8.4030	0.0347	8.4580	-0.0369
8.5330	-0.0344	8.5960	-0.0104	8.6380	-0.0260	8.7350	0.1534
8.8180	-0.0028	8.8600	0.0233	8.8820	-0.0261	8.9150	-0.0022
8.9560	-0.1849	9.0530	0.1260	9.0950	0.0320	9.1230	0.0955
9.1500	0.1246	9.2530	-0.0328	9.2890	-0.0451	9.4270	0.1301
9.4410	-0.1657	9.5100	0.0419	9.6350	-0.0936	9.7040	0.0816
9.8150	-0.0881	9.8980	0.0064	9.9390	-0.0006	9.9950	0.0586
10.0200	-0.0713	10.0500	-0.0448	10.0800	-0.0221	10.1000	0.0093
10.1500	0.0024	10.1900	0.0510				

5.9 Summary

Response spectra are plots that give the maximum response for a single-degree-of-freedom system subjected to a specified excitation. The construction of these plots requires the solution of single-degree-of-freedom systems for a sequence of values of the natural frequency and of the damping ratio in the range of interest. Every solution provides only one point (the maximum value) of the response spectrum. In solving the single-degree-of-freedom systems, use is made of Duhamel's integral or of the direct method (Chapter 4) for elastic systems and of the step-by-step linear acceleration method for inelastic behavior (Chapter 6). Since a large number of systems must be analyzed to fully plot each response spectrum, the task is lengthy and time consuming even with the use of a computer. However, once these curves are constructed and are available for the excitation of interest, the analysis for the design of structures subjected to dynamic loading is reduced to a simple calculation of the natural frequency of the system and the use of the response spectrum.

In the chapters of Part II dealing with structures that are modeled as systems with many degrees of freedom, it will be shown that the dynamic analysis of a system with n degrees of freedom can be transformed to the problem of solving n systems in which each one is a single-degree-of-freedom system. Consequently, this transformation extends the usefulness of response spectra for single-degree-of-freedom systems to the solution of systems of any number of degrees of freedom.

The reader should thus realize the full importance of a thorough understanding and mastery of the concepts and methods of solutions for single-degree-of-freedom systems, since these same methods are also applicable to systems of many degrees of freedom after the problem has been transformed to independent single-degree-of-freedom systems.

5.10 Problems

Problem 5.1

The steel frame shown in Fig. P5.1 is subjected to horizontal force at the girder level of $(1000 \sin 10t \text{ lb})$ for a time duration of half a cycle of the forcing sine function. Use the appropriate response spectral chart to obtain the maximum displacement. Neglect damping.

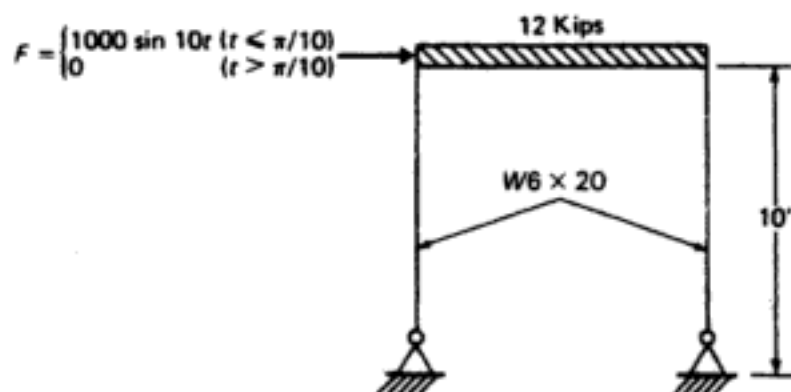


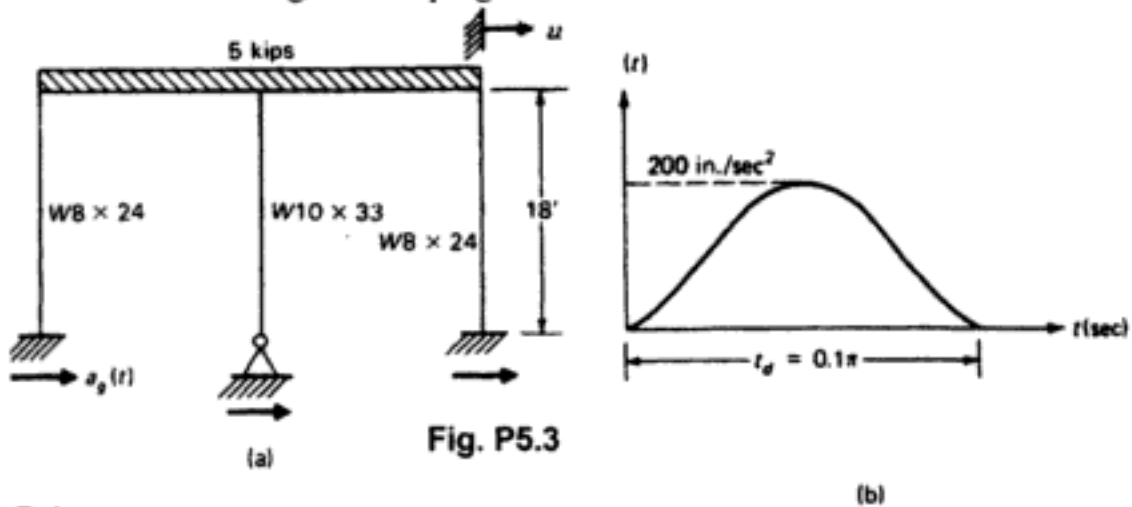
Fig. P5.1

Problem 5.2

Determine the maximum stresses in the columns of the frame of Problem 5.1.

Problem 5.3

Consider the frame shown in Fig. P5.3(a) subjected to a foundation excitation produced by a half cycle of the function $a_g = 200 \sin 10t$ in./sec² as shown in Fig. P5.3(b). Determine the maximum horizontal displacement of the girder relative to the motion of the foundation. Neglect damping.

**Problem 5.4**

Determine the maximum stress in the columns of the frame of Problem 5.3.

Problem 5.5

The frame shown in Fig. P5.1 is subjected to the excitation produced by the El Centro earthquake of 1940. Assume 10% damping and from the appropriate chart determine the spectral values for displacement, velocity, and acceleration. Assume elastic behavior.

Problem 5.6

Repeat Problem 5.5 using the basic design spectra given in Fig. 5.9 to determine the spectral values for acceleration, velocity, and displacement. (Scale down spectral values by factor 0.32, the estimated peak ground acceleration)

Problem 5.7

A structure modeled as the spring-mass system shown in Fig. P5.7 is assumed to be subjected to a support motion produced by the El Centro earthquake of 1940. Assuming elastic behavior and using the appropriate response spectral chart find the maximum relative displacement between the mass and the support. Also compute the maximum force acting on the spring. Neglect damping.

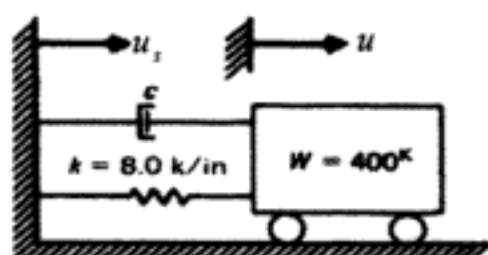


Fig. P5.7

Problem 5.8

Repeat Problem 5.7 assuming that the system has 10% of the critical damping.

Problem 5.9

Determine the force transmitted to the foundation for the system analyzed in Problem 5.8.

Problem 5.10

Consider the spring-mass system of Problem 5.7 and assume that the spring element follows an elastoplastic behavior with a maximum value for the restoring force in tension or in compression equal to half the value of the elastic maximum force in the spring calculated in Problem 5.7. Determine the spectral value for the displacement. Neglect damping. (Hint: Start by assuming $\mu = 2$, find spectral value S_D , calculate μ and use this calculated value of μ to determine new spectral values, etc.)

Problem 5.11

Repeat Problem 5.10 for 10% damping.

Problem 5.12

A structure modeled as a single-degree-of-freedom system has a natural period $T = 0.5$ sec. Use the response spectral method to determine in the elastic range the maximum absolute acceleration, the maximum relative displacement, and the maximum pseudovelocity for: (a) a foundation motion equal to the El Centro earthquake of 1940 and (b) the design spectrum with a maximum ground acceleration equal to $0.3 g$. Neglect damping.

Problem 5.13

Solve Problem 5.12 assuming elastoplastic behavior of the system with ductility ratio $\mu = 4$.

Problem 5.14

Use Program 6 to develop a table having the spectral values for displacements, velocities, and accelerations in the range of frequency from 0.10 cps to 1.0 cps. The excitation is a constant acceleration of magnitude $0.01 g$ applied for 10 sec. Neglect damping.

Problem 5.15

Use Program 6 to develop the response spectra for elastic systems subjected to the first 10 sec of the 1940 El Centro earthquake. Neglect damping. The digitized values corresponding to the accelerations recorded for the first 10 sec of the El Centro earthquake are given in Table 5.3.

Problem 5.16

Repeat Problem 5.15 corresponding to a system with 10% of the critical damping.

6 Nonlinear Structural Response

In discussing the dynamic behavior of single-degree-of-freedom systems, we assumed that in the model representing the structure, the restoring force was proportional to the displacement. We also assumed the dissipation of energy through a viscous damping mechanism in which the damping force was proportional to the velocity. In addition, the mass in the model was always considered to be unchanging with time. As a consequence of these assumptions, the equation of motion for such a system resulted in a linear, second order ordinary differential equation with constant coefficients, namely,

$$m\ddot{u} + c\dot{u} + ku = F(t) \quad (6.1)$$

In the previous chapters it was illustrated that for particular forcing functions such as harmonic functions, it was relatively simple to solve this equation (6.1) and that a general solution always existed in terms of Duhamel's integral. Equation (6.1) thus represents the dynamic behavior of many structures modeled as a single-degree-of-freedom system. There are, however, physical situations for which this linear model does not adequately represent the dynamic characteristics of the structure. The analysis in such cases requires the introduction of a model in which the spring force or the damping force may not remain proportional, respectively, to the displacement or to the velocity. Consequently, the resulting equation of motion will no longer be linear and its mathematical solution, in general, will have a much greater complexity, often requiring a numerical procedure for its integration.

6.1 Nonlinear Single-Degree-of-Freedom Model

Figure 6.1(a) shows the model for a single-degree-of-freedom system and in Fig. 6.1 (b) the corresponding free body diagram. The dynamic equilibrium in the system is

established by equating to zero the sum of the inertial force $F_I(t)$, the damping force $F_D(t)$, the spring force $F_s(t)$, and the external force $F(t)$. Hence, at time t , the equilibrium of these forces is expressed as

$$F_I(t) + F_D(t) + F_s(t) = F(t) \quad (6.2)$$

Considering the case that in this equation, the mass is constant, $F_I(t) = m\ddot{u}$; that the damping force is proportional to the velocity with the damping coefficient also constant, $F_D(t) = c\dot{u}$; and the resisting force or spring force is a function of the displacement, $F_s(t) = F_s(u)$, we may then express eq. (6.2) as

$$m\ddot{u} + c\dot{u} + F_s(u) = F(t) \quad (6.3)$$

Hence, at time t_i , using the notation $u_i = u(t_i)$, $\dot{u}_i = \dot{u}(t_i)$ and $\ddot{u}_i = \ddot{u}(t_i)$,

$$m\ddot{u}_i + c\dot{u}_i + F_s(u_i) = F(t_i) \quad (6.4)$$

and at short time later, $t_{i+1} = t_i + \Delta t$

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + F_s(u_{i+1}) = F(t_{i+1}) \quad (6.5)$$

Subtracting eq. (6.4) from eq. (6.5) results in the difference equation of motion in terms of increments, namely

$$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + \Delta F_s(u_i) = \Delta F_i \quad (6.6)$$

Furthermore, we assume that the incremental resisting or spring force is proportional to the incremental displacement, that is $\Delta F_s(u_i) = k_r\Delta u_i$,

$$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + k_r\Delta u_i = \Delta F_i \quad (6.7)$$

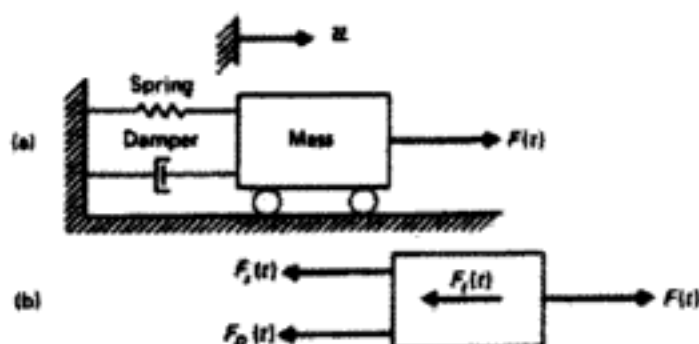


Fig. 6.1 (a) Model for a single-degree-of-freedom system. (b) Free body diagram showing the inertial force, the damping force, the spring force, and the external force.

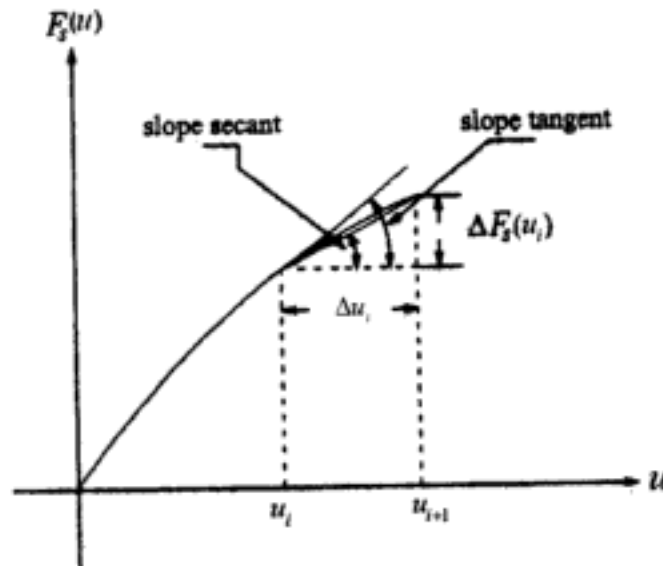


Fig. 6.2 Nonlinear stiffness function, spring force v.s. displacement.

The coefficient k_i is defined as the current evaluation for the resisting force per unit displacement (stiffness coefficient) which may be taken as the slope of the tangent of the force-displacement function, at the initiation of the time step Δt for the interval Δt or as the slope of the secant line as shown in Fig. 6.2 for the plot of the resisting force $F_s(u)$. The value of the coefficient k_i is calculated at a displacement corresponding to time t_i and assumed to remain constant during the increment of time Δt . Since, in general, this coefficient does not remain constant during that time increment, eq. (6.7) is an approximate equation.

The incremental displacement Δu_i , incremental velocity $\Delta \dot{u}_i$ and incremental acceleration $\Delta \ddot{u}_i$ are given by

$$\Delta u_i = u(t_i + \Delta t) - u(t_i) \quad (6.8)$$

$$\Delta \dot{u}_i = \dot{u}(t_i + \Delta t) - \dot{u}(t_i) \quad (6.9)$$

$$\Delta \ddot{u}_i = \ddot{u}(t_i + \Delta t) - \ddot{u}(t_i) \quad (6.10)$$

6.2 Integration of the Nonlinear Equation of Motion

Among the many methods available for the solution of the nonlinear equation of motion, probably one of the most effective is the step-by-step integration method. In this method, the response is evaluated at successive increments Δt of time, usually taken of equal lengths of time for computational convenience. At the beginning of each interval, the condition of dynamic equilibrium is established. Then, the response for a time increment Δt is evaluated approximately on the basis that the coefficients $k(u)$ and $c(\dot{u})$ remain constant during the interval Δt . The nonlinear characteristics of these coefficients are considered in the analysis by reevaluating these coefficients at the beginning of each time increment. The response is then obtained using the displacement and velocity calculated at the end of the time interval as the initial conditions for the next time step.

As we have said for each time interval, the stiffness coefficient $k(u)$ and the damping coefficient $c(\dot{u})$ are evaluated at the initiation of the interval but are assumed to remain constant until the next step; thus the nonlinear behavior of the system is approximated by a sequence of successively changing linear systems. It should also be obvious that the assumption of constant mass is unnecessary; it could just as well also be represented by a variable coefficient.

There are many procedures available for performing the step-by-step integration of eq. (6.12). Two of the most popular methods are the constant acceleration method and the linear acceleration method. As the names of these methods imply, in the first method the acceleration is assumed to remain constant during the time interval Δt , while in the second method, the acceleration is assumed to vary linearly during the interval. As may be expected, the constant acceleration method is simpler but less accurate when compared with the linear acceleration method for the same value of the time increment. We shall present here in detail both methods.

6.3 Constant Acceleration Method

In the constant acceleration method, it is assumed that acceleration remains constant for the time step between times t_i and $t_{i+1} = t_i + \Delta t$ as shown in Fig. 6.3. The value of the constant acceleration during the interval Δt is taken as the average of the values of the acceleration \ddot{u}_i at the initiation of the time step and \ddot{u}_{i+1} , the acceleration at the end of the time step. Thus, the acceleration $\ddot{u}(t)$ at any time t during the time interval Δt is given by

$$\ddot{u}(t) = \frac{1}{2}(\ddot{u}_i + \ddot{u}_{i+1}) \quad (6.11)$$

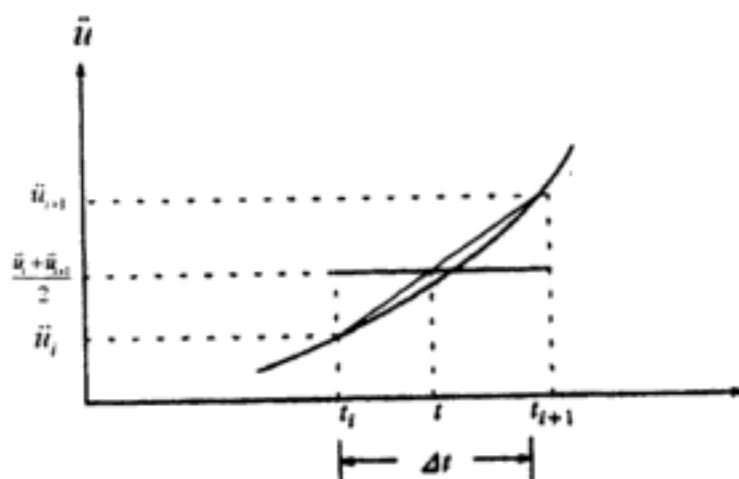


Fig 6.3 Constant acceleration assumed during time interval Δt .

Integrating this equation twice with respect to the time between the limits t_i and t results in

$$\dot{u}(t) = \dot{u}_i + \frac{1}{2}(\ddot{u}_i + \ddot{u}_{i+1})(t - t_i) \quad (6.12)$$

and

$$u(t) = u_i + \dot{u}_i(t - t_i) + \frac{1}{4}(\ddot{u}_i + \ddot{u}_{i+1})(t - t_i)^2 \quad (6.13)$$

The evaluation of eqs. (6.12) and (6.13) at time $t_{i+1} = t_i + \Delta t$ gives

$$\Delta \dot{u}_i = \frac{\Delta t}{2}(\ddot{u}_i + \ddot{u}_{i+1}) \quad (6.14)$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{\Delta t^2}{4}(\ddot{u}_i + \ddot{u}_{i+1}) \quad (6.15)$$

where Δu_i and $\Delta \dot{u}_i$ are respectively the incremental displacement and incremental velocity defined by eqs (6.8) and (6.9).

To use the incremental displacement in the analysis, eq. (6.15) is solved for \ddot{u}_{i+1} and substituted into eq. (6.14) to obtain:

$$\ddot{u}_{i+1} = \frac{4}{\Delta t^2} \Delta u_i - \frac{4}{\Delta t} \dot{u}_i - \ddot{u}_i \quad (6.16)$$

and

$$\Delta \dot{u}_i = \frac{2}{\Delta t} \Delta u_i - 2\dot{u}_i \quad (6.17)$$

Now subtracting \ddot{u}_i on both sides of eq (6.16) results in

$$\Delta \ddot{u}_i = \frac{4}{\Delta t^2} \Delta u_i - \frac{4}{\Delta t} \dot{u}_i - 2\ddot{u}_i \quad (6.18)$$

The substitution into eq. (6.7) of $\Delta \dot{u}_i$ and $\Delta \ddot{u}_i$, respectively, from eqs. (6.17) and (6.18) gives

$$m\left(\frac{4}{\Delta t^2} \Delta u_i - \frac{4}{\Delta t} \dot{u}_i - 2\ddot{u}_i\right) + c\left(\frac{2}{\Delta t} \Delta u_i - 2\dot{u}_i\right) + k_i \Delta u_i = \Delta F_i \quad (6.19)$$

Equation (6.19) is then solved for the incremental displacement Δu_i to obtain

$$\Delta u_i = \frac{\Delta \bar{F}_i}{\bar{k}_i} \quad (6.20)$$

in which the effective stiffness \bar{k}_i is

$$\bar{k}_i = \frac{4m}{\Delta t^2} + \frac{2c}{\Delta t} + k_i \quad (6.21)$$

and the effective incremental force $\Delta \bar{F}_i$ is

$$\Delta \bar{F}_i = \Delta F_i + \frac{4m}{\Delta t} \dot{u}_i + 2m\Delta \ddot{u}_i + 2c\dot{u}_i \quad (6.22)$$

The displacement $u_{i+1} = u(t_i + \Delta t)$ at time $t_{i+1} = t_i + \Delta t$ is obtained from eq. (6.8) after solving for incremental displacement Δu_i in eq. (6.20). The incremental velocity is calculated by eq. (6.17) and the velocity at time $t_{i+1} = t_i + \Delta t$ from eq. (6.9) as

$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i \quad (6.23)$$

Finally, the acceleration \ddot{u}_{i+1} at the end of the time step, $t_{i+1} = t_i + \Delta t$, is obtained directly from the differential equation of motion, eq. (6.3), rather than using eq. (6.16). Hence, from eq. (6.3):

$$\ddot{u}_{i+1} = \frac{1}{m} [F(t_{i+1}) - c\dot{u}_{i+1} - F_s(u_{i+1})] \quad (6.24)$$

in which $F_s(u_{i+1})$ is the restoring force evaluated at time $t_{i+1} = t_i + \Delta t$

After the displacement, velocity, and acceleration have been determined at time $t_{i+1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the following time step $t_{i+2} = t_{i+1} + \Delta t$, and the process is continued to any desired final value of time.

6.4 Linear Acceleration Step-by-Step Method

In the linear acceleration method, it is assumed that the acceleration may be expressed by a linear function of time during the time interval Δt . Let t_i and $t_{i+1} = t_i + \Delta t$ be, respectively, the designation for the time at the beginning and at the end of the time interval Δt . In this type of analysis, the material properties of the system c_i and k_i may include any form of nonlinearity. Thus it is not necessary for the spring force to be only a function of displacement or for the damping force to be specified only as a function of velocity. The only restriction in the analysis is that we evaluate these coefficients at an instant of time t_i and then assume that they remain constant during the increment of time Δt . When the acceleration is assumed to be a linear function of time for the interval of time t_i or $t_{i+1} = t_i + \Delta t$ as depicted in Fig. 6.4, we may express the acceleration as

$$\ddot{u}(t) = \ddot{u}_i + \frac{\Delta \ddot{u}_i}{\Delta t} (t - t_i) \quad (6.25)$$

where $\Delta \ddot{u}_i$ is given by eq. (6.10). Integrating eq. (6.25) twice with respect to time between the limits t_i and t yields

$$\dot{u}(t) = \dot{u}_i + \ddot{u}_i (t - t_i) + \frac{1}{2} \frac{\Delta \ddot{u}_i}{\Delta t} (t - t_i)^2 \quad (6.26)$$

and

$$u(t) = u_i + \dot{u}_i (t - t_i) + \frac{1}{2} \ddot{u}_i (t - t_i)^2 + \frac{1}{6} \frac{\Delta \ddot{u}_i}{\Delta t} (t - t_i)^3 \quad (6.27)$$

The evaluation of eqs. (6.26) and (6.27) at time $t = t_i + \Delta t$ gives

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \frac{1}{2} \Delta \ddot{u}_i \Delta t \quad (6.28)$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2} \ddot{u}_i \Delta t^2 + \frac{1}{6} \Delta \ddot{u}_i \Delta t^3 \quad (6.29)$$

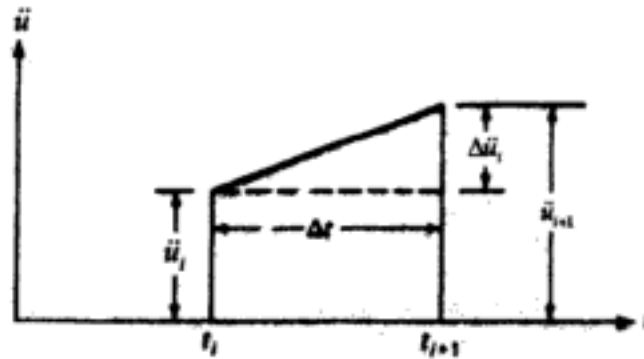


Fig. 6.4 Assumed linear variation of the acceleration during a time interval.

where Δu_i and $\Delta \dot{u}_i$ are defined in eqs. (6.8) and (6.9), respectively. Now, to use the incremental displacement Δu as the basic variable in the analysis, eq. (6.29) is solved for the incremental acceleration $\Delta \ddot{u}_i$ and then substituted into eq. (6.28) to obtain

$$\Delta \ddot{u}_i = \frac{6}{\Delta t^2} \Delta u_i - \frac{6}{\Delta t} \dot{u}_i - 3\ddot{u}_i \quad (6.30)$$

and

$$\Delta \dot{u}_i = \frac{3}{\Delta t} \Delta u_i - 3\dot{u}_i - \frac{\Delta t}{2} \ddot{u}_i \quad (6.31)$$

The substitution of eqs. (6.30) and (6.31) into eq. (6.7) leads to the following form of the equation of motion:

$$m \left\{ \frac{6}{\Delta t^2} \Delta u_i - \frac{6}{\Delta t} \dot{u}_i - 3\ddot{u}_i \right\} + c_i \left\{ \frac{3}{\Delta t} \Delta u_i - 3\dot{u}_i - \frac{\Delta t}{2} \ddot{u}_i \right\} + k_i \Delta u_i = \Delta F_i \quad (6.32)$$

Finally, transferring in eq. (6.32) all the terms containing the unknown incremental displacement Δu_i to the left-hand side gives

$$\bar{k}_i \Delta u_i = \Delta \bar{F}_i \quad (6.33)$$

in which \bar{k}_i is the effective spring constant, given by

$$\bar{k}_i = k_i + \frac{6m}{\Delta t^2} + \frac{3c_i}{\Delta t} \quad (6.34)$$

and $\Delta \bar{F}_i$ is the effective incremental force, expressed by

$$\Delta \bar{F}_i = \Delta F_i + m \left\{ \frac{6}{\Delta t} \dot{u}_i + 3\ddot{u}_i \right\} + c_i \left\{ 3\dot{u}_i + \frac{\Delta t}{2} \ddot{u}_i \right\} \quad (6.35)$$

It should be noted that eq. (6.33) is equivalent to the static incremental equilibrium equation, and may be solved for the incremental displacement by simply dividing the effective incremental force $\Delta \bar{F}_i$ by the effective spring constant \bar{k}_i , that is,

$$\Delta u_i = \frac{\Delta \bar{F}_i}{\bar{k}_i} \quad (6.36)$$

To obtain the displacement $u_{i+1} = u(t_i + \Delta t)$ at time $t_{i+1} = t_i + \Delta t$, this value of Δu_i is substituted into eq. (6.8) yielding

$$u_{i+1} = u_i + \Delta u_i \quad (6.37)$$

Then the incremental velocity $\Delta \dot{u}_i$ is obtained from eq. (6.31) and the velocity at time $t_{i+1} = t_i + \Delta t$ from eq. (6.9) as

$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i \quad (6.38)$$

Finally, the acceleration \ddot{u}_{i+1} at the end of the time step is obtained directly from the differential equation of motion, eq. (6.2), where the equation is written for time $t_{i+1} = t_i + \Delta t$. Hence, after setting $F_1 = m \ddot{u}_{i+1}$ in eq. (6.2), it follows that

$$\ddot{u}_{i+1} = \frac{1}{m} \{ F(t_{i+1}) - F_D(t_{i+1}) - F_s(t_{i+1}) \} \quad (6.39)$$

where that damping force $F_D(t_{i+1})$ and the spring force $F_S(t_{i+1})$ are now evaluated at time $t_{i+1} = t_i + \Delta t$.

After the displacement, velocity, and acceleration have been determined at time $t_{i+1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the following time step $t_{i+2} = t_{i+1} + \Delta t$, and the process is continued to any desired final value of time. The reader should, however, realize that this numerical procedure involves two significant approximations: (1) the acceleration is assumed to vary linearly during the time increment Δt ; and (2) the damping and stiffness properties of the system are evaluated at the initiation of each time increment and assumed to remain constant during the time interval. In general, these two assumptions introduce errors that are small if the time step is short. However, these errors generally might tend to accumulate from step to step. This accumulation of errors should be avoided by imposing a total dynamic equilibrium condition at each step in the analysis. This is accomplished by expressing the acceleration at each step using the differential equation of motion in which the displacement and velocity as well as the stiffness and damping forces are evaluated at that time step.

There still remains the problem of the selection of the proper time increment Δt . As in any numerical method, the accuracy of the step-by-step integration method depends upon the magnitude of the time increment selected. The following factors should be considered in the selection of Δt : (1) the natural period of the structure; (2) the rate of variation of the loading function; and (3) the complexity of the stiffness and damping functions.

In general, it has been found that sufficiently accurate results can be obtained if the time interval is taken to be no longer than one-tenth of the natural period of the structure. The second consideration is that the interval should be small enough to represent properly the variation of the load with respect to time. The third point that should be considered is any abrupt variation in the rate of change of the stiffness or damping function. For example, in the usual assumption of elastoplastic materials, the stiffness suddenly changes from linear elastic to a yielding plastic phase. In this case, to obtain the best accuracy, it would be desirable to select smaller time steps in the neighborhood of such drastic changes.

6.5 The Newmark Beta Method

The Newmark Beta Method includes, in its formulation, several time-step methods used for the solution of linear or nonlinear equations. It uses a numerical parameter designated as β . The method, as originally proposed by Newmark (1959), contained in addition to β , a second parameter γ . Particular numerical values for these parameters leads to well-known methods for the solution of the differential equation of motion, the constant acceleration method, and the linear acceleration method.

The Newmark equation can be written in incremental quantities for a constant time step Δt , as

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \gamma \Delta \ddot{u}_i \Delta t \quad (6.40)$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2} \ddot{u}_i \Delta t^2 + \beta \Delta \ddot{u}_i \Delta t^2 \quad (6.41)$$

in which the incremental displacement Δu_i and incremental velocity $\Delta \dot{u}_i$ are defined, respectively, by eq (6.8) and (6.9).

It has been found that for values of γ different than $1/2$, the method introduced a superfluous damping in this system. For this reason, this parameter is generally set as $\gamma = 1/2$. The solution of eq. (6.41) for $\Delta \ddot{u}_i$ and its subsequent substitution into eq. (6.40) after setting $\gamma = 1/2$ yield

$$\Delta \ddot{u}_i = \frac{1}{\beta \Delta t^2} \Delta u_i - \frac{1}{\beta \Delta t} \dot{u}_i - \frac{1}{2\beta} \ddot{u}_i \quad (6.42)$$

and

$$\Delta \dot{u}_i = \frac{1}{2\beta \Delta t} \Delta u_i - \frac{1}{2\beta} \dot{u}_i + (1 - \frac{1}{4\beta}) \Delta t \cdot \ddot{u}_i \quad (6.43)$$

Now, the substitution of eqs. (6.42) and (6.43) into the incremental equation of motion, eq. (6.7), results in an equation to calculate the incremental displacement Δu_i , namely,

$$\bar{k}_i \Delta u_i = \Delta \bar{F}_i \quad (6.44)$$

where the effective stiffness \bar{k}_i , and the effective incremental force $\Delta \bar{F}_i$ are given respectively by

$$\bar{k}_i = k_i + \frac{m}{\beta \Delta t^2} + \frac{c_i}{2\beta \Delta t} \quad (6.45)$$

and

$$\Delta \bar{F}_i = \Delta F_i + \frac{m}{\beta \Delta t} \dot{u}_i + \frac{c_i}{2\beta} \dot{u}_i + \frac{m}{2\beta} \ddot{u}_i - c_i \Delta t (1 - \frac{1}{4\beta}) \ddot{u}_i \quad (6.46)$$

In these equations, c_i and k_i are respectively the damping and stiffness coefficients evaluated at the initial time t_i of the time step $\Delta t = t_{i+1} - t_i$.

In the implementation of the Newmark Beta Method, a numerical value for the parameter β is selected. Newmark suggested a value in the range $1/6 \leq \beta \leq 1/2$. For $\beta = 1/4$, the method corresponds to the constant acceleration method and for $\beta = 1/6$ to the linear acceleration method.

6.6 Elastoplastic Behavior

If any structure modeled as a single-degree-of-freedom system (spring-mass system) is allowed to yield plastically, then the restoring force exerted is likely to be of the form shown in Fig. 6.5(a). There is a portion of the curve in which linear elastic behavior occurs, whereupon, for any further deformation, plastic yielding takes place. When the structure is unloaded, the behavior is again elastic until further reverse

loading produces compressive plastic yielding. The structure may be subjected to cyclic loading and unloading in this manner. Energy is dissipated during each cycle by an amount that is proportional to the area under the curve (hysteresis loop) as indicated in Fig. 6.5(a). This behavior is often simplified by assuming a definite yield point beyond which additional displacement takes place at a constant value for the restoring force without any further increase in the load. Such behavior is known as elastoplastic behavior; the corresponding force-displacement curve is shown in Fig 6.5(b)

For the structure modeled as a spring-mass system, expressions of the restoring force for a system with elastoplastic behavior are easily written.

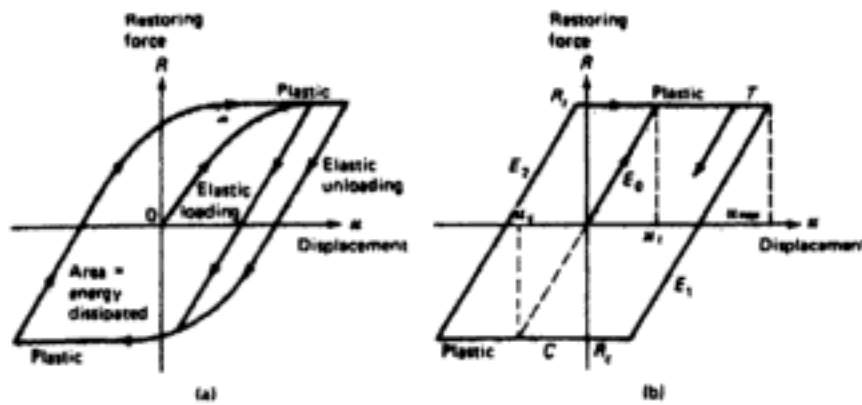


Fig.6.5 Elastic-plastic structural models. (a) General plastic behavior. (b) Elastoplastic behavior.

These expressions depend on the magnitude of the restoring force as well as upon whether the motion is such that the displacement is increasing ($\dot{u} > 0$) or decreasing ($\dot{u} < 0$). Referring to Fig. 6.5(b) in which a general elastoplastic cycle is represented, we assume that the initial conditions are zero ($u_0 = 0$, $\dot{u} = 0$) for the unloaded structure. Hence, initially, as the load is applied, the system behaves elastically along curve E_0 . The displacement u_t at which plastic behavior in tension may be initiated, and the displacement u_c , at which plastic behavior in compression may be initiated, are calculated, respectively, from

$$u_t = R_t / k \quad (6.47)$$

and

$$u_c = R_c / k$$

where R_t and R_c are the respective values of the forces that produce yielding in tension and compression and k is the elastic stiffness of the structure. The system will remain on curve E_0 as long as the displacement u satisfies

$$u_c < u < u_t \quad (6.48)$$

If the displacement u increases to u_t the system begins to behave plastically in tension along curve T on Fig. 6.5(b); it remains on curve T as long as the velocity $\dot{u} > 0$.

When $\dot{u} < 0$, the system reverses to elastic behavior on a curve such as E_1 with new yielding points given by

$$\begin{aligned} u_t &= u_{\max} \\ u_c &= u_{\max} - (R_t - R_c)/k \end{aligned} \quad (6.49)$$

in which u_{\max} is the maximum displacement along curve T , which occurs when $\dot{u} = 0$.

Conversely, if u decreases to u_c the system begins a plastic behavior in compression along curve C and it remains on this curve as long as $\dot{u} < 0$. The system returns to an elastic behavior when the velocity again changes direction and $\dot{u} > 0$. In this case, the new yielding limits are given by

$$\begin{aligned} u_c &= u_{\min} \\ u_t &= u_{\min} + (R_t - R_c)/k \end{aligned} \quad (6.50)$$

in which u_{\min} is the minimum displacement along curve C , which occurs when $\dot{u} = 0$. The same condition given by eq. (6.48) is valid for the system to remain operating along any elastic segment such as E_0, E_1, E_2, \dots as shown in Fig. 6.5(b).

We are now interested in calculating the restoring force at each of the possible segments of the elastoplastic cycle. The restoring force on an elastic phase of the cycle (E_0, E_1, E_2, \dots) may be calculated as

$$R = R_t - (u_t - u)k \quad (6.51)$$

on a plastic phase in tension as

$$R = R_t \quad (6.52)$$

and on the plastic compressive phase as

$$R = R_c \quad (6.53)$$

The algorithm for the step-by-step linear acceleration method of a single degree-of-freedom system assuming an elastoplastic behavior is outlined in the following section.

6.7 Algorithm for Step-by-Step Solution for Elastoplastic Single-Degree-of-Freedom System

Initialize and input data:

- (1) Input values for k, m, c, R_t, R_c , and a table giving the time t_i and magnitude of the excitation F_j .
- (2) Set $u_0 = 0$ and $\dot{u}_0 = 0$.

(3) Calculate initial acceleration:

$$\ddot{u}_0 = \frac{F(t=0)}{m} \quad (6.54)$$

(4) Select time step Δt and calculate constants:

$$a_1 = 3/\Delta t, a_2 = 6/\Delta t, a_3 = \Delta t/2, a_4 = 6/\Delta t^2$$

(5) Calculate initial yield points:

$$\begin{aligned} u_t &= R_t / k \\ u_c &= R_c / k \end{aligned} \quad (6.55)$$

For each time step:

(1) Use the following code to establish the elastic or plastic state of the system:

$$\begin{aligned} \text{KEY} &= 0 \text{ (elastic behavior)} \\ \text{KEY} &= -1 \text{ (plastic behavior in compression)} \\ \text{KEY} &= 1 \text{ (plastic behavior in tension)} \end{aligned} \quad (6.56)$$

(2) Calculate the displacement \dot{u} and velocity u at the end of the time step and set the value of KEY according to the following conditions:

(a) When the system is behaving elastically at the beginning of the time step and

$$\begin{aligned} u_c &< u < u_t & \text{KEY} &= 0 \\ u &> u_t & \text{KEY} &= 1 \\ u &< u_c & \text{KEY} &= -1 \end{aligned}$$

(b) When the system is behaving plastically in tension at the beginning of the time step and

$$\begin{aligned} \dot{u} &> 0 \text{ KEY} = 1 \\ \dot{u} &< 0 \text{ KEY} = 0 \end{aligned}$$

(c) When the system is behaving plastically in compression at the beginning of the time step and

$$\begin{aligned} \dot{u} &< 0 \text{ KEY} = -1 \\ \dot{u} &> 0 \text{ KEY} = 0 \end{aligned}$$

(3) Calculate the effective stiffness:

$$\bar{k}_i = k_p + a_4 m + a_1 c_i \quad (6.57)$$

where

$$\begin{aligned} k_p &= k & \text{for elastic behavior (KEY = 0)} \\ k_p &= 0 & \text{for plastic behavior (KEY = 1 or -1)} \end{aligned} \quad (6.58)$$

(4) Calculate the incremental effective force:

$$\Delta \bar{F}_i = \Delta F_i + (a_2 m + 3c_i) + (3m + a_3 c_i) \ddot{u}_i \quad (6.59)$$

(5) Solve for the incremental displacement:

$$\Delta u_i = \Delta \bar{F}_i / \bar{k}_i \quad (6.60)$$

(6) Calculate the incremental velocity:

$$\Delta \dot{u}_i = a_1 \Delta u_i - 3\dot{u}_i - a_3 \ddot{u}_i \quad (6.61)$$

(7) Calculate displacement and velocity at the end of time interval:

$$u_{i+1} = u_i + \Delta u_i \quad (6.62)$$

$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i \quad (6.63)$$

(8) Calculate acceleration \ddot{u}_{i+1} at the end of time interval using the dynamic equation of equilibrium:

$$\ddot{u}_{i+1} = \frac{1}{m} [F(t_{i+1}) - c_{i+1} \dot{u}_{i+1} - R] \quad (6.64)$$

in which

$$\begin{aligned} R &= R_t - (u_i - u_{i+1})k & \text{if KEY} = 0 \\ R &= R_t & \text{if KEY} = 1 \end{aligned} \quad (6.65)$$

or

$$R = R_c \quad \text{if KEY} = -1$$

Illustrative Example 6.1

To illustrate the hand calculations in applying the step-by-step integration method described above, consider the single-degree-of-freedom system in Fig. 6.6 with elastoplastic behavior subjected to the loading history as shown. For this example, we assume that the damping coefficient remains constant ($\xi = 0.087$). Hence the only nonlinearities in the system arise from the changes in stiffness as yielding occurs.

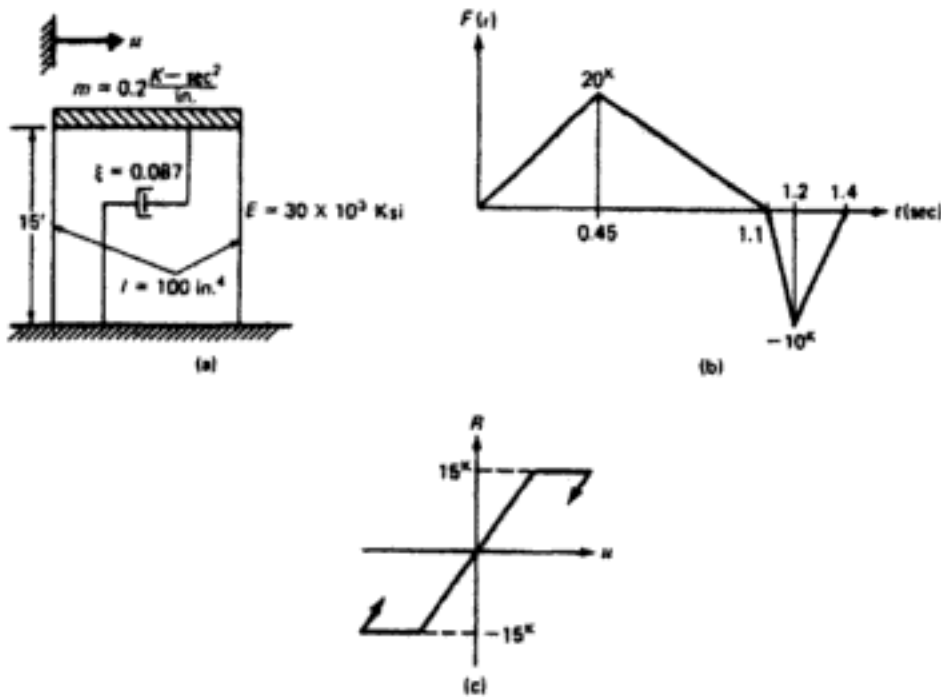


Fig. 6.6 Frame with elastoplastic behavior subjected to dynamic loading. (a) Frame. (b) Loading. (c) Elastoplastic behavior.

Solution:

The stiffness of the system during elastic behavior is

$$k = \frac{12E(2I)}{L^3} = \frac{12 \times 30 \times 10^3 \times 2 \times 100}{(15 \times 12)^3} = 12.35 \text{ Kip/in}$$

and the damping coefficient

$$c = \xi c_{cr} = (0.087)(2)\sqrt{0.2 \times 12.35} = 0.274 \text{ kip-sec/in}$$

Initial displacement and initial velocity are $u_0 = \dot{u}_0 = 0$.

The initial acceleration is

$$\ddot{u}_0 = \frac{F(0)}{k} = 0$$

Yield displacements are

$$u_t = \frac{R_t}{k} = \frac{15}{12.35} = 1.215 \text{ in}$$

and

$$u_c = -1.215 \text{ in}$$

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The natural period is $T = 2\pi\sqrt{m/k} = 0.8$ (for the elastic system). For numerical convenience, we select $\Delta t = 0.1$ sec. The effective stiffness from eq. (6.57) is

$$\bar{k} = k_p + \frac{6}{0.1^2} 0.2 + \frac{3}{0.1} 0.274$$

or

$$\bar{k} = k_p + 128.22$$

where

$$\begin{aligned} k_p &= k = 12.35 && \text{(elastic behavior)} \\ k_p &= 0 && \text{(plastic behavior)} \end{aligned}$$

The effective incremental loading from eq. (6.59) is

$$\begin{aligned} \Delta \bar{F}_i &= \Delta F_i + \left(\frac{6}{\Delta t} m + 3c\right) \dot{u}_i + \left(3m + \frac{\Delta t}{2}\right) \ddot{u}_i \\ \Delta \bar{F}_i &= \Delta F_i + 12.822 \dot{u}_i + 0.613 \ddot{u}_i \end{aligned}$$

The velocity increment given by eq. (6.61) becomes

$$\Delta \dot{u}_i = 30 \Delta u_i - 3 \dot{u}_i - 0.05 u_i$$

The necessary calculations may be conveniently arranged as illustrated in Table 6.1. In this example with elastoplastic behavior, the response changes abruptly as the yielding starts and stops. To obtain better accuracy, it would be desirable to subdivide the time step in the neighborhood of the change of state; however, an iterative procedure would be required to establish the length of the subintervals. This refinement has not been used in the present analysis or in the computer program described in the next section. The stiffness computed at the initiation of the time step has been assumed to remain constant during the entire time increment. The reader is again cautioned that a significant error may arise during phase transitions unless the time step is selected relatively small.

Table 6.1 Nonlinear Response -- Linear Acceleration Step-by-Step Method for Illustrative Example 6.1

t (sec)	F (Kip)	u (in)	KEY	\dot{u} (in/sec)	R (Kip)	\ddot{u} (in/sec ²)	k_p Kip/in	\bar{k} Kip/in	ΔF (Kip)	$\Delta \bar{F}$ (Kip)	Δu (in)	$\Delta \dot{u}$ (in/sec)
0	0		0	0	0	0	12.35	140.57	4.444	4.444	0.0316	0.9485
0.1	4.444	0.0316	0	0.9485	0.390	18.972	12.35	140.57	4.444	28.249	0.2010	2.2359
0.2	8.888	0.2326	0	3.1844	2.871	25.723	12.35	140.57	4.444	61.050	0.4343	2.193
0.3	13.333	0.6669	0	5.3760	8.233	18.134	12.35	140.57	4.444	84.510	0.6012	1.000
0.4	17.777	1.2681	1	6.3768	15.00	5.152	0	128.22	0.685	85.609	0.6677	0.6422
0.5	18.462	1.9358	1	7.0190	15.00	7.691	0	128.22	-3.077	91.641	0.7147	-0.0001
0.6	15.238	2.6505	1	7.0189	15.00	-7.693	0	128.22	-3.077	82.199	0.6409	-1.440
0.7	12.308	3.2916	1	5.5791	15.00	-21.105	0	128.22	-3.077	55.506	0.4329	-2.695
0.8	9.231	3.7244	1	2.8840	15.00	-32.797	0	128.22	-3.077	13.773	0.1074	-3.789
0.9	6.154	3.8319	0	-0.9054	15.00	-42.990	12.35	140.57	-3.077	41.069	-0.2922	-3.899
1.0	3.077	3.5397	0	-4.8048	11.39	-34.998	12.35	140.57	-3.077	-86.162	-0.6130	-2.225
1.1	0	2.9268	0	-7.0295	3.825	-9.497	12.35	140.57	-10	-105.96	-0.7538	-1.051
1.2	-10	2.1729	0	-8.0806	-5.481	-11.525	12.35	140.57	5	-105.68	-0.7518	2.263
1.3	-5	1.4211	0	-5.8177	-14.76	56.784	12.35	140.57	5	-34.746	-0.2472	7.198
1.4	0	1.1739	-1	1.3860	-15.00	73.109	0	128.22	0	62.568	0.4880	6.842
1.5	0	1.6619	0	8.2227	-15.00	63.735	12.35	140.57	0	144.55	1.0283	2.995

6.8 Program 5–Response for Elastoplastic Behavior

The same as for the other programs presented in this book, Program 5 initially requests information about the data file: New, Modify, or Use existent data file. After the user has selected one of these options, the program requests the name of the file and the necessary input data. The program continues by reading and printing the data and by setting the initial values to the various constants and variables in the equations. Then by linear interpolation, values of the forcing function are computed at time increments equal to the selected time step Δt for the integration process. In the main body of the program, the displacement, velocity, and acceleration are computed at each time step. The nonlinear behavior of the restoring force is appropriately considered in the calculation by the variable KEY which is tested through a series of conditional statements in order to determine the correct expressions for the yield points and the magnitude of the restoring force in the system.

The output consists of a table giving the displacement, velocity, and acceleration at time increments Δt . The last column of the table shows the value of the index KEY which provides information about the state of the elastoplastic system. As indicated before, KEY = 0 for elastic behavior and KEY = 1 or KEY = -1 for plastic behavior, respectively, in tension or in compression.

Illustrative Example 6.2

Using the computer Program 5, find the response of the structure in Example 6.1. Then repeat the calculation assuming elastic behavior. Plot and compare results for the elastoplastic behavior with the elastic response.

Solution:

Problem Data (from Illustrative Example 6.1)

Spring constant: $k = 12.35 \text{ Kip/in}$

Damping coefficient: $c = 0.274 \text{ Kip. sec/in}$

Mass: $m = 0.2 \text{ (Kip. sec}^2/\text{in)}$

Max. restoring force (tension): $R_t = 15 \text{ Kip}$

Max. restoring force (compression): $R_c = -15 \text{ Kip}$

Natural period: $T = 2\pi\sqrt{m/k} = 2\pi\sqrt{0.2/12.35} = 0.8 \text{ sec}$

Select time step: $\Delta t = 0.1 \text{ sec}$

Gravitational index: $G = 0 \text{ (force on the mass)}$

The same computer program is used to obtain the response for a completely elastic behavior. It is only necessary to assign a large value to the maximum restoring forces in tension and in compression. These assigned values should be large enough in order for the structure to remain in the elastic range. For the present example, $R_t = 100$ Kip and $R_c = -100$ Kip are deemed adequate in this case. In order to visualize and facilitate a comparison between the elastic and inelastic responses for this example, the displacements are plotted in Fig. 6.7 with the response during yielding shown as dashed lines.

Input Data and Output Results

PROGRAM 5: RESPONSE FOR ELASTOPLASTIC BEHAVIOIR DATA FILE: D5

INPUT DATA:

NUMBER OF POINTS DEFINING THE EXCITATION	NE = 6
MASS	AM = .2
SPRING CONSTANT	AK = 12.35
DAMPING COEFFICIENT	C = .274
TIME STEP OF INTEGRATION	H = .1
MAX. RESTORING FORCE IN TENSION	RT = 15
MIN. RESTORING FORCE COMPRESSION	RC = -15
INDEX (GRAVITY OR ZERO)	G = 0

TIME	EXCITATION	TIME	EXCITATION	TIME	EXCITATION	TIME	EXCITATION
0.000	0.00	0.450	20.00	1.100	0.00	1.200	-10.00
1.400	0.00	2.000	0.00				

OUTPUT RESULTS:

TIME	DISPL.	VELOC.	ACC.	KEY
0.000	0.0000	0.0000	0.0000	0
0.100	0.0316	0.9485	18.9704	0
0.200	0.2326	3.1831	25.7221	0
0.300	0.6668	5.3755	18.1251	0
0.400	1.2679	6.3749	5.1553	1
0.500	1.9354	7.0173	7.6940	1
0.600	2.6500	7.0175	-7.6909	1
0.700	3.2909	5.5778	-21.1031	1
0.800	3.7237	4.8828	-32.7956	1
0.900	3.8310	-0.9064	-42.9890	0
1.000	3.5387	-4.8052	-34.9861	0
1.1.00	2.9258	-7.0283	-9.4766	0
1.200	2.1722	-8.0774	-11.5050	0
1.300	1.4208	-5.8129	56.7935	0
1.400	1.1741	1.3851	73.1025	-1
1.500	1.6625	8.2267	63.7295	0
1.600	2.6911	11.2188	-3.8869	0
1.700	3.6974	7.9471	-61.5465	0
1.800	4.1356	0.8688	-76.1903	1
1.900	3.8758	-6.2618	-66.4214	0
2.000	3.0147	-9.9897	-8.1384	0

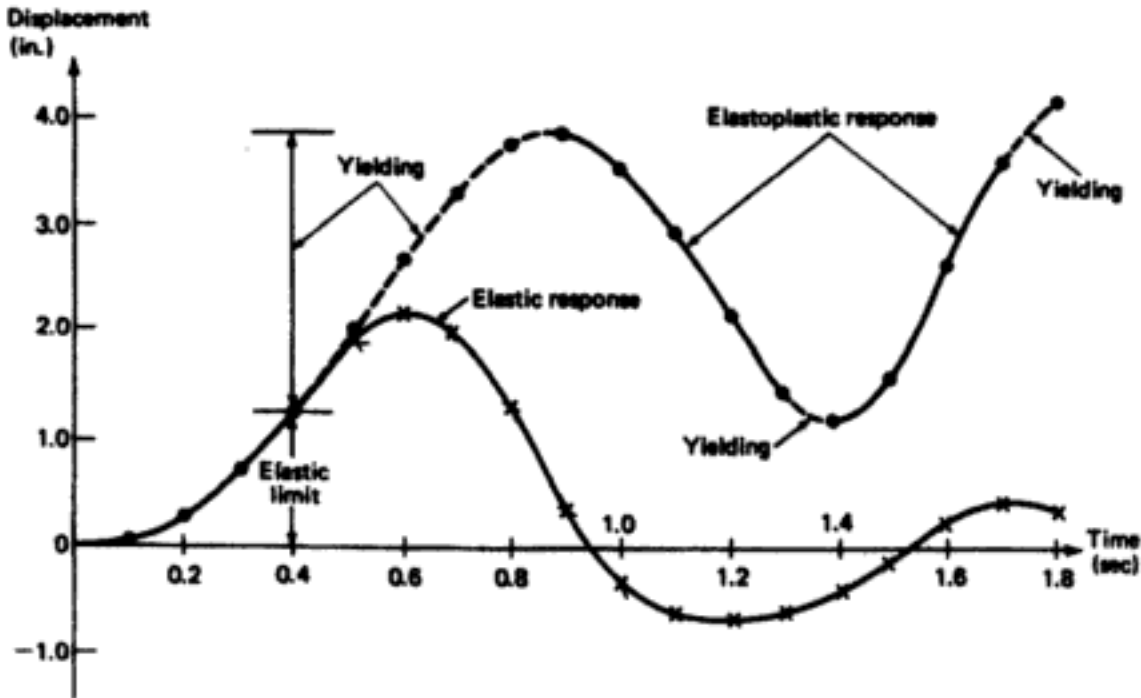


Fig. 6.7 Comparison of elastoplastic behavior with elastic response for Illustrative Example 6.2.

6.9 Summary

Structures are usually designed on the assumption that the structure is linearly elastic and that it remains linearly elastic when subjected to an expected dynamic excitation. However, there are situations in which the structure has to be designed for an eventual excitation of large magnitude such as the strong motion of an earthquake or the effects of nuclear explosion. In these cases, it is not realistic to assume that the structure will remain linearly elastic and it is then necessary to design the structure to withstand deformation beyond the elastic limit. The simplest and most accepted assumption for the design beyond the elastic limit is to assume an elastoplastic behavior. In this type of behavior, the structure is elastic until the restoring force reaches a maximum value (tension or compression) at which it remains constant until the motion reverses its direction and returns to an elastic behavior.

There are many methods to solve numerically the differential equation of this type of motion. The step-by-step linear acceleration presented in this chapter provides satisfactory results with relatively simple calculations. However, these calculations are tedious and time consuming when performed by hand. The use of a computer and the availability of a computer program, such as the one described in this chapter, reduce the effort to a simple routine of data preparation.

6.10 Problems

Problem 6.1

The single-degree-of-freedom of Fig. P6.1 (a) is subjected to the foundation acceleration history in Fig. P6.1 (b) Determine the maximum relative displacement of the columns. Assume elastoplastic behavior of Fig. P6.1 (c).

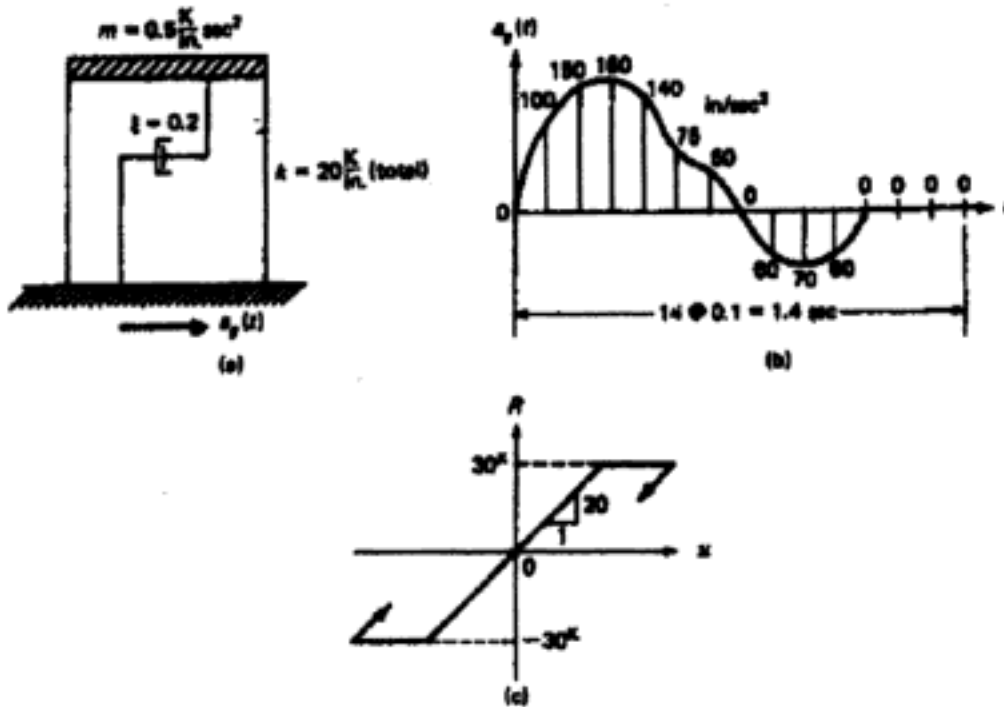


Fig. P6.1

Problem 6.2

Determine the displacement history for the structure in Fig. P6.1 when it is subjected to the impulse loading of Fig. P6.2 applied horizontally at the mass.

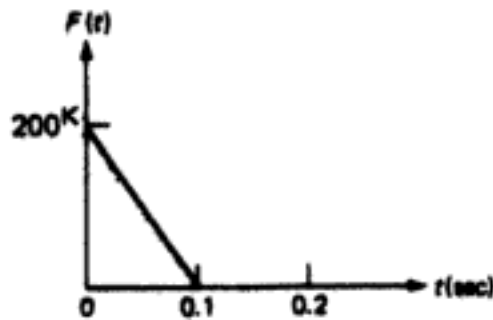


Fig. P6.2

Problem 6.3

Repeat Problem 6.2 for the impulse loading shown in Fig. P6.3 applied horizontally at the mass.

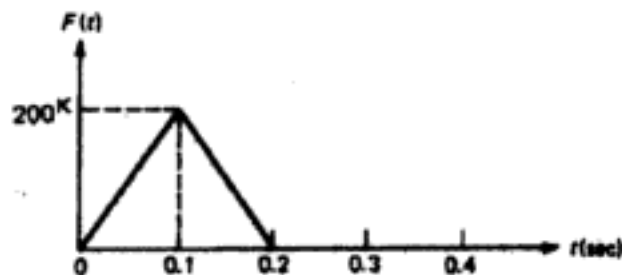


Fig. P6.3

Problem 6.4

Repeat Problem 6.2 for the acceleration history shown in Fig. P6.4 applied horizontally to the foundation.

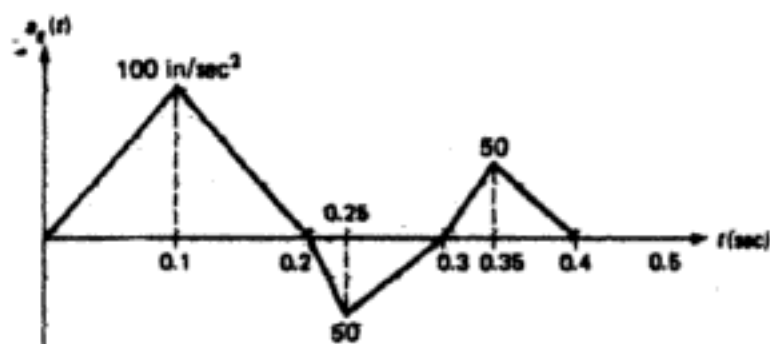


Fig. P6.4

Problem 6.5

Solve Problem 6.1 assuming elastic behavior of the structure. (Hint: Use computer Program 5 with $R_t = 200$ Kip and $R_c = -200$ Kip.)

Problem 6.6

Solve Problem 6.2 for elastic behavior of the structure. Plot the time-displacement response and compare with results from Problem 6.2.

Problem 6.7

Determine the ductility ratio from the results of Problem 6.2 (Ductility ratio is defined as the ratio of the maximum displacement to the displacement at the yield point).

Problem 6.8

A structure modeled as spring-mass shown in Fig. P6.8 (b) is subjected to the loading force depicted in Fig. P6.8 (a). Assume elastoplastic behavior of Fig. P6.8(c). Determine the response.

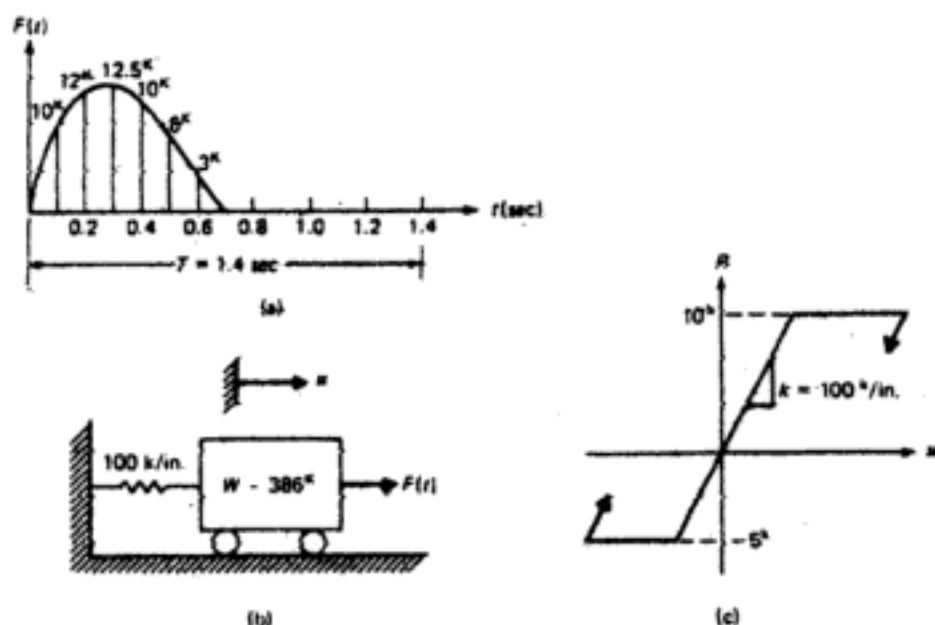


Fig. P6.8

Problem 6.9

Repeat Problem 6.8 assuming damping in the system equal to 20% of the critical damping.

Problem 6.10

Solve Problem 6.8 assuming elastic behavior of the system. (Hint: Use Program 5 with $R_T = 1000$ Kip and $R_c = -1000$ Kip.)

Problem 6.11

Solve Problem 6.9 assuming elastic behavior of the system.

Problem 6.12

A structure modeled as the damped spring-mass system shown in Fig. P6.12(a) is subjected to the time-acceleration excitation acting at its support. The excitation function is expressed as $a(t) = a_0 f(t)$, where $f(t)$ is depicted in Fig. P6.12(b). Determine the maximum value that the factor a_0 may have for the structure to remain elastic. Assume that the structure has an elastoplastic behavior of Fig. P6.12(c).

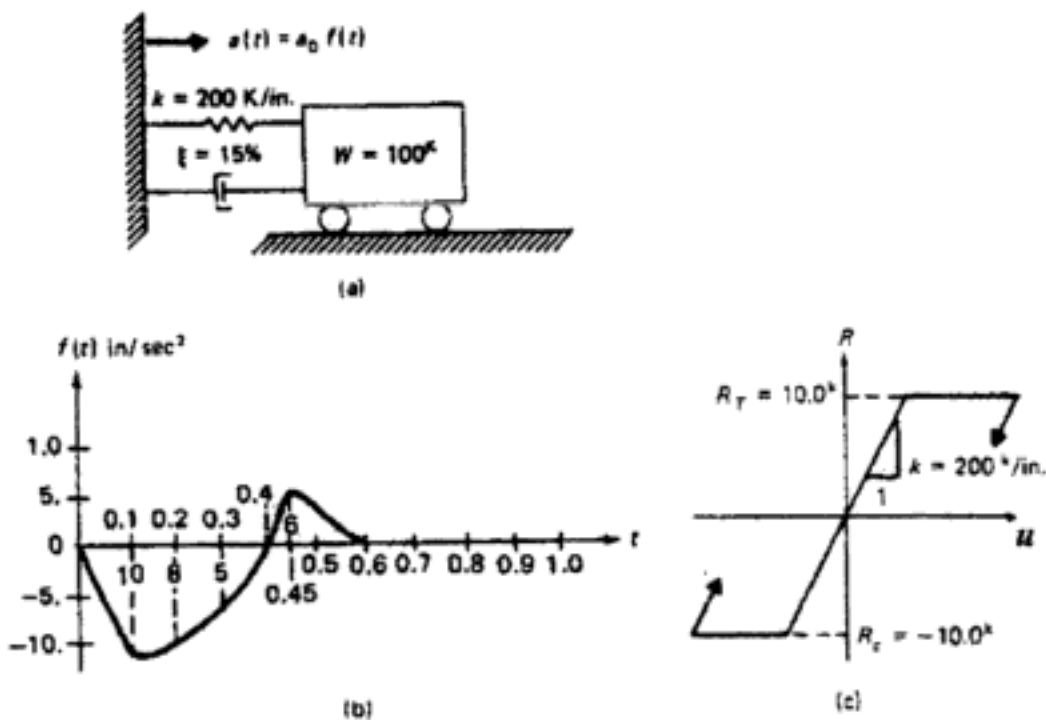


Fig. P6.12

PART II

Structures Modeled as Shear Buildings

7 Free Vibration of a Shear Building

In Part I we analyzed and obtained the dynamic response for structures modeled as a single-degree-of-freedom system. Only if the structure can assume a unique shape during its motion will the single-degree model provide the exact dynamic response. Otherwise, when the structure takes more than one possible shape during motion, the solution obtained from a single-degree model will be at best, only an approximation to the true dynamic behavior.

Structures cannot always be described by a single-degree-of-freedom model and, in general, have to be represented by multiple-degree models. In fact, structures are continuous systems and as such possess an infinite number of degrees of freedom. There are analytical methods to describe the dynamic behavior of continuous structures that have uniform material properties and regular geometry. These methods of analysis, though interesting in revealing information for the discrete modeling of structures, are rather complex and are applicable only to relatively simple actual structures. They require considerable mathematical analysis, including the solution of partial differential equations which will be presented in Part IV. For the present, we shall consider one of the most instructive and practical types of structure which involve many degrees of freedom, the multistory shear building.

7.1 Stiffness Equations for the Shear Building

A shear building may be defined as a structure in which there is no rotation of a horizontal section at the level of the floors. In this respect, the deflected building will have many of the features of a cantilever beam that is deflected by shear forces only, hence the name shear building. To accomplish such deflection in a building, we must assume that: 1) the total mass of the structure is concentrated at the levels of the

floors; 2) the slabs or girders on the floors are infinitely rigid as compared to the columns; and 3) the deformation of the structure is independent of the axial forces present in the columns. These assumptions transform the problem from a structure with an infinite number of degrees of freedom (due to the distributed mass) to a structure that has only as many degrees as it has lumped masses at the floor levels. A three-story structure modeled as a shear building [Fig. 7.1(a)] will have three degrees of freedom, that is, the three horizontal displacements at the floor levels. The second assumption introduces the requirement that the joints between girders and columns are fixed against rotation. The third assumption leads to the condition that the rigid girders will remain horizontal during motion.

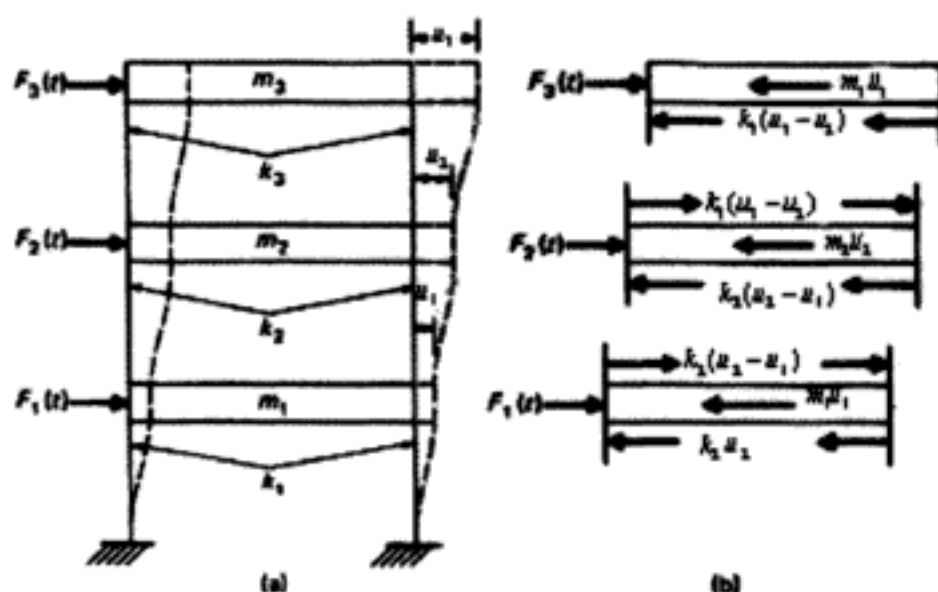


Fig. 7.1 (a) Single-bay model representation of a shear building.
(b) Free body diagram

It should be noted that the building may have any number of bays and that it is only a matter of convenience that we represent the shear building solely in terms of a single bay. Actually, we can further idealize the shear building as a single column [Fig. 7.2(a)], having concentrated masses at the floor levels with the understanding that only horizontal displacements of these masses are possible. Another alternative is to adopt a multimass-spring system shown in Fig. 7.3(a) to represent the shear building. In any of the three representations depicted in these figures, the stiffness coefficient, or spring constant k_i , shown between any two consecutive masses is the force required to produce a relative unit displacement of the two adjacent floor levels.

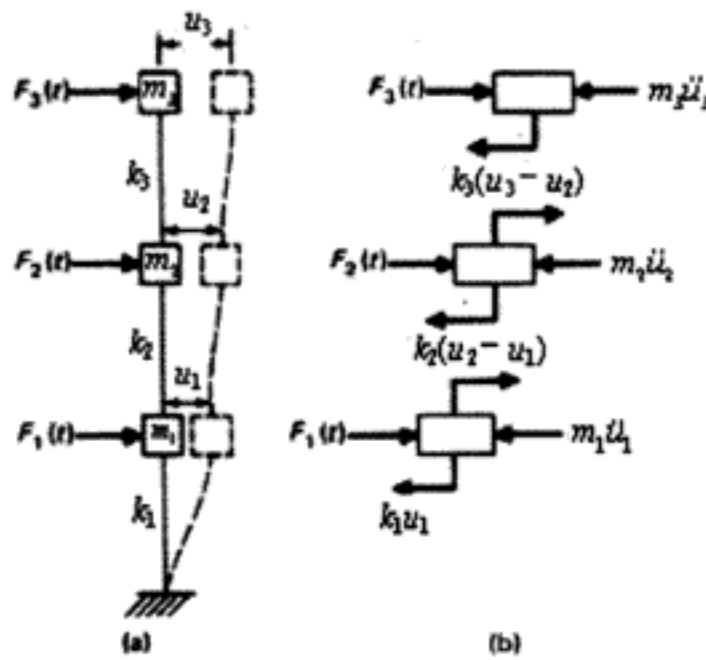


Fig. 7.2 (a) Single-column model representation of a shear building.
(b) Free body diagram.

For a uniform column with the two ends fixed against rotation, the stiffness or spring constant, k , is given by

$$k = \frac{12EI}{L^3} \quad (7.1a)$$

and for a column with one end fixed and the other pinned by

$$k = \frac{3EI}{L^3} \quad (7.1b)$$

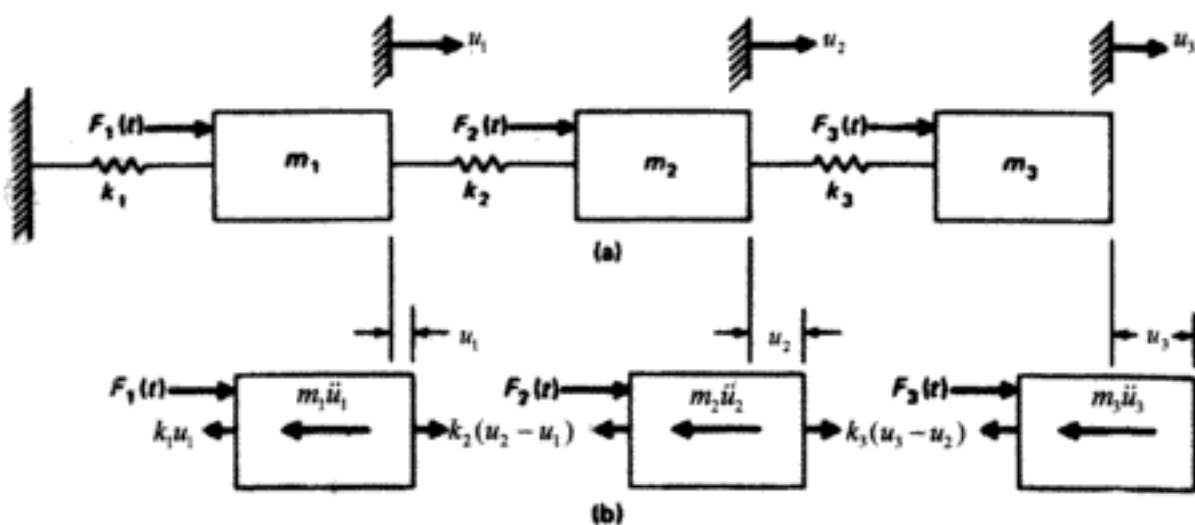


Fig. 7.3 Multimass-spring model representation of a shear building.

where E is the material modulus of elasticity, I the cross-sectional moment of inertia, and L the length of the column.

It should be clear that all of the three representations shown in Figs. 7.1 to 7.3 for the shear building are equivalent. Consequently, the following equations of motion for the three-story shear building are obtained from any of the corresponding free body diagrams shown in these figures by equating to zero the sum of the forces acting on each mass. Hence

$$\begin{aligned} m_1 \ddot{u}_1 + k_1 u_1 - k_2 (u_2 - u_1) - F_1(t) &= 0 \\ m_2 \ddot{u}_2 + k_2 (u_2 - u_1) - k_3 (u_3 - u_2) - F_2(t) &= 0 \\ m_3 \ddot{u}_3 + k_3 (u_3 - u_2) - F_3(t) &= 0 \end{aligned} \quad (7.2)$$

This system of equations constitutes the stiffness formulation of the equations of motion for a three-story shear building. It may conveniently be written in matrix notation as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F\} \quad (7.3)$$

where $[M]$ and $[K]$ are the mass and stiffness matrices given, respectively, by

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad (7.4)$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \quad (7.5)$$

and $\{u\}$, $\{\ddot{u}\}$ and $\{F\}$ are, respectively, the displacement, acceleration and force vectors given by

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \quad \{\ddot{u}\} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}, \quad \{F\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix} \quad (7.6)$$

It should be noted that the mass matrix, eq. (7.4), corresponding to the shear building is a diagonal matrix (the non-zero elements are only in the main diagonal). The elements of the stiffness matrix, eq. (7.5), are designated *stiffness coefficients*. In general, the stiffness coefficient, k_{ij} , is defined as the force at coordinate i when a unit displacement is given at j , all other coordinates being fixed. For example, the coefficient in the second row and second column of eq. (7.5), $k_{22} = k_2 + k_3$, is the force required at the second floor when a unit displacement is given to this floor.

7.2 Natural Frequencies and Normal Modes

The problem of free vibration requires that the force vector $\{F\}$ be equal to zero in eq. (7.3). Namely,

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \quad (7.7)$$

For free vibrations of the undamped structure, we seek solutions of eq. (7.7) in the form

$$u_i = a_i \sin(\omega t - \alpha), \quad i = 1, 2, \dots, n$$

or in vector notation

$$\{u\} = \{a\} \sin(\omega t - \alpha) \quad (7.8)$$

where a_i is the amplitude of motion of the i th coordinate and n is the number of degrees of freedom. The substitution of eq. (7.8) into eq. (7.7) gives

$$-\omega^2 [M]\{a\} \sin(\omega t - \alpha) + [K]\{a\} \sin(\omega t - \alpha) = 0$$

or factoring out $\sin(\omega t - \alpha)$ and rearranging terms

$$[[K] - \omega^2 [M]]\{a\} = \{0\} \quad (7.9)$$

which for the general case, is set for n homogenous (right-hand side equal to zero) algebraic system of linear equations with n unknown displacements a_i and an unknown parameter ω^2 . The formulation of eq. (7.9) is an important mathematical problem known as an eigenproblem. Its nontrivial solution, that is, the solution for which not all $a_i = 0$, requires that the determinant of the matrix factor of $\{a\}$ be equal to zero; in this case

$$|[K] - \omega^2 [M]| = 0 \quad (7.10)$$

In general, the expansion of the determinant in eq. (7.10) results in a polynomial equation of degree n in ω^2 which should be satisfied for n values of ω^2 . This polynomial is known as the characteristic equation of the system. For each of these values of ω^2 satisfying the characteristic eq. (7.10) we can solve eq. (7.9) for a_1, a_2, \dots, a_n in terms of an arbitrary constant. The necessary calculations are better explained through a numerical example.

Illustrative Example 7.1

The structure to be analyzed is the two-story steel rigid frame shown in Fig. 7.4. The weights of the floors and walls are indicated in the figure and are assumed to include the structural weight as well. The building consists of a series of frames spaced 15 ft apart. It is further assumed that the structural properties are uniform along the length of the building and, therefore, the analysis to be made of an interior frame yields the response of the entire building. Determine (a) the natural frequencies and corresponding modal shapes, (b) the equations of motion with initial conditions for displacements u_{01} , u_{02} , and for velocities \dot{u}_{01} , and \dot{u}_{02} , respectively, for the first and second stories of the building.

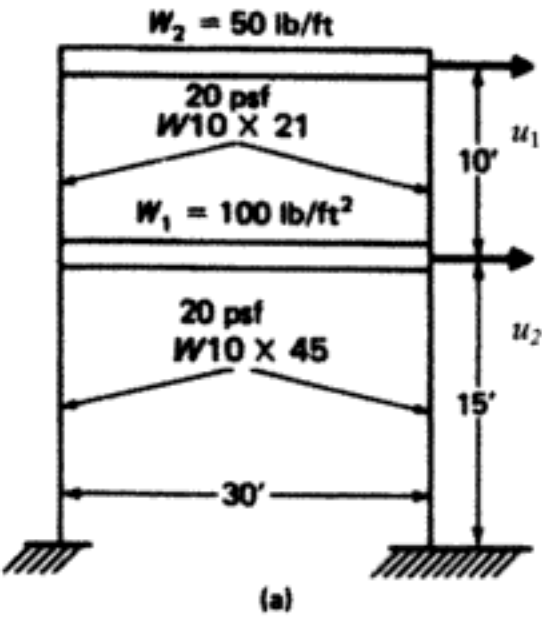


Fig. 7.4 Two-story shear building for Illustrative Example 7.1.

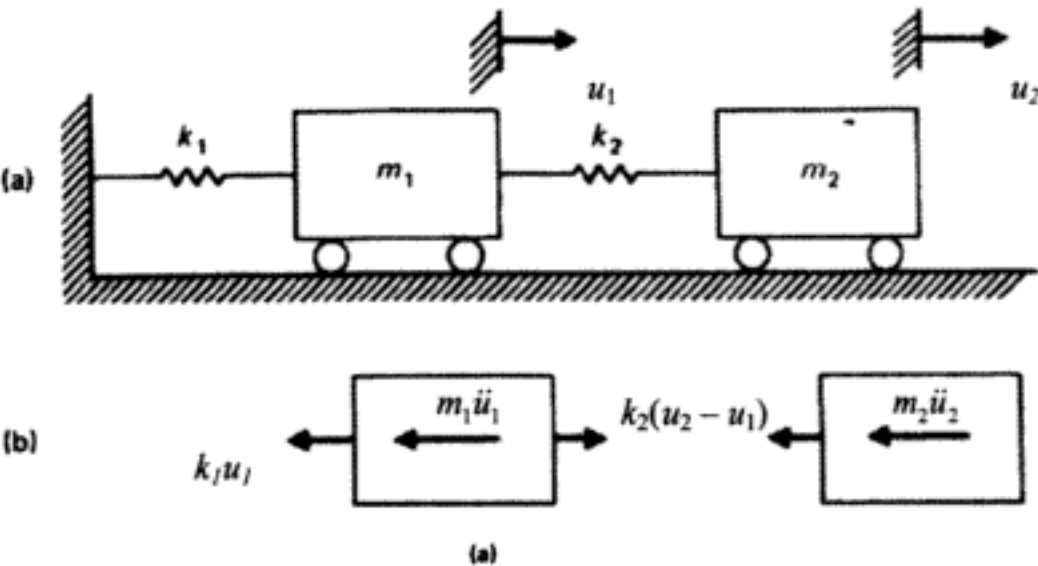


Fig. 7.5 Multimass-spring model for a two-story shear building of Illustrative Example 7.1. (a) Model, (b) Free body diagram.

Solution:

(a) Natural Frequencies and Modal Shapes

The building is modeled as a shear building and, under the assumptions stated, the entire building may be represented by the spring-mass system shown in Fig. 7.5. The concentrated weights, which are each taken as the total floor weight plus that of the tributary walls, are computed as follows:

$$W_1 = 100 \times 30 \times 15 + 20 \times 12.5 \times 15 \times 2 = 52,500 \text{ lb}$$

$$m_1 = 136 \text{ lb} \cdot \text{sec}^2 / \text{in}$$

$$W_2 = 50 \times 30 \times 15 + 20 \times 5 \times 15 \times 2 = 25,500 \text{ lb}$$

$$m_2 = 66 \text{ lb} \cdot \text{sec}^2 / \text{in}$$

Since the girders are assumed to be rigid and fixed at the two ends, the stiffness (spring constant) of each story is given by eq. (7.1a) as

$$k = \frac{12E(2I)}{L^3}$$

and the individual values for the steel column sections indicated are thus

$$k_1 = \frac{12 \times 30 \times 10^6 \times 248 \times 2}{(15 \times 12)^3} = 30,700 \text{ lb/in}$$

$$k_2 = \frac{12 \times 30 \times 10^6 \times 118 \times 2}{(10 \times 12)^3} = 44,300 \text{ lb/in}$$

The equations of motion for the system, which are obtained by considering in Fig. 7.5(b) the dynamic equilibrium of each mass in free vibration, are

$$m_1 \ddot{u}_1 + k_1 u_1 - k_2 (u_2 - u_1) = 0$$

$$m_2 \ddot{u}_2 + k_2 (u_2 - u_1) = 0$$

In the usual manner, these equations of motion are solved for free vibration by substituting

$$\begin{aligned} u_1 &= a_1 \sin(\omega t - \alpha) \\ u_2 &= a_2 \sin(\omega t - \alpha) \end{aligned} \quad (\text{a})$$

for the displacements and

$$\ddot{u}_1 = -a_1 \omega^2 \sin(\omega t - \alpha)$$

$$\ddot{u}_2 = -a_2 \omega^2 \sin(\omega t - \alpha)$$

for the accelerations. In matrix notation, we obtain

$$\begin{bmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{b})$$

For a nontrivial solution, we require that the determinant of the coefficients be equal to zero, that is,

$$\begin{vmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{vmatrix} = 0 \quad (c)$$

The expansion of this determinant gives a quadratic equation in ω^2 , namely

$$m_1 m_2 \omega^4 - [(k_1 - k_2)m_2 + m_1 k_2] \omega^2 + k_1 k_2 = 0 \quad (d)$$

or by introducing the numerical values for this example, we obtain

$$8976 \omega^4 - 10,974,800 \omega^2 + 1.36 \times 10^6 = 0 \quad (e)$$

The roots of this quadratic are

$$\omega_1^2 = 140$$

$$\omega_2^2 = 1082$$

Therefore, the natural frequencies of the structure are

$$\omega_1 = 11.83 \text{ rad/sec}$$

$$\omega_2 = 32.89 \text{ rad/sec}$$

or in cycles per second (cps)

$$f_1 = \omega_1 / 2\pi = 1.88 \text{ cps}$$

$$f_2 = \omega_2 / 2\pi = 5.24 \text{ cps}$$

and the corresponding natural periods:

$$T_1 = \frac{1}{f_1} = 0.532 \text{ sec}$$

$$T_2 = \frac{1}{f_2} = 0.191 \text{ sec}$$

To solve eq. (b) for the amplitudes a_1 and a_2 , we note that by equating the determinant to zero in eq.(c), the number of independent equations is one less. Thus in the present case, the system of two equations is reduced to one independent equation. Considering the first equation in eq. (b) and substituting the first natural frequency, $\omega_1 = 11.8 \text{ rad/sec}$, we obtain

$$55,960 a_{11} - 44,300 a_{21} = 0 \quad (f)$$

We have introduced a second sub-index in a_1 and a_2 to indicate that the value ω_1 has been used in this equation. Since in the present case there are two unknowns and only one equation, we can solve eq. (f) only for the relative value of a_{21} to a_{11} . This relative value is known as the normal mode or modal shape corresponding to the first frequency. For this example, eq. (f) gives

$$\frac{a_{21}}{a_{11}} = 1.263$$

It is customary to describe the normal modes by assigning a unit value to one of the amplitudes; thus, for the first mode we set a_{11} equal to unity so that

$$\begin{aligned} a_{11} &= 1.000 \\ a_{21} &= 1.263 \end{aligned} \quad (g)$$

Similarly, substituting the second natural frequency, $\omega_2 = 32.9$ rad/sec into eq. (b) we obtain the second normal mode as

$$\begin{aligned} a_{12} &= 1.000 \\ a_{22} &= -1.629 \end{aligned} \quad (h)$$

It should be noted that although we obtained only ratios, the amplitudes of motion could, of course, be found from initial conditions.

We have now arrived at two possible simple harmonic motions of the structure which can take place in such a way that all the masses move in phase in the same frequency, either ω_1 or ω_2 . Such a motion of an undamped system is called a normal or natural mode of vibration. The shapes for these modes (a_{21}/a_{11} and a_{22}/a_{12}) for this example are called normal mode shapes of simply modal shapes for the corresponding natural frequencies ω_1 and ω_2 . These two modes for this example are depicted in Fig. 7.6.

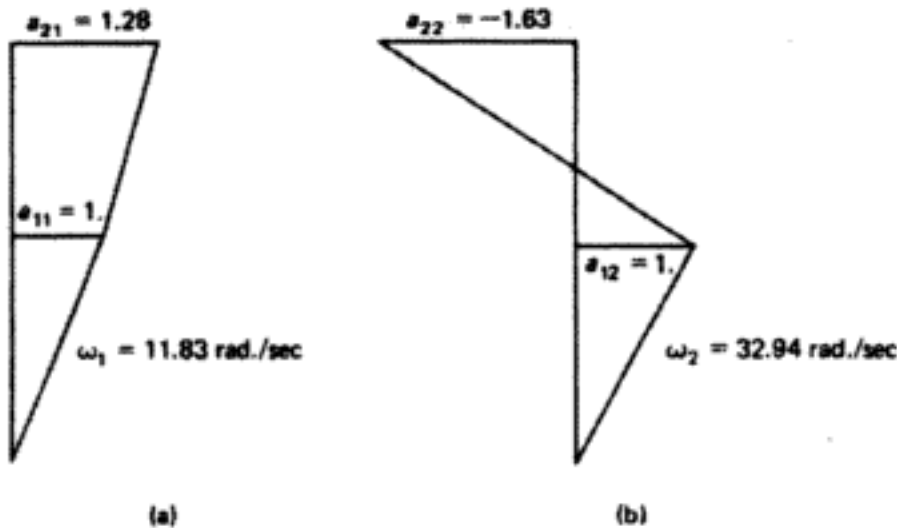


Fig. 7.6 Normal modes for Illustrative Example 7.1
(a) First mode (b) Second mode.

We often use the designation first mode or fundamental mode to refer to the mode associated with the lowest frequency. The other modes are sometimes called harmonics or higher harmonics. It is evident that the modes of vibration, each having its own frequency, behave essentially as single-degree-of-freedom systems.

(b) Equations of Motion

The total motion of the system, that is, the total solution of the equations of motion, eq. (7.7), is given by the superposition of the modal harmonic

vibrations which in terms of arbitrary constants of integration may be written as

$$\begin{aligned} u_1(t) &= C_1' a_{11} \sin(\omega_1 t - \alpha_1) + C_2' a_{12} \sin(\omega_2 t - \alpha_2) \\ u_2(t) &= C_1' a_{21} \sin(\omega_1 t - \alpha_1) + C_2' a_{22} \sin(\omega_2 t - \alpha_2) \end{aligned} \quad (i)$$

Here C_1' and C_2' as well as α_1 and α_2 are four constants of integration to be determined from four initial conditions which are the initial displacement and velocity for each mass in the system. For a two-degree-of-freedom system, these initial conditions are

$$\begin{aligned} u_1(0) &= u_{01}, & \dot{u}_1(0) &= \dot{u}_{01} \\ u_2(0) &= u_{02}, & \dot{u}_2(0) &= \dot{u}_{02} \end{aligned} \quad (j)$$

For computational purposes, it is convenient to eliminate the phase angles $[\alpha_1 \text{ and } \alpha_2]$ in eq.(i) in favor of other constants. Expanding the trigonometric functions in eq.(i) and renaming the constants, we obtain

$$\begin{aligned} u_1(t) &= C_1 a_{11} \sin \omega_1 t + C_2 a_{11} \cos \omega_1 t + C_3 a_{12} \sin \omega_2 t + C_4 a_{12} \cos \omega_2 t \\ u_2(t) &= C_1 a_{21} \sin \omega_1 t + C_2 a_{21} \cos \omega_1 t + C_3 a_{22} \sin \omega_2 t + C_4 a_{22} \cos \omega_2 t \end{aligned} \quad (k)$$

in which C_1 , C_2 , C_3 and C_4 are the new renamed constants of integration. From the first two initial conditions in eq. (j), we obtain the following two equations:

$$\begin{aligned} u_{01} &= C_2 a_{11} + C_4 a_{12} \\ u_{02} &= C_2 a_{21} + C_4 a_{22} \end{aligned} \quad (l)$$

Since the modes are independent, these equations can always be solved for C_2 and C_4 . Similarly, by expressing in eq.(k) the velocities at time equal to zero, we find

$$\begin{aligned} \dot{u}_{01} &= \omega_1 C_1 a_{11} + \omega_2 C_3 a_{12} \\ \dot{u}_{02} &= \omega_1 C_1 a_{21} + \omega_2 C_3 a_{22} \end{aligned} \quad (m)$$

The solution of these two sets of equations, (l) and (m), allows us to express the motion of the system in terms of the two modal vibrations, each proceeding at its own frequency, completely independent of the other, the amplitudes and phases being determined by the initial conditions.

7.3 Orthogonality Property of the Normal Modes

We shall now introduce an important property of the normal modes, the orthogonality property. This property constitutes the basis of the most important method for solving

dynamic problems, the Modal Superposition Method of multi-degree-of-freedom systems. We begin by rewriting the equations of motion in free vibration, eq. (7.7) as

$$[K]\{a\} = \omega^2[M]\{a\} \quad (7.11)$$

For the two-degree-of-freedom system, we obtain from eq.(b) of Illustrative Example 7.1

$$\begin{aligned} (k_1 + k_2)a_1 - k_2a_2 &= \omega^2 m_1 a_1 \\ -k_2a_1 + k_2a_2 &= \omega^2 m_2 a_2 \end{aligned} \quad (7.12)$$

These equations are exactly the same as eq. (b) of Illustrative Example 7.1 but written in this form they may be given a static interpretation as the equilibrium equations for the system acted on by forces of magnitude $\omega^2 m_1 a_1$ and $\omega^2 m_2 a_2$ applied to masses m_1 and m_2 , respectively. The modal shapes may then be considered as the static deflections resulting from the forces on the right-hand side of eq. (7.12) for any of the two modes. This interpretation, as a static problem, allows us to use the results of the general static theory of linear structures. In particular, we may use of Betti's theorem which states: For a structure acted upon by two systems of loads and corresponding displacements, the work done by the first system of loads moving through the displacements of the second system is equal to the work done by this second system of loads undergoing the displacements produced by the first load system. The two systems of loading and corresponding displacements which we shall consider are as follows:

System I:

Forces	$\omega_1^2 m_1 a_{11}$, $\omega_1^2 m_2 a_{21}$
Displacements	a_{11} , a_{21}

System II:

Forces	$\omega_2^2 m_1 a_{12}$, $\omega_2^2 m_2 a_{22}$
Displacements	a_{12} , a_{22}

The application of Betti's theorem for these two systems yields

$$\omega_1^2 m_1 a_{11} a_{12} + \omega_1^2 m_2 a_{21} a_{22} = \omega_2^2 m_1 a_{12} a_{11} + \omega_2^2 m_2 a_{22} a_{21}$$

or

$$(\omega_1^2 - \omega_2^2)(m_1 a_{11} a_{12} + m_2 a_{21} a_{22}) = 0$$

If the natural frequencies are different ($\omega_1 \neq \omega_2$), it follows that

$$m_1 a_{11} a_{12} + m_2 a_{21} a_{22} = 0$$

which is the so-called orthogonality relationship between modal shapes of a two degree-of-freedom system. For an n -degree-of-freedom system in which the mass

matrix is diagonal, the orthogonality condition between any two modes i and j may be expressed as

$$\sum_{k=1}^n m_k a_{ki} a_{kj} = 0, \quad \text{for } i \neq j \quad (7.13)$$

and in general for any n -degree-of-freedom system as

$$\{a\}_i^T [M] \{a\}_j = 0 \quad \text{for } i \neq j \quad (7.14)$$

in which $\{a_i\}$ and $\{a_j\}$ are any two modal vectors and $[M]$ is the mass matrix of the system.

As mentioned before, the amplitudes of vibration in a normal mode are only relative values which may be scaled or normalized to some extent as a matter of choice. The following is an especially convenient normalization for a general system:

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\{a\}_i^T [M] \{a\}_j}} \quad (7.15)$$

which, for a system having a diagonal mass matrix, may be written as

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^n m_k a_{kj}^2}} \quad (7.16)$$

in which ϕ_{ij} is the normalized i component of the j modal vector. For normalized eigenvectors, the orthogonality condition is given by

$$\begin{aligned} \{\phi\}_i^T [M] \{\phi\}_j &= 0 & \text{for } i \neq j \\ &= 1 & \text{for } i = j \end{aligned} \quad (7.17)$$

Another orthogonality condition is obtained by writing eq. (7.9) for the normalized j mode as

$$[K] \{\phi\}_j = \omega_j^2 [M] \{\phi\}_j \quad (7.18)$$

Then pre-multiplying eq. (7.18) by $\{\phi\}_i^T$ we obtain, in view of eq. (7.17), the following orthogonality condition between eigenvectors:

$$\begin{aligned} \{\phi\}_i^T [K] \{\phi\}_j &= 0 & \text{for } i \neq j \\ &= \omega_j^2 & \text{for } i = j \end{aligned} \quad (7.19)$$

Illustrative Example 7.2

For the two-story shear building of Illustrative Example 7.1 determine a) the normalized modal shapes of vibration, and b) verify the orthogonality condition between the modes.

Solution:

The substitution of eqs. (g) and (h) from Illustrative Example 7.1 together with the values of the masses from Illustrative Example 7.1 into the normalization factor required in eq. (7.16) gives

$$\begin{aligned}\sqrt{(136)(1.00)^2 + (66)(1.263)^2} &= \sqrt{241.31} \\ \sqrt{(136)(1.00)^2 + (66)(-1.629)^2} &= \sqrt{311.08}\end{aligned}$$

Consequently, the normalized modes are

$$\begin{aligned}\phi_{11} &= \frac{1.00}{\sqrt{241.31}} = 0.06437, & \phi_{12} &= \frac{1.00}{\sqrt{311.08}} = 0.0567 \\ \phi_{21} &= \frac{1.263}{\sqrt{241.31}} = 0.0813, & \phi_{22} &= \frac{-1.629}{\sqrt{311.08}} = -0.0924\end{aligned}$$

The normal modes may be conveniently arranged in the columns of a matrix known as the modal matrix of the system. For the general case of n degrees of freedom, the modal matrix is written as

$$[\Phi] = \begin{bmatrix} \phi_{11} & \phi_{12} \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} \cdots & \phi_{2n} \\ \vdots & \vdots & \vdots \\ \phi_{n1} & \phi_{n2} \cdots & \phi_{nn} \end{bmatrix} \quad (7.20)$$

The orthogonality condition may then be expressed in general as

$$[\Phi]^T [M] [\Phi] = [I] \quad (7.21)$$

where $[\Phi]^T$ is the matrix transpose of $[\Phi]$, and $[M]$ the mass matrix of the system. For this example of two degrees of freedom, the modal matrix is

$$[\Phi] = \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} \quad (a)$$

To check the orthogonality condition, we simply substitute the normal modes from eq. (a) into eq. (7.21) and obtain

$$\begin{bmatrix} 0.06437 & 0.0813 \\ 0.0567 & -0.0924 \end{bmatrix} \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix} \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We have seen that to determine the natural frequencies and normal modes of vibration of a structural system, we have to solve an eigenvalue problem. The direct method of solution based on the expansion of the determinant and the solution of the resulting characteristic equation is limited in practice to systems having only a few degrees of freedom. For a system of many degrees of freedom, the algebraic and numerical work required for the solution of an eigenproblem becomes so immense as

to make the direct method impossible. However, there are many numerical methods available for the calculation of eigenvalues and eigenvectors of an eigenproblem. The discussion of these methods belongs in a mathematical text on numerical methods rather than in a text such as this on structural dynamics. One of the most popular methods for the numerical solution of an eigenproblem is the Jacobi Method, which is an iterative method to calculate the eigenvalues and eigenvectors of the system. The basic Jacobi solution method has been developed for the solution of standard eigenproblems (i.e., $[M]$ being the identity matrix). The method was proposed over a century ago and has been used extensively. This method can be applied to all symmetric matrices $[K]$ with no restriction on the eigenvalues. It is possible to transform the generalized eigenproblem, $[[K] - \omega^2[M]] = \{\Phi\} = \{0\}$ into the standard form and still maintain the symmetry required for the Jacobi Method. However, this transformation can be dispensed with by using a generalized Jacobi solution method (Bathe, K. J. 1982) which operates directly on $[K]$ and $[M]$.

7.4 Rayleigh's Quotient

Several iterative methods for the solution of an eigenproblem make use of the Rayleigh's quotient. The Rayleigh's quotient may be obtained by pre-multiplying eq. (7.18) by the transpose of the modal vector $\{\phi\}_j^T$. Hence,

$$\{\phi\}_j^T [K] \{\phi\}_j = \omega_j^2 \{\phi\}_j^T [M] \{\phi\}_j$$

The property of the mass matrix $[M]$ being positive definite¹ renders the product $\{\phi\}_j^T [M] \{\phi\}_j \neq 0$, thus, it is permissible to solve for ω_j^2 :

$$\omega^2 = \frac{\{\phi\}^T [K] \{\phi\}}{\{\phi\}^T [M] \{\phi\}} \quad (7.22)$$

in which for convenience the sub-index j has been omitted.

The ratio given by eq. (7.22) is known as the Rayleigh's quotient. This quotient has the following properties: 1) It provides the eigenvalue ω_j^2 when the corresponding eigenvector $\{\phi\}_j$ is introduced in eq. (7.22). 2) When a vector $\{\phi\}$ different from an eigenvector is used, then eq. (7.22) provides a value ω^2 that lies between the smallest eigenvalue, ω_1^2 and the largest eigenvalue ω_N^2 . 3) Finally, if a vector $\{\phi\}$ that is an approximation to eigenvector $\{\phi\}_j$, correct to d decimals is used, then the value of ω^2 obtained from eq. (7.22) is accurate to $2d$ number of decimals as an approximation to ω_j^2 .

¹ Matrix $[A]$ is defined as positive definite if it satisfies the condition that for any arbitrary nonzero vector $\{v\}$, the product $\{v\}^T [A] \{v\} > 0$.

Illustrative Example 7.3

Use Rayleigh's quotient to calculate an approximate value for the first eigenvalue of the structure in Illustrative Example 7.1 beginning with the approximate eigenvector for the first mode $\{\phi\}^T = \{1.00 \ 1.50\}$, then iterate using eqs. (7.12) and (7.22) to converge to the eigenvalue and eigenvector for the first mode.

Solution:

The substitution of the given vector $\{\phi\}^T = \{1.00 \ 1.50\}$ and the matrices $[K]$ and $[M]$ into the numerator of eq. (7.22) results in

$$\{1.00 \ 1.50\} \begin{bmatrix} 75,000 & -44,300 \\ -44,300 & 44,300 \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.50 \end{Bmatrix} = 42,025$$

and also into the denominator

$$\{1.00 \ 1.50\} \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.50 \end{Bmatrix} = 284.5$$

which substituted into eq.(7.22) yields

$$\omega^2 = \frac{42,025}{284.5} = 147.9$$

The use of the calculated value $\omega^2 = 147.9$ together with $a_1 = 1.00$ into the first eq. (7.12) yields

$$a_1 = 1.00 \quad \text{and} \quad a_2 = 1.24$$

A second iteration of eq. (7.22) with $\{\Phi\}^T = \{1.00 \ 1.24\}$ yields

$$\omega^2 = 140.02$$

This value of ω^2 is virtually equal to the solution $\omega_1 = 140.02$ obtained for the first mode in Illustrative Example 7.1

Another popular iterative method to solve an eigenproblem, that is, for structural dynamics, to calculate natural frequencies and modal shapes, is the Subspace Iteration Method. Commercial computer programs such as SAP2000 used in this book, usually provide several other methods in addition to the Jacobi and the Subspace Iteration methods. The program SAP2000 gives the user the option of solving for natural frequencies and modal shapes by any of the following methods: Jacobi, Subspace

Iteration, Lanczos, Inverse Iteration Power methods and Complex Eigenvalue Analysis.

7.5 Program 8—Natural Frequencies and Normal Modes

The program presented in this section uses the generalized Jacobi method to determine the natural frequencies and corresponding modal shapes for a structure modeled as a discrete system.

Illustrative Example 7.4.

Use Program 8 to solve the eigenproblem corresponding to a system having the following stiffness and mass matrices:

$$[K] = \begin{bmatrix} 3000 & -1500 & 0 \\ -1500 & 3000 & -1500 \\ 0 & -1500 & 1500 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

The execution of Program 8 to calculate natural frequencies and modal shapes requires the previous preparation of a file containing the stiffness and mass matrices of the system. This file is created during the execution of one of the programs to model the structure or by execution of the auxiliary Program X1. This program accepts as input the stiffness and mass matrices of the structure and creates the file required to execute Program 8. In the solution of Example 7.4, the required file was created by executing Program X1.

Input Data and Output Results

PROGRAM 8: NATURAL FREQUENCIES AND NORMAL MODES

DATA FILE: SK

INPUT DATA:

STIFFNESS MATRIX

```
0.30000E+04  -.15000E+04  0.00000E+00
-.15000E+04  0.30000E+04  -.15000E+04
0.00000E+00  -.15000E+04  0.15000E+04
```

MASS MATRIX

```
0.10000E+01  0.00000E+00  0.00000E+00
0.00000E+00  0.10000E+01  0.00000E+00
0.00000E+00  0.00000E+01  1.00000E+00
```

OUTPUT RESULTS:

EIGENVALUES:

297.1	2332.4	4870.5
-------	--------	--------

NATURAL FREQUENCIES (C.P.S.):

2.74	7.69	11.11
------	------	-------

EIGENVECTORS BY ROWS:

0.32799	0.59101	0.73698
0.73698	0.32799	0.59101
0.59101	0.73698	0.32799

7.6 Free Vibration of a Shear Building Using SAP2000

Illustrative Example 7.5

Determine the natural frequencies and modal shapes for the six-story steel shear building modeled as shown in Fig. 7.7.

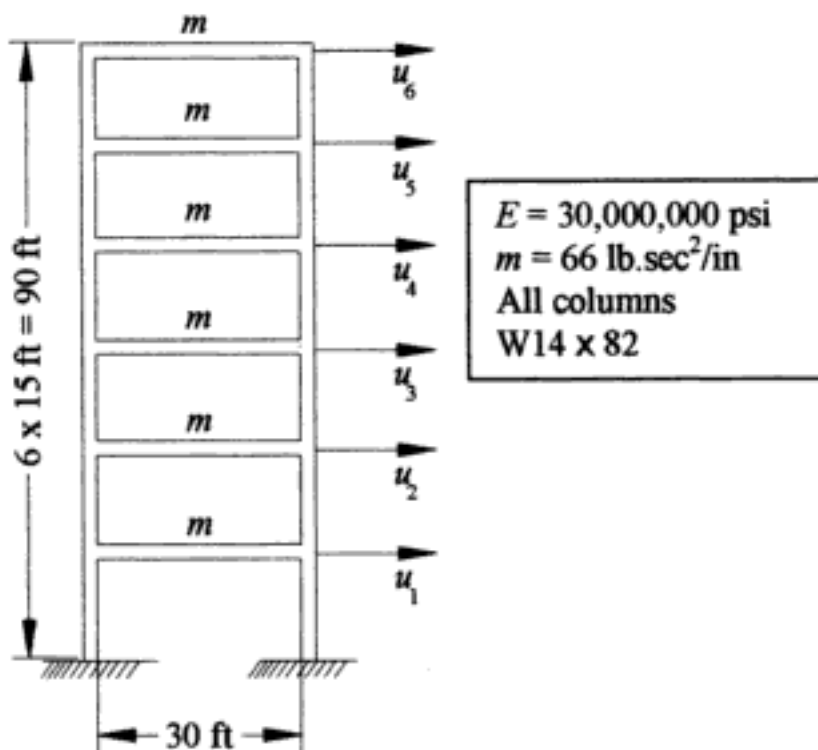


Fig. 7.7 Six-story shear building of Illustrative Example 7.5.

Solution:

Analytical Model

The analysis is performed by modeling the structure as a six-story plane frame with rigid horizontal members. The following commands are implemented in SAP2000:

Begin: Open SAP2000 (already installed)

Hint: Maximize both windows for a full view of the screen.

Select: Use the drop-down menu located in the lower right-hand corner of the screen to select "lb-in".

Model: From the Main Menu bar:
FILE>NEW MODEL FROM TEMPLATE
Select the button for the Plane Frame (top of second column) and enter:

Number of Stories	6
Number of Bays	1
Story Height	180
Bay Width	360

Accept all other default values.

Edit: Click on XZ View icon in the Menu Bar
Maximize the XZ View window and use the PAN icon in the toolbar to drag the figure to the center of the screen.

Label: From the Main Menu enter:
VIEW>SET ELEMENTS
Click on Joint Labels and Frame Labels. Then OK.
Use the PAN icon on the toolbar to center the figure on the screen.

Supports: Select joints 1 and 8 by clicking on them and enter:
ASSIGN>JOINT>RESTRAINTS
Click on restraints in all directions
Then OK.

Material: DEFINE>MATERIALS
Under "Materials" select STEEL
Click to Modify/Show Material
Enter:

Material Name =	STEEL
Mass per Unit Volume =	0
Weight per Unit Volume =	0
Modulus of Elasticity =	30E6

Accept remaining default values. Then OK, OK.

Frame Sections: DEFINE>FRAME SECTIONS
Click on Import I/Wide Flange
In the new window opened select "Sections"
The window C:\SAP2000\Sections.pro will open
Select W14 x 82 from the list of I beams
Then OK, Ok, OK.

DEFINE>FRAME SECTIONS
Click on "Add I/Wide Flange"
Click on "Add General" from the drop down menu
In the "Property Data" window enter:

Cross Sectional Area = 1E6
 Moment of Inertia about 3 axis = 1E6
 Accept remaining default values.
 Material select STEEL Then OK, OK, OK.

Assign: Select all the columns of the frame by clicking on them.

ASSIGN>FRAME>SECTIONS
 Select W14 x 82. Then OK.
 Select all girders by clicking on them
 ASSIGN>FRAME>SECTIONS
 Select FSEC2. Then OK.

Masses: Select joints 2 through 7 by clicking on them

ASSIGN>JOINT>MASSES
 Direction 1 = 66
 Accept all remaining default values.
 Then OK.

Analyze: ANALYZE>SET OPTIONS

Check only UX in Available DOF's (Degrees of Freedom)
 Check Dynamic Analysis and then click on Set Dynamic Parameters
 Enter: Number of Modes = 6.
 Then OK, OK.

ANALYZE>RUN

Enter filename: EXD 7.5. Then SAVE.
 At the conclusion of the calculations, after checking for errors, press OK.
 Select XZ view.

Print Input Tables: FILE>PRINT INPUT TABLES

CLICK "Print to File" and accept all other default values.
 Then OK.
 Use Notepad or other word editor to retrieve, edit and print the input table
 file C:/SAP2000/EXD 7.5.txt.

(The edited Input Tables of Illustrative Example 7.5 are reproduced in Table 7.1)

Table 7.1 Edited Input Tables for Illustrative Example 7.5.

STATIC LOAD CASES		
STATIC	CASE	SELF WT
CASE	TYPE	FACTOR
LOAD1	DEAD	0.0000

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-180.00000	0.00000	0.00000	1 1 1 1 1
2	-180.00000	0.00000	180.00000	0 0 0 0 0
3	-180.00000	0.00000	360.00000	0 0 0 0 0
4	-180.00000	0.00000	540.00000	0 0 0 0 0
5	-180.00000	0.00000	720.00000	0 0 0 0 0
6	-180.00000	0.00000	900.00000	0 0 0 0 0
7	-180.00000	0.00000	1080.00000	0 0 0 0 0
8	180.00000	0.00000	0.00000	1 1 1 1 1
9	180.00000	0.00000	180.00000	0 0 0 0 0
10	180.00000	0.00000	360.00000	0 0 0 0 0
11	180.00000	0.00000	540.00000	0 0 0 0 0
12	180.00000	0.00000	720.00000	0 0 0 0 0
13	180.00000	0.00000	900.00000	0 0 0 0 0
14	180.00000	0.00000	1080.00000	0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
2	66.000	0.000	0.000	0.000	0.000	0.000
3	66.000	0.000	0.000	0.000	0.000	0.000
4	66.000	0.000	0.000	0.000	0.000	0.000
5	66.000	0.000	0.000	0.000	0.000	0.000
6	66.000	0.000	0.000	0.000	0.000	0.000
7	66.000	0.000	0.000	0.000	0.000	0.000

FRAME ELEMENT DATA

FRAME	JNT-1	JNT-2	SECTION	ANG	RELEASES	SEG	R1	R2	FACTOR	LENGTH
1	1	2	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
2	2	3	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
3	3	4	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
4	4	5	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
5	5	6	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
6	6	7	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
7	8	9	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
8	9	10	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
9	10	11	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
10	11	12	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
11	12	13	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
12	13	14	W14X82	0.000	000000	2	0.0	0.0	1.000	180.000
13	2	9	FSEC2	0.000	000000	4	0.0	0.0	1.000	360.000
14	3	10	FSEC2	0.000	000000	4	0.0	0.0	1.000	360.000
15	4	11	FSEC2	0.000	000000	4	0.0	0.0	1.000	360.000
16	5	12	FSEC2	0.000	000000	4	0.0	0.0	1.000	360.000
17	6	13	FSEC2	0.000	000000	4	0.0	0.0	1.000	360.000

Modal Periods and Natural Frequencies:

Using Notepad or other word editor to open the following file:

C:/SAP2000/EXD 7.5.OUT

The Natural Period and Frequency table may be found in this file.

(Table 7.2 contains the edited values for the natural periods and frequencies of the structure in Illustrative Example 7.5)

Table 7.2 Modal periods and natural frequencies of Illustrative Example 7.5.

MODAL PERIODS AND NATURAL FREQUENCIES				
MODE	PERIOD (sec)	FREQUENCY (cps)	FREQUENCY (rad/sec)	EIGENVALUE (rad/sec)**2
1	0.684	1.463	9.191	84.474
2	0.245	4.085	25.667	658.802
3	0.162	6.158	38.693	1497.140
4	0.129	7.752	48.707	2372.376
5	0.112	8.911	55.988	3134.665
6	0.104	9.622	60.459	3655.307

Mode shapes: DISPLAY>SHOW MODE SHAPE

Mode number = 1

Click on wire Shadow and Cubic Curve

Then OK.

To view values for the deflection components at any node in Mode 1 right-click on a joint.

Modes 1 through 6 may also be viewed and read on the screen by clicking on the + sign in the lower right hand corner of the screen

The eigenvectors for this structure are arranged in the columns of the modal matrix $[\Phi]$ given by eq.(a).

$$[\Phi] = \begin{bmatrix} 0.016 & 0.045 & 0.064 & 0.068 & 0.056 & 0.032 \\ 0.032 & 0.068 & 0.045 & -0.016 & -0.064 & -0.056 \\ 0.045 & 0.056 & -0.032 & -0.064 & 0.016 & 0.068 \\ 0.056 & 0.016 & 0.068 & 0.032 & 0.045 & -0.064 \\ 0.064 & -0.032 & -0.016 & 0.056 & -0.068 & 0.045 \\ 0.068 & -0.064 & 0.056 & 0.056 & 0.032 & -0.016 \end{bmatrix} \quad (a)$$

7.7 Summary

The motion of an undamped dynamic system in free vibration is governed by a homogenous system of differential equations which in matrix notation is

$$[M]\{\ddot{u}\} + [K]\{u\} = 0$$

The process of solving this system of equations leads to a homogenous system of linear algebraic equations of the form

$$[[K] - \omega^2[M]]\{a\} = \{0\}$$

which mathematically is known as an eigenproblem.

For a nontrivial solution of this problem, it is required that the determinant of the coefficients of the unknown $\{a\}$ be equal to zero, that is,

$$[K] - \omega^2[M] = 0$$

The roots of this equation provide the natural frequencies ω_i , ($i = 1, 2, \dots, n$). It is then possible to solve for the unknowns $\{a\}_i$ in terms of relative values. The vectors $\{a\}_i$, corresponding to the roots ω_i^2 are the modal shapes (eigenvectors) of the dynamic system. The arrangement in matrix format of the modal shapes constitutes the modal matrix $[\Phi]$ of the system. It is particularly convenient to normalize the eigenvectors to satisfy the following condition:

$$\{\phi\}_i^T [M] \{\phi\}_j = [I] \quad i = 1, 2, \dots, n$$

where the normalized modal vectors $\{\phi\}_i$ are obtained by dividing the components of the vector $\{a\}_{ij}$ by $\sqrt{\{a\}_i^T [M] \{a\}_i}$.

The normalized modal vectors satisfy the following important conditions of orthogonality:

$$\{\phi\}_i^T [M] \{\phi\}_j = 0 \quad \text{for } i \neq j$$

$$\{\phi\}_i [M] \{\phi\}_j = 1 \quad \text{for } i = j$$

and

$$\{\phi\}_i^T [K] \{\phi\}_j = 0 \quad \text{for } i \neq j$$

$$\{\phi\}_i [K] \{\phi\}_j = \omega_i^2 \quad \text{for } i = j$$

The above relations are equivalent to

$$[\Phi]^T [M] [\Phi] = [I]$$

and

$$[\Phi]^T [K] [\Phi] = [\Omega]$$

in which $[\Phi]$ is the modal matrix of the system and $[\Omega]$ is a diagonal matrix containing the eigenvalues ω_i^2 in the main diagonal.

For a dynamic system with only a few degrees of freedom, the natural frequencies and modal shapes may be determined by expanding the determinant and calculating the roots of the resulting characteristic equation. However, for a system with a large number of degrees of freedom, this direct method of solution becomes impractical. It is then necessary to resort to other numerical methods which usually require an iteration process.

7.8 Problems

Problem 7.1

Determine the natural frequencies and normal modes for the two-story shear building shown in Fig. P7.1

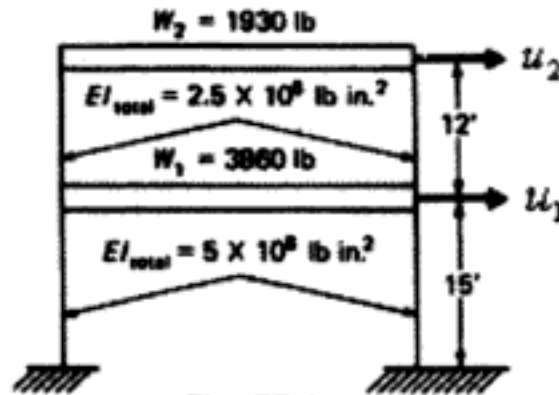


Fig. P7.1

Problem 7.2

A certain structure has been modeled as a three-degree-of-freedom system having the numerical values indicated in Fig. P7.2. Determine the natural frequencies of the structure and the corresponding normal modes. Check your answer using SAP2000.

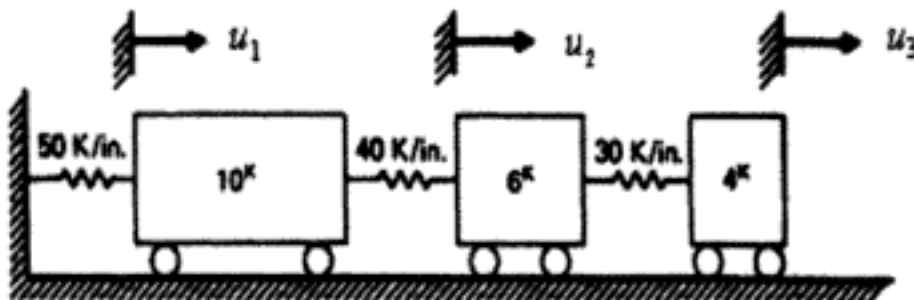


Fig. P7.2

Problem 7.3

Assume a shear building model for the frame shown in Fig. P7.3 and determine the natural frequencies and normal modes.

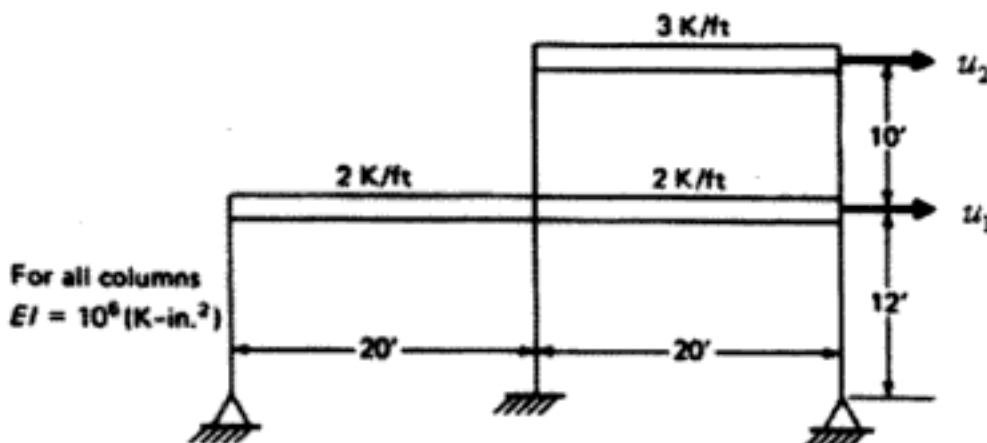


Fig. P7.3

Problem 7.4

A movable structural frame is supported on rollers as shown in Fig. P7.4. Determine natural periods and corresponding normal modes. Assume shear building model and check answers using SAP2000.

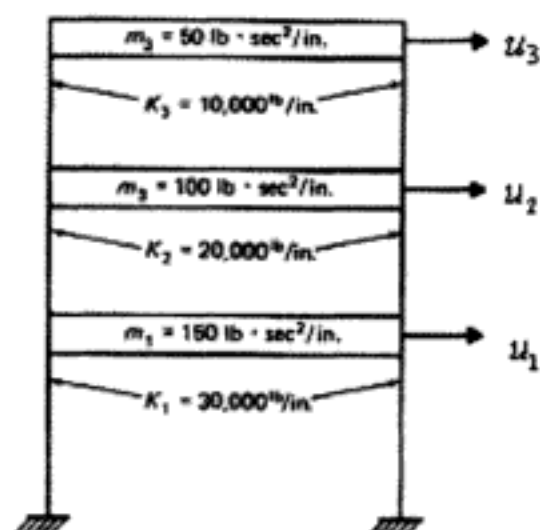


Fig. P7.4

Problem 7.5

Consider the uniform shear building in which the mass of each floor is m and the stiffness of each story is k . Determine the general form of the system of differential equations for a uniform shear building of N stories.

Problem 7.6

Find the natural frequencies and modal shapes for the three-degree-of-freedom shear building in Fig. P7.6.

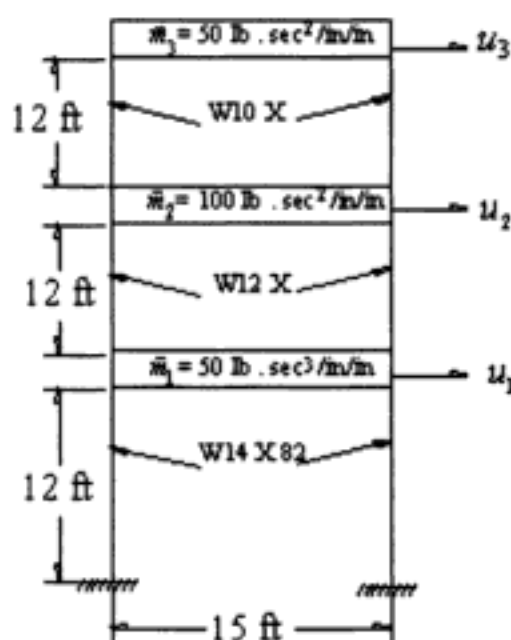


Fig. P7.6

Problem 7.7

Use the results of Problem 7.6 to write the expressions for the free vibration displacements u_1 , u_2 , and u_3 of the shear building in Fig. P7.6 in terms of constants of integration.

Problem 7.8

Use SAP2000 to determine the natural frequencies for the six-story uniform shear building modeled as a column shown in Fig. P7.8.

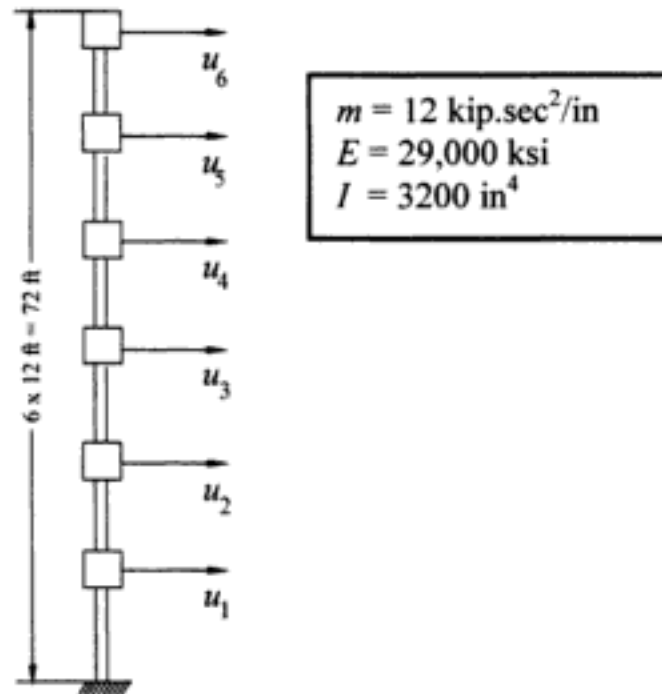


Fig. P7.8

8 Forced Motion of Shear Buildings

In the preceding chapter, we have shown that the free motion of a dynamic system may be expressed in terms of the normal modes of vibration. Our present interest is to demonstrate that the forced motion of such a system may also be expressed in terms of the normal modes of vibration and that the total response may be obtained as the superposition of the solution of independent modal equations. In other words, our aim in this chapter is to show that the normal modes may be used to transform the system of coupled differential equations into a set of uncoupled differential equations in which each equation contains only one dependent variable. Thus the modal superposition method reduced the problem of finding the response of a multi-degree-of-freedom system to the determination of the response of single-degree-of-freedom systems.

8.1 Modal Superposition Method

In Chapter 6, we have shown that any free motion of a multi-degree-of-freedom system may be expressed in terms of normal modes of vibration. It will now be demonstrated that the forced motion of such a system may also be expressed in terms of the normal modes of vibration. We return to the equations of motion, eq.(3.40), which for the particular case of a two-degree-of-freedom shear building are

$$\begin{aligned} m_1 \ddot{u}_1 + (k_1 + k_2)u_1 - k_2 u_2 &= F_1(t) \\ m_2 \ddot{u}_2 - k_2 u_1 + k_2 u_2 &= F_2(t) \end{aligned} \tag{8.1}$$

We seek to transform this coupled system of differential equations into a system of independent or uncoupled equations in which each equation contains only one unknown function of time. It is first necessary to express the solution in terms of the normal modes multiplied by some factors determining the contributions of each

mode. In the case of free motion, these factors were sinusoidal functions of time; in the present case, for forced motion they are general functions of time which we designate as $z_i(t)$. Hence, the solution of eq.(8.1) is assumed to be of the form

$$\begin{aligned} u_1(t) &= a_{11}z_1(t) + a_{12}z_2(t) \\ u_2(t) &= a_{21}z_1(t) + a_{22}z_2(t) \end{aligned} \quad (8.2)$$

Upon substitution into eq.(8.1), we obtain

$$\begin{aligned} m_1 a_{11} \ddot{z}_1 + (k_1 + k_2) a_{11} z_1 - k_2 a_{21} z_1 + m_1 a_{12} \ddot{z}_2 + (k_1 + k_2) a_{12} z_2 - k_2 a_{22} z_2 &= F_1(t) \\ m_2 a_{21} \ddot{z}_1 - k_2 a_{11} z_1 + k_2 a_{21} z_1 + m_2 a_{22} \ddot{z}_2 - k_2 a_{12} z_2 + k_2 a_{22} z_2 &= F_2(t) \end{aligned} \quad (8.3)$$

To determine the appropriate functions $z_1(t)$ and $z_2(t)$ that will uncouple eq.(8.3), it is necessary to make use of the orthogonality relations to separate the modes. The orthogonality relations are used by multiplying the first of eq.(8.3) by a_{11} and the second by a_{21} . Addition of these equations after multiplication and simplification by using eq.(6.3) and (6.5) yields

$$(m_1 a_{11}^2 + m_2 a_{21}^2) \ddot{z}_1 + \omega_1^2 (m_1 a_{11}^2 + m_2 a_{21}^2) z_1 = a_{11} F_1(t) + a_{21} F_2(t) \quad (8.4a)$$

Similarly, multiplying the first of eqs.(8.3) by a_{12} and the second by a_{22} , we obtain

$$(m_1 a_{12}^2 + m_2 a_{22}^2) \ddot{z}_2 + \omega_2^2 (m_1 a_{12}^2 + m_2 a_{22}^2) z_2 = a_{12} F_1(t) + a_{22} F_2(t) \quad (8.4b)$$

The results obtained in eqs.(8.4) permit a simple physical interpretation. The force that is effective in exciting a mode is equal to the work done by the external force displaced by the modal shape in question. From the mathematical point of view, what we have accomplished is to separate or uncouple, by a change of variables, the original system of differential equations. Consequently, each of these equations, eq.(8.4a) and (8.4b), corresponds to a single-degree-of-freedom system which may be written as

$$\begin{aligned} M_1 \ddot{z}_1 + K_1 z_1 &= P_1(t) \\ M_2 \ddot{z}_2 + K_2 z_2 &= P_2(t) \end{aligned} \quad (8.5)$$

where $M_1 = m_1 a_{11}^2 + m_2 a_{21}^2$ and $M_2 = m_1 a_{12}^2 + m_2 a_{22}^2$ are the modal masses; $K_1 = \omega_1^2 M_1$ and $K_2 = \omega_2^2 M_2$, the modal spring constants; $P_1(t) = a_{11} F_1(t) + a_{21} F_2(t)$ and $P_2(t) = a_{12} F_1(t) + a_{22} F_2(t)$ the modal forces. Alternatively, if we make use of the previous normalization in eq.(7.15) or (7.16), these equation may be written simply as

$$\begin{aligned}\ddot{z}_1 + \omega_1^2 z_1 &= P_1(t) \\ \ddot{z}_2 + \omega_2^2 z_2 &= P_2(t)\end{aligned}\quad (8.6)$$

where P_1 and P_2 are now given by

$$\begin{aligned}P_1 &= \phi_{11} F_1(t) + \phi_{21} F_2(t) \\ P_2 &= \phi_{12} F_1(t) + \phi_{22} F_2(t)\end{aligned}\quad (8.7)$$

The solution for the uncoupled differential equations, eqs.(8.5) or eqs.(8.6), may now be found by any of the methods presented in the previous chapters for the solution of a single degree of freedom system. In particular, Duhamel's integral provides a general solution for these equations regardless of the functions describing the forces acting on the structure. Also, maximum values of the response for each modal equation may readily be obtained using available response spectra. However, the superposition of modal maximum responses presents a problem. The fact is that these modal maximum values will in general not occur simultaneously as the transformation of coordinates, eq.(8.2), requires. To obviate the difficulty, it is necessary to use an approximate method. An upper limit for the maximum response may be obtained by adding the absolute values of the maximum modal contributions, that is, by substituting z_1 and z_2 in eqs.(8.2) for the maximum modal responses, $z_{1\max}$ and $z_{2\max}$, and adding the absolute values of the terms in these equations, so that

$$\begin{aligned}u_{1\max} &= |\phi_{11} z_{1\max}| + |\phi_{12} z_{2\max}| \\ u_{2\max} &= |\phi_{21} z_{1\max}| + |\phi_{22} z_{2\max}|\end{aligned}\quad (8.8)$$

The results obtained by this method will generally overestimate the maximum response. Another method, which is widely accepted and which generally gives a reasonable estimate of the maximum response from these spectral values, is the Square Root of the Sums of Squares of the modal contributions (SRSS method). Thus the maximum displacements may be approximated by

$$u_{1\max} = \sqrt{(\phi_{11} z_{1\max})^2 + (\phi_{12} z_{2\max})^2}$$

and

$$u_{2\max} = \sqrt{(\phi_{21} z_{1\max})^2 + (\phi_{22} z_{2\max})^2}\quad (8.9)$$

The results obtained by application of the SRSS method (Square Root of the Sum of the Squares of modal contributions) may substantially underestimate or overestimate the total response when two or more modes are closely spaced. In this case, another method known as the Complete Quadratic Combination (CQC) for combining modal responses to obtain the total response is recommended. The discussion of such a method is presented in this chapter in Section 8.6

The transformation from a system of two coupled differential equations, eqs.(8.1), to a set of two uncoupled differential equations, eqs.(8.6), may be extended to a general

system of N degrees of freedom. For such a system, it is particularly convenient to use matrix notation. With such notation, the equation of motion for a linear system of N degrees of freedom is given by eq.(7.3) as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F(t)\} \quad (8.10)$$

where $[M]$ and $[K]$ are respectively the mass and the stiffness matrices of the system, $\{F(t)\}$ the vector of external forces, and $\{u\}$ the vector of unknown displacements at the nodal coordinates.

Introducing into eq.(8.10) the linear transformation of coordinates

$$\{u\} = [\Phi]\{z\} \quad (8.11)$$

in which $[\Phi]$ is the modal matrix of the system, yields

$$[M][\Phi]\{\ddot{z}\} + [K][\Phi]\{z\} = \{F(t)\} \quad (8.12)$$

The pre-multiplication of eq.(8.12) by the transpose of the i th modal vector, $\{\phi\}_i^T$, results in

$$\{\phi\}_i^T [M][\Phi]\{\ddot{z}\} + \{\phi\}_i^T [K][\Phi]\{z\} = \{\phi\}_i^T \{F(t)\} \quad (8.13)$$

The orthogonality conditions between normalized modes, eqs. (7.17) and (7.19), imply that

$$\{\phi\}_i^T [M][\Phi] = 1 \quad (8.14)$$

and

$$\{\phi\}_i^T [K][\Phi] = \omega_i^2 \quad (8.15)$$

Consequently, eq.(8.13) may be written as

$$\ddot{z}_i + \omega_i^2 z_i = P_i(t) \quad i = 1, 2, 3, \dots, N \quad (8.16)$$

where the modal force $P_i(t)$ is given by

$$P_i(t) = \phi_{1i} F_1(t) + \phi_{2i} F_2(t) + \dots + \phi_{Ni} F_N(t) \quad (8.17)$$

Equation (8.16) constitutes a set of N uncoupled or independent equations of motion in terms of the modal coordinates z_i . These uncoupled equations, as may be observed, may readily be written after the natural frequencies ω_i and the modal vectors, $\{\phi\}_i$ have been determined in the solution of the corresponding eigenproblem as presented in Chapter 7.

Illustrative Example 8.1

The two-story frame of Illustrative Example 7.1 is acted upon at the floor levels by horizontal triangular impulsive forces as shown in Fig. 8.1. For this frame, determine the maximum floor displacements and the maximum shear forces in the columns.

Solution:

The results obtained in Illustrative Examples 7.1 and 7.2 for the free vibration of this frame gave the following values for the natural frequencies and normalized modes:

$$\omega_1 = 11.83 \text{ rad/sec}, \quad \omega_2 = 32.89 \text{ rad/sec}$$

$$\phi_{11} = 0.06437, \quad \phi_{12} = 0.0567$$

$$\phi_{21} = 0.08130, \quad \phi_{22} = -0.0924$$

The forces acting on the frame which are shown in Fig. 8.1(b) may be expressed by

$$\begin{aligned} F_1(t) &= 10,000(1 - t/t_d) \text{ lb} \\ F_2(t) &= 20,000(1 - t/t_d) \text{ lb} \end{aligned} \quad \text{for } t \leq 0.1 \text{ sec}$$

in which $t_d = 0.1 \text{ sec}$ and

$$F_1(t) = F_2(t) = 0, \quad \text{for } t > 0.1 \text{ sec}$$

The substitution of these values into the uncoupled equations of motion, eqs.(8.6) and (8.7) gives

$$\ddot{z}_1 + 140z_1 = 2270f(t)$$

$$\ddot{z}_2 + 1082.41z_2 = -1281f(t)$$

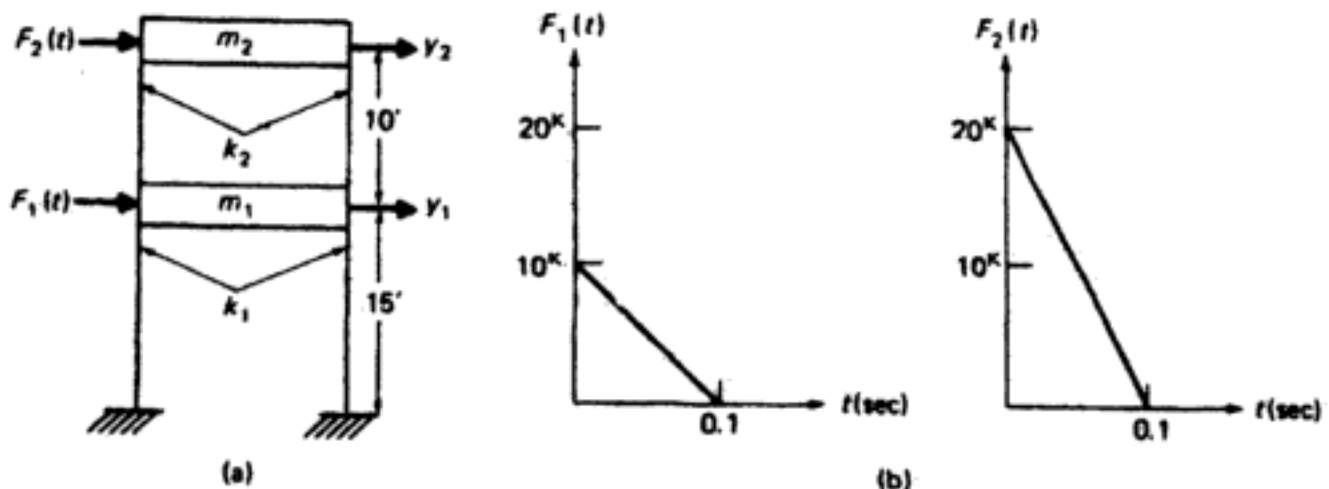


Fig. 8.1 Shear building with impulsive loadings. (a) Two-story shear building. (b) Impulsive loadings

in which $f(t) = 1 - t/t_d$ for $t \leq 0.1$ and $f(t) = 0$ for $t > 0.1$. The maximum values for z_1 and z_2 are then obtained from available spectral charts such as the one shown in Fig. 4.5. For this example,

$$\frac{t_d}{T_1} = \frac{0.1}{0.532} = 0.188$$

and

$$\frac{t_d}{T_2} = \frac{0.1}{0.191} = 0.524$$

in which the modal natural periods are calculated as

$$T_1 = \frac{2\pi}{\omega_1} = 0.532 \text{ sec} \quad \text{and} \quad T_2 = \frac{2\pi}{\omega_2} = 0.191 \text{ sec}$$

From Fig. 4.5, we obtain:

$$(DLF)_{1\max} = \frac{z_{1\max}}{z_{1st}} = 0.590$$

$$(DLF)_{2\max} = \frac{z_{2\max}}{z_{2st}} = 1.22$$

where the static deflections, z_{1st} and z_{2st} , are calculated as

$$z_{1st} = \frac{F_{01}}{\omega_1^2} = \frac{2270}{140} = 16.3, \quad z_{2st} = \frac{F_{02}}{\omega_2^2} = \frac{-1281}{1082.41} = -1.18$$

Then the maximum values of the modal response are:

$$z_{1\max} = 0.590 \times 16.3 = 9.62, \quad z_{2\max} = 1.22 \times 1.18 = 1.44$$

As indicated above these maximum modal values do not occur simultaneously and therefore cannot simply be superimposed to obtain the maximum response of the system. However, an upper limit for the absolute maximum displacement may be calculated with eqs.(8.8) as

$$u_{1\max} = |0.06437 \times 9.62| + |0.0567 \times 1.44| = 0.70 \text{ in}$$

$$u_{2\max} = |0.08130 \times 9.62| + |-0.0924 \times 1.44| = 0.92 \text{ in}$$

A second acceptable estimate of the maximum response is obtained by taking the square root of the sum of the squared modal contributions as indicated by eqs.(8.9). For this example, we have

$$\begin{aligned} u_{1\max} &= \sqrt{(0.06437 \times 9.62)^2 + (0.0567 \times 1.44)^2} = 0.62 \text{ in} \\ u_{2\max} &= \sqrt{(0.08130 \times 9.62)^2 + (-0.0924 \times 1.44)^2} = 0.79 \text{ in} \end{aligned} \quad (a)$$

The maximum shear force V_{\max} in the columns is given by

$$V_{\max} = k\Delta_u \quad (8.18)$$

in which k is the stiffness of the story and Δ_u the difference between the displacements at the two ends of the column. Since the maximum displacements calculated as in eq.(a) may have positive or negative values, the relative displacement Δ_u cannot be determined as the difference of the absolute displacements of the two ends of the column. The maximum positive value for Δ_u could be estimated as the sum of the absolute maximum displacements at the ends of the columns. However, this procedure will in most cases greatly overestimate the actual forces in the columns. The recommended procedure is to calculate first the shear force in the columns for each mode, separately, and then combine these modal forces by a suitable method, such as the square root of the sum of the squares of modal contributions. This procedure is based on the fact that modal displacements are known with their correct relative sign and not as absolute values.

The maximum shear force V_{ij} at story i corresponding to mode j is given by

$$V_{ij} = z_{j\max} (\phi_{ij} - \phi_{i-1j}) k_i \quad (8.19)$$

where $z_{j\max}$ is the maximum modal response, $(\phi_{ij} - \phi_{i-1j})$ the relative modal displacement of story i (with $\phi_{0j} = 0$), and k_i the stiffness of the story. For this example we have for the first story

$$\begin{aligned} k_1 &= \frac{12EI_1}{L_1^3} = \frac{12 \times 30 \times 10^6 \times 248.6}{(15 \times 12)^3} = 15,345 \text{ lb/in} \\ V_{11} &= 9.62 \times 0.06437 \times 15,345 = 9502 \text{ lb} \\ V_{12} &= 1.44 \times 0.0567 \times 15,345 = 1253 \text{ lb} \\ V_{1\max} &= \sqrt{9502^2 + 1253^2} = 9584 \text{ lb} \end{aligned}$$

and for a column in the second story

$$k_2 = \frac{12EI_2}{L_2^3} = \frac{12 \times 30 \times 10^6 \times 106.3}{(10 \times 12)^3} = 22,146 \text{ lb}$$

$$V_{21} = 9.62 \times (0.08130 - 0.06437) \times 22,146 = 3607 \text{ lb}$$

$$V_{22} = 1.44 \times (-0.0924 - 0.0567) \times 22,146 = -4755 \text{ lb}$$

$$V_{2\max} = \sqrt{3607^2 + 4755^2} = 5968 \text{ lb}$$

8.2 Response of a Shear Building to Base Motion

The response of a shear building to the base or foundation motion is conveniently obtained in terms of floor displacements relative to the base motion. For the two-story shear building of Fig. 8.2(a), which is modeled as shown in Fig. 8.2(b), the equations of motion obtained by equating to zero the sum of forces in the free body diagrams of Fig. 8.2(c) are the following:

$$\begin{aligned} m_1 \ddot{u}_1 + k_1(u_1 - u_s) - k_2(u_2 - u_1) &= 0 \\ m_2 \ddot{u}_2 + k_2(u_2 - u_1) &= 0 \end{aligned} \quad (8.20)$$

where $u_s = u_s(t)$ is the displacement imposed at the foundation of the structure. Expressing the floor displacements relative to the base motion, we have

$$\begin{aligned} u_{r1} &= u_1 - u_s \\ u_{r2} &= u_2 - u_s \end{aligned} \quad (8.21)$$

The differentiation yields

$$\begin{aligned} \ddot{u}_1 &= \ddot{u}_{r1} + \ddot{u}_s \\ \ddot{u}_2 &= \ddot{u}_{r2} + \ddot{u}_s \end{aligned} \quad (8.22)$$

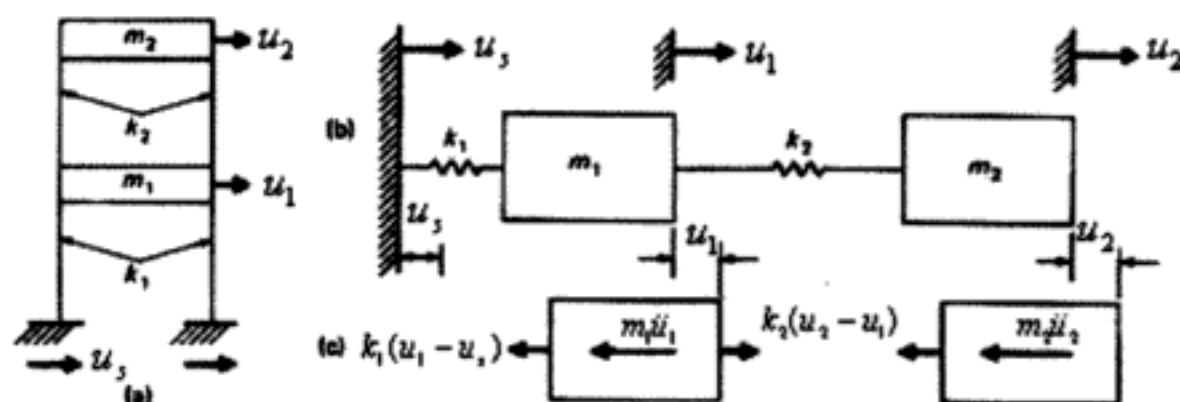


Fig. 8.2 Shear building with base motion. (a) Two-story shear building. (b) Mathematical model. (c) Free-body diagrams.

Substitution of eqs.(8.21) and (8.22) into eqs.(8.20) results in

$$\begin{aligned} m_1 \ddot{u}_{r1} + (k_1 + k_2)u_{r1} - k_2 u_{r2} &= -m_1 \ddot{u}_s \\ m_2 \ddot{u}_{r2} - k_2 u_{r1} + k_2 u_{r2} &= m_2 \ddot{u}_s \end{aligned} \quad (8.23)$$

We note that the right-hand sides of eqs.(8.23) are proportional to the same function of time, $\ddot{u}_s(t)$. This fact leads to a somewhat simpler solution compared to the solution of eqs.(8.6), which may contain different functions of time in each equation. For the base motion of the shear building, eqs.(8.4) may be written as

$$\begin{aligned} \ddot{z}_1 + \omega_1^2 z_1 &= -\frac{m_1 a_{11} + m_2 a_{211}}{m_1 a_{11}^2 + m_2 a_{21}^2} \ddot{u}_s(t) \\ \ddot{z}_2 + \omega_2^2 z_2 &= -\frac{m_1 a_{12} + m_2 a_{222}}{m_1 a_{12}^2 + m_2 a_{22}^2} \ddot{u}_s(t) \end{aligned} \quad (8.24)$$

or

$$\begin{aligned} \ddot{z}_1 + \omega_1^2 z_1 &= \Gamma_1 \ddot{u}_s(t) \\ \ddot{z}_2 + \omega_2^2 z_2 &= \Gamma_2 \ddot{u}_s(t) \end{aligned} \quad (8.25)$$

where Γ_1 and Γ_2 are called participation factors and are given by

$$\begin{aligned} \Gamma_1 &= -\frac{m_1 a_{11} + m_2 a_{21}}{m_1 a_{11}^2 + m_2 a_{21}^2} \\ \Gamma_2 &= -\frac{m_1 a_{12} + m_2 a_{22}}{m_1 a_{12}^2 + m_2 a_{22}^2} \end{aligned} \quad (8.26)$$

The relation between the modal displacements z_1, z_2 and the relative displacement u_{r1}, u_{r2} is given from eqs.(8.2) as

$$\begin{aligned} u_{r1} &= a_{11} z_1 + a_{12} z_2 \\ u_{r2} &= a_{21} z_1 + a_{22} z_2 \end{aligned} \quad (8.27)$$

In practice, it is convenient to introduce a change of variables in eqs.(8.25) such that the second members of these equations equal $\ddot{u}_s(t)$. The required change of variables to accomplish this simplification is

$$\begin{aligned} z_1 &= \Gamma_1 g_1 \\ z_2 &= \Gamma_2 g_2 \end{aligned} \quad (8.28)$$

which when introduced into eqs.(8.25) gives

$$\begin{aligned}\ddot{g}_1 + \omega_1^2 g_1 &= \ddot{u}_s(t) \\ \ddot{g}_2 + \omega_2^2 g_2 &= \ddot{u}_s(t)\end{aligned}\quad (8.29)$$

Finally, solving for $g_1(t)$ and $g_2(t)$ in the uncoupled eqs.(8.29) and substituting the solution into eqs.(8.27) and (8.28) give the response as

$$\begin{aligned}u_{r1}(t) &= \Gamma_1 a_{11} g_1(t) + \Gamma_2 a_{12} g_2(t) \\ u_{r2}(t) &= \Gamma_1 a_{21} g_1(t) + \Gamma_2 a_{22} g_2(t)\end{aligned}\quad (8.30)$$

When the maximum modal response $g_{1\max}$ and $g_{2\max}$ are obtained from spectral charts, we may estimate the maximum values $u_{r1\max}$ and $u_{r2\max}$ by the SRSS combination method as

$$\begin{aligned}u_{r1\max} &= \sqrt{(\Gamma_1 a_{11} g_{1\max})^2 + (\Gamma_2 a_{12} g_{2\max})^2} \\ u_{r2\max} &= \sqrt{(\Gamma_1 a_{21} g_{1\max})^2 + (\Gamma_2 a_{22} g_{2\max})^2}\end{aligned}\quad (8.31)$$

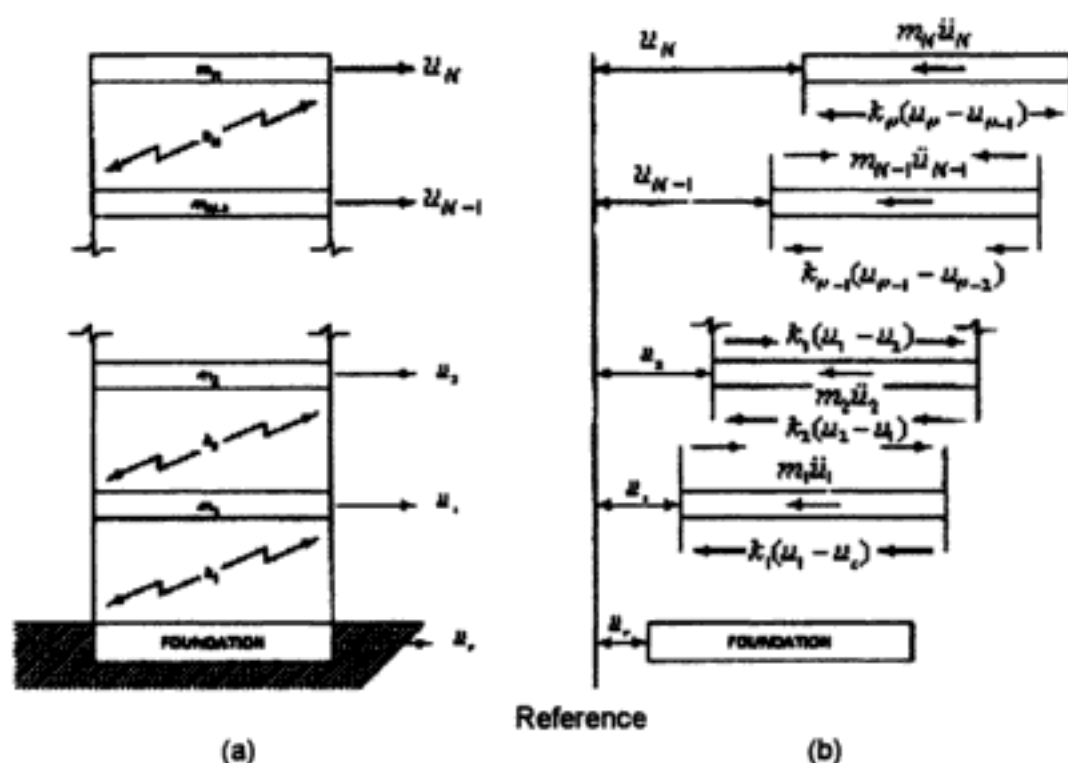


Fig. 8.3 Multistory shear building excited at the foundation
(a) Structural Model. (b) Free body diagrams.

The equations of motion for an N -story shear building [Fig. 8.3(a)] subjected to excitation motion at its base are obtained by equating to zero the sum of forces shown in the free body diagrams of Fig. 8.3(b), namely

$$\begin{aligned} m_1 \ddot{u}_1 + k_1(u_1 - u_s) - k_2(u_2 - u_1) &= 0 \\ m_2 \ddot{u}_2 + k_2(u_2 - u_1) - k_3(u_3 - u_2) &= 0 \\ &\dots \\ m_{N-1} \ddot{u}_{N-1} + k_{N-1}(u_{N-1} - u_{N-2}) - k_N(u_N - u_{N-1}) &= 0 \\ m_N \ddot{u}_N + k_N(u_N - u_{N-1}) &= 0 \end{aligned} \quad (8.32)$$

Introducing into eq. (8.32)

$$u_{ri} = u_i - u_s \quad (i = 1, 2, \dots, N) \quad (8.33)$$

results in

$$\begin{aligned} m_1 \ddot{u}_{r1} + k_1 u_{r1} - k_2(u_{r2} - u_{r1}) &= -m_1 \ddot{u}_s \\ m_2 \ddot{u}_{r2} + k_2 u_{r2} - k_3(u_{r3} - u_{r2}) &= -m_2 \ddot{u}_s \\ &\dots \\ m_{N-1} \ddot{u}_{rN-1} + k_{N-1}(u_{rN-1} - u_{rN-2}) - k_N(u_{rN} - u_{rN-1}) &= -m_{N-1} \ddot{u}_s \\ m_N \ddot{u}_{rN} + k_N(u_{rN} - u_{rN-1}) &= -m_N \ddot{u}_s \end{aligned} \quad (8.34)$$

where $\ddot{u}_s = \ddot{u}_s(t)$ is the acceleration function exciting the base of the structure.

Equations (8.34) may be conveniently be written in matrix notation as

$$[M]\{\ddot{u}_r\} + [K]\{u_r\} = -[M]\{1\}\ddot{u}_s(t) \quad (8.35)$$

in which $[M]$, the mass matrix, is a symmetric matrix, $\{1\}$ is a vector with all its elements equal to 1, $\ddot{u}_s = \ddot{u}_s(t)$ is the applied acceleration at the foundation of the building, and $\{u_r\}$ and $\{\ddot{u}_r\}$ are, respectively, the displacement and acceleration vectors relative to the motion of the foundation.

As has been demonstrated, the system of differential equations (8.35) can be uncoupled through the transformation given by eq.(8.11) as

$$\{u_r\} = [\Phi]\{z\} \quad (8.36)$$

where $[\Phi]$ is the modal matrix obtained in the solution of corresponding eigenproblem $[[K] - \omega^2[M]]\{\phi\} = \{0\}$.

The substitution of eq.(8.36) into eq.(8.35) followed by premultiplication by the transpose of the i th eigenvector, $\{\phi\}_i^T$ (the i th modal shape), results in

$$\{\phi\}_i^T [M][\Phi]\{\ddot{z}\} + \{\phi\}_i^T [K][\Phi]\{z\} = -\{\phi\}_i^T [M]\{1\}\ddot{u}_s(t) \quad (8.37)$$

which upon introduction of orthogonality property of the normalized eigenvectors [eq.(8.14) and (8.15)] results in the modal equations

$$\ddot{z}_i + \omega_i^2 z_i = \Gamma_i \ddot{u}_s(t) \quad (i = 1, 2, \dots, N) \quad (8.38)$$

Where the modal participation factor Γ_i is given in general by

$$\Gamma_i = \frac{\sum_{j=1}^N m_j \phi_{ji}}{\sum_{j=1}^N m_j \phi_{ji}^2} \quad (8.39)$$

and for normalized eigenvectors by

$$\Gamma_i = \sum_{j=1}^N m_j \phi_{ji} \quad (i = 1, 2, \dots, N) \quad (8.40)$$

The maximum response in terms of maximum values for displacements ($u_{ri \max}$) or for acceleration ($\ddot{u}_{i \max}$) at the modal coordinates calculated by the SRSS method, is then given, respectively, by

$$u_{ri \max} = \sqrt{\sum_{j=1}^N (\Gamma_j \phi_{ij} S_{Dj})^2} \quad (8.41)$$

and

$$\ddot{u}_{i \max} = \sqrt{\sum_{j=1}^N (\Gamma_j \phi_{ij} S_{Aj})^2} \quad (8.42)$$

where S_{Dj} and S_{Aj} are, respectively, the spectral displacement and spectral acceleration for the j th mode.

The participation factors Γ_j indicated in eqs.(8.39) and (8.40) are the coefficients of the excitation function $\ddot{u}_s(t)$ in eq.(8.38). As presented in Chapter 5, response spectral charts are prepared as the solution of eq.(8.38) (with $\Gamma_i = 1$). Therefore, the spectral values obtained from these charts S_{Dj} or S_{Aj} should be multiplied as indicated in eqs.(8.41) and (8.42) by the participation factor T_j , which was omitted in the calculation of spectral values.

Illustrative Example 8.2

Determine the response of the frame of Illustrative Example 8.1 shown in Fig. 8.2 when it is subjected to a suddenly applied constant acceleration $\ddot{u}_s = 0.28 \text{ g}$ at its base.

Solution:

The natural frequencies and corresponding normal modes from calculations in Examples 7.1 and 7.2 are

$$\begin{aligned}\omega_1 &= 11.83 \text{ rad/sec}, & \omega_2 &= 32.89 \text{ rad/sec} \\ \phi_{11} &= 0.06437, & \phi_{12} &= 0.0567 \\ \phi_{21} &= 0.08130, & \phi_{22} &= -0.0924\end{aligned}$$

The acceleration acting at the base of this structure is

$$\ddot{u}_s = 0.28 \times 386 = 108.47 \text{ in/sec}^2$$

The participation factors are calculated from eqs.(8.39) with the denominators set equal to unity since the modes are normalized. These factors are then

$$\begin{aligned}\Gamma_1 &= -(136 \times 0.06437 + 66 \times 0.08130) = -14.120 \\ \Gamma_2 &= -(136 \times 0.0567 - 66 \times 0.0924) = -1.613\end{aligned}\tag{a}$$

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The modal equations (8.29) are

$$\begin{aligned}\ddot{g}_1 + 140g_1 &= 108.47 \\ \ddot{g}_2 + 1082g_2 &= 108.47\end{aligned}\tag{b}$$

and their solution, assuming zero initial conditions for velocity and displacement, is given by eqs.(4.5) as

$$\begin{aligned}g_1(t) &= \frac{108.47}{140}(1 - \cos 11.83t) \\ g_2(t) &= \frac{108.47}{1082}(1 - \cos 32.89t)\end{aligned}\tag{c}$$

The response in terms of the relative motion of the stories at the floor levels with respect to the displacement of the base is given as a function of time by eq.(8.27) and (8.28), as

$$\begin{aligned}u_{r1}(t) &= -14.120 \times 0.06437 \times 0.775(1 - \cos 11.83t) - 1.613 \times 0.0567 \times 0.100(1 - \cos 32.89t) \\ u_{r2}(t) &= -14.120 \times 0.08130 \times 0.775(1 - \cos 11.83t) + 1.613 \times 0.0924 \times 0.100(1 - \cos 32.89t)\end{aligned}$$

or, upon simplification, as

$$\begin{aligned}u_{r1} &= -0.7135 + 0.704 \cos 11.83t + 0.009 \cos 32.89t \\u_{r2} &= -0.874 + 0.900 \cos 11.83t - 0.015 \cos 32.89t\end{aligned}\tag{d}$$

In this example, due to the simple excitation function (a constant acceleration), it was possible to obtain a closed solution of the problem as a function of time. For a complex excitation function such as the one produced by an actual earthquake, it would be necessary to resort to numerical integration to obtain the response or to use response spectra if available. The maximum modal response is obtained for the present example when the cosine functions in eqs.(c) are set equal to -1 . In this case the maximum modal response is then

$$\begin{aligned}g_{1\max} &= 1.55 \\g_{2\max} &= 0.20\end{aligned}\tag{e}$$

and the maximum response, calculated from the approximate formulas (8.31), is

$$\begin{aligned}u_{r1\max} &= 1.409 \text{ in} \\u_{r2\max} &= 1.800 \text{ in}\end{aligned}\tag{f}$$

The possible maximum values for the response calculated from eqs. (d) by setting the cosines function to their maximum value results in

$$\begin{aligned}u_{1\max} &= 1.426 \text{ in} \\u_{2\max} &= 1.789 \text{ in}\end{aligned}\tag{g}$$

which for this particular example certainly compares very well with the approximate results obtained in eqs.(f) above.

8.3 Program 9-Response by Modal Superposition

Program 9 calculates the response of a linear system by superposition of the solutions of the modal equations. Before one can use this program, it is necessary to solve an eigenproblem to determine the natural frequencies and modal shapes of the structure. The program determines the response of the structure excited either by time-dependent forces applied at nodal coordinates or a time-dependent acceleration at the support of the structure.

Illustrative Example 8.3.

Use computer Program 9 to find the response of the frame analyzed in Illustrative Example 8.2.

Solution:

The natural frequencies and the modal matrix for this structure as calculated in Example 10.1 are

$$\begin{aligned}\omega_1 &= 11.83 \text{ rad/sec} \\ \omega_2 &= 32.89 \text{ rad/sec}\end{aligned}$$

and

$$[\Phi] = \begin{bmatrix} 0.0644 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix}$$

The mass matrix is

$$[M] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}$$

Previous to the execution of Program 9, it is necessary to execute one of the programs to model the structure (for example, Program 7 to model the structure as a shear building) followed by the execution of Program 8 to solve the eigenproblem. The execution of these programs will prepare the file required to execute Program 9. Alternately, when the eigensolution is known, we would execute the auxiliary Program X2 to prepare the required file. Since the eigensolution in the present example is known, we proceed to execute Program X2:

Input Data and Output Results

PROGRAM X2: EIGENSOLUTION DATA

DATA FILE: DX2

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM
EIGENVALUES:

ND = 2

0.140E+03 0.108E+04

MODE SHAPES BY ROWS:

0.6437 0.0813
0.05670 -0.0924

PROGRAM 9: MODAL SUPERPOSITION

DATA FILE: D9

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM

ND=2

NUMBER OF EXTERNAL FORCES

NF=1

TIME STEP OF INTEGRATION

H= 0.05

GRAVITATIONAL INDEX

G=386

PRINT TIME HISTORY NPRT=1; ONLY MAX. VALUES NPRT=0

NPRT=0

FORCE #, COORD. # OF APPLIED. NUM. OF POINTS DEFINING THE FORCE

1 0 2

EXCITATION FUNCTION FOR FORCE # 1:

TIME EXCITATION

0.00 0.280
1.00 0.280

FLOOR MASSES OF THE SHEAR BUILDING:

136 66

MODAL DAMPING RATIOS:

0 0

246 Program 9-Response by Modal Superposition

MAXIMUM RESPONSE:

COORD	MAX. DISPL.	MAX. VELOC.	MAX. ACC.
1	1.405	8.3.94	212.48
2	1.728	10.650	239.71

Illustrative Example 8.4.

Solve Illustrative Example 8.1 using Program 9.

Solution:

Input Data and Output Results

PROGRAM 9: MODAL

DATA FILE: d9b

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM	ND=2
NUMBER OF EXTERNAL FORCES	NF=2
TIME STEP OF INTEGRATION	H=.01
GRAVITATIONAL INDEX	G=0
PRINT TIME HISTORY NPRT=1; ONLY MAX. VALUES NPRT=0	NPRT=1

FORCE #, COORD. #, WHERE FORCE IS APPLIED, NUM. OF POINTS DEFINING THE FORCE

1	1	3
2	2	3

EXCITATION FUNCTION FOR FORCE # 1 :

TIME	EXCITATION
0.0000	10000.0000
0.1000	0.0000
0.4000	0.0000

EXCITATION FUNCTION FOR FORCE # 2 :

TIME	EXCITATION
0.0000	20000.0000
0.1000	0.0000
0.4	0.0000

MODAL DAMPING RATIOS:

0	0
1	

MAXIMUM RESPONSE:

COORD.	MAX.DISPL.	MAX. VELOC.	MAX.ACC.
1	0.67328	7.6438	159.4700
2	0.75652	12.9851	303.0303

8.4 Harmonic Force Excitation

When the excitation, that is, the external forces or base motion, is harmonic (sine or cosine function), the analysis is quite simple and the response can readily be found without the use of modal analysis. Let us consider the two-story shear building as

shown in Fig. 8.4 subjected to a single harmonic force $F = F_0 \sin \bar{\omega}t$ which is applied at the level of the second floor. In this case, eqs. (8.1) with $F_1(t) = 0$ and $F_2 = F_0 \sin \bar{\omega}t$ become

$$\begin{aligned} m_1 \ddot{u}_1 + (k_1 + k_2)u_1 - k_2 u_2 &= 0 \\ m_2 \ddot{u}_2 - k_2 u_1 + k_2 u_2 &= F_0 \sin \bar{\omega}t \end{aligned} \quad (8.43)$$

For the steady-state response we seek a solution of the form

$$\begin{aligned} u_1 &= U_1 \sin \bar{\omega}t \\ u_2 &= U_2 \sin \bar{\omega}t \end{aligned} \quad (8.44)$$

After substitution of eqs.(8.44) into eqs.(8.43) and cancellation of the common factor $\sin \bar{\omega}t$, we obtain

$$\begin{aligned} (k_1 + k_2 - m_1 \bar{\omega}^2)U_1 - k_2 U_2 &= 0 \\ -k_2 U_1 + (k_2 - m_2 \bar{\omega}^2)U_2 &= F_0 \end{aligned} \quad (8.45)$$

which is a system of two equation in two unknowns, U_1 , and U_2 . This system always has a unique solution except in the case when the determinant formed by the coefficients of the unknowns is equal to zero. The reader should remember that in this case the forced frequency $\bar{\omega}$ would equal one of the natural frequencies, since this determinant when equated to zero is precisely the condition used for determining the natural frequencies. In other words, unless the structure is forced to vibrate at one of the resonant frequencies, the algebraic system of eqs.(8.43) has a unique solution for U_1 , and U_2 .

Illustrative Example 8.5

Determine the steady-state response of the two-story shear building of Illustrative Example 7.1 when a force $F_2(t) = 10,000 \sin 20t$ is applied to the second story as shown in Fig. 8.4.

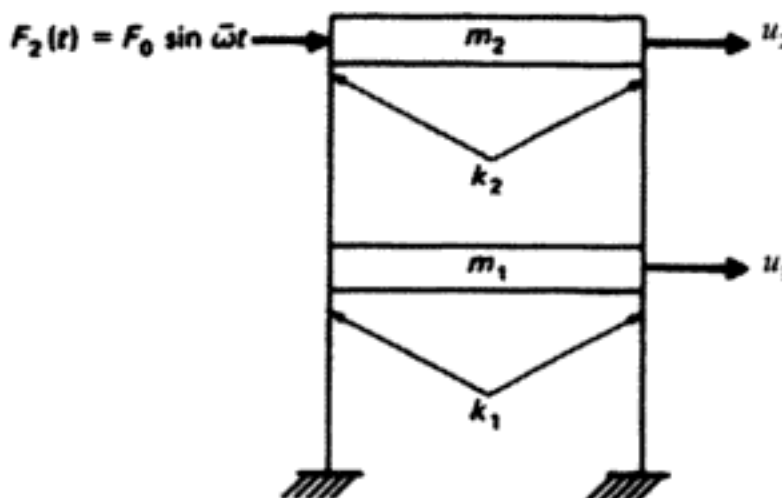


Fig. 8.4 Shear building with harmonic load.

Solution:

The natural frequencies for this frame were determined in Illustrative Example 7.1 to be

$$\begin{aligned}\omega_1 &= 11.83 \text{ rad/sec} \\ \omega_2 &= 32.89 \text{ rad/sec}\end{aligned}$$

Since the forcing frequency is 20 rad/sec, the system is not at resonance. The steady-state response is then given by solving eqs.(8.45) for U_1 and U_2 . substituting numerical values in this system of equations, we have

$$\begin{aligned}(75,000 - 136 \times 20^2)U_1 - 44,300U_2 &= 0 \\ -44,300U_1 + (44,300 - 66 \times 20^2)U_2 &= 10,000\end{aligned}$$

Solving these equations simultaneously results in

$$U_1 = -0.28 \text{ in}, \quad U_2 = -0.13 \text{ in}$$

Therefore, according to eqs.(8.44), the steady-state response is

$$\begin{aligned}u_1 &= -0.28 \sin 20t \text{ in} \\ u_2 &= -0.13 \sin 20t \text{ in}\end{aligned} \quad \text{(Ans.)}$$

b

Damping may be considered in the analysis by simply including damping elements in the model as it is shown in Fig. 8.5 for a two-story shear building. The equations of motion which are obtained by equating to zero the sum of the forces in the free body diagram shown in Fig. 8.5 (c) are

$$\begin{aligned}m_1 \ddot{u}_1 + (c_1 + c_2) \dot{u}_1 + (k_1 + k_2)u_1 - c_2 \dot{u}_2 - k_2 u_2 &= F_1(t) \\ m_2 \ddot{u}_2 - c_2 \dot{u}_1 - k_2 u_1 + c_2 \dot{u}_2 + k_2 u_2 &= F_2(t)\end{aligned} \quad (8.46)$$

Now, considering the general case of applied forces of the form given by

$$F(t) = F_c \cos \bar{\omega}t + F_s \sin \bar{\omega}t \quad (8.47)$$

we, conveniently, express such force in complex form as

$$F(t) = (F_c - iF_s)e^{i\bar{\omega}t} \quad (8.48)$$

with the tacit understanding that only the real part of the eq.(8.48) is the applied force. We show that the real part of the complex force in eq.(8.48) is precisely the force in eq.(8.47). Using Euler's formula $e^{i\bar{\omega}t} = \cos \bar{\omega}t + i \sin \bar{\omega}t$, we obtain

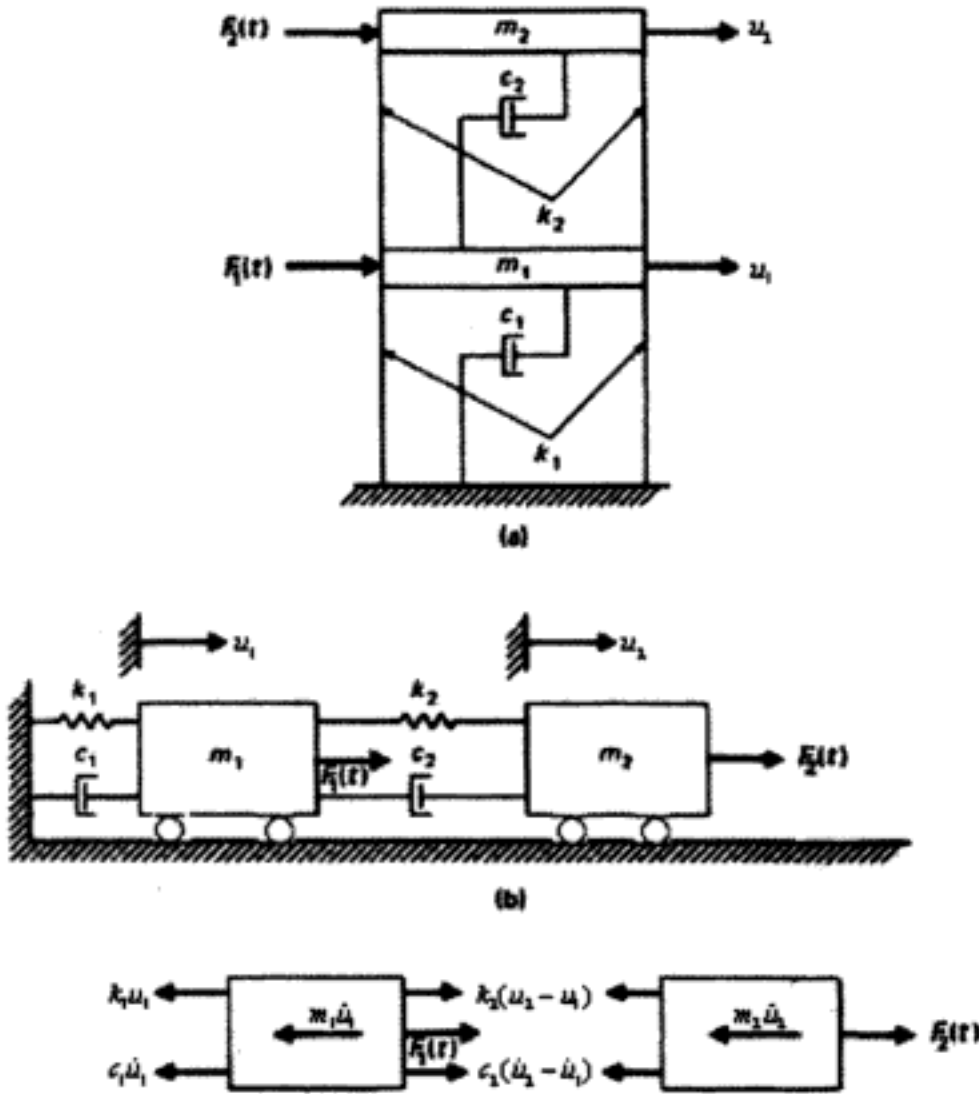


Fig. 8.5 (a) Damped shear building with harmonic load.
(b) Multi-degree mass-spring model. (c) Free body diagrams.

$$\begin{aligned} \text{Real} \{ (F_c - iF_s) e^{i\bar{\omega}t} \} &= \text{Real} \{ (F_c - iF_s) (\cos \bar{\omega}t + i \sin \bar{\omega}t) \} \\ &= F_c \cos \bar{\omega}t + F_s \sin \bar{\omega}t \end{aligned} \quad (8.49)$$

which is equal to the expression in eq.(8.47).

Assuming that the forces $F_1(t)$ and $F_2(t)$ in eq.(8.46) are in the form given by eq.(8.47), we substitute eq.(8.48) into eq.(8.46) to obtain

$$\begin{aligned} m_1 \ddot{u}_1 + (c_1 + c_2) \dot{u}_1 + (k_1 + k_2) u_1 - c_2 \dot{u}_2 - k_2 u_2 &= (F_{c1} - iF_{s1}) e^{i\bar{\omega}t} \\ m_2 \ddot{u}_2 - c_2 \dot{u}_1 - k_2 u_1 + c_2 \dot{u}_2 + k_2 u_2 &= (F_{c2} - iF_{s2}) e^{i\bar{\omega}t} \end{aligned} \quad (8.50)$$

The solution of the complex system of eqs.(8.50) will, in general, be of the form of

$$\begin{aligned} u_1(t) &= (U_{c1} + iU_{s1}) \cdot e^{i\bar{\omega}t} \\ u_2(t) &= (U_{c2} + iU_{s2}) e^{i\bar{\omega}t} \end{aligned} \quad (8.51)$$

The substitution of eqs.(8.51) together with the first and second derivatives of u_1 and u_2 into eq.(8.50) results in the following system of complex algebraic equations:

$$\begin{aligned} \{(k_1+k_2 - m \bar{\omega}^2)+i \bar{\omega} (c_1 + c_2)\} (U_{c1} + iU_{s1}) - \{(k_2 + i \bar{\omega} c_2)(U_{c2} + iU_{s2}) = F_{c1}-F_{s1} \\ -(k_2 + i \bar{\omega} c_2)(U_{c1} + iU_{s1}) + \{(k_2 - m_2 \bar{\omega}^2)+i \bar{\omega} c_2\}(U_{c2} + iU_{s2}) = F_{c2}-F_{s2} \end{aligned} \quad (8.52)$$

As already stated, the response is then found by solving the complex system of equations (11.54) and retaining only the real part of the solution. Hence, analogously to eq.(8.49),

$$\begin{aligned} u_1(t) &= U_{c1} \cos \bar{\omega} t - U_{s1} \sin \bar{\omega} t \\ u_2(t) &= U_{c2} \cos \bar{\omega} t - U_{s2} \sin \bar{\omega} t \end{aligned} \quad (8.53)$$

in which U_{c1} , U_{s2} , U_{c2} , U_{s2} , is the solution of the complex equations (8.52). The necessary calculations are better explained through the use of a numerical example.

Illustrative Example 8.6

Determine the steady state response for the two-story shear building of Illustrative Example 8.5 in which damping is considered in the analysis (Fig. 8.5). Assume for this example that the damping constants c_1 and c_2 are, respectively, proportional to the magnitude of spring constants k_1 and k_2 in which the factor of proportionality, $a_0 = 0.01$.

Solution:

The damping constants are calculated as

$$\begin{aligned} c_1 &= a_0 k_1 = 307 \text{ lb.sec/in} \\ c_2 &= a_0 k_2 = 443 \end{aligned} \quad (a)$$

The substitution of numerical values for this example into eqs.(8.54) results in the following system of equations:

$$\begin{aligned} (20,600 + 15,000i)U_1 - (44,300 + 8860i)U_2 &= 0 \\ -(44,300 + 8860i)U_1 + (17,900 + 8860i)U_2 &= -10,000i \end{aligned} \quad (b)$$

The solution of this system of equations is

$$\begin{aligned} U_1 &= 0.000681 + i 0.26865 \\ U_2 &= -0.06390 + i 0.13777 \end{aligned}$$

The, it follows from eq. (8.55),

$$\begin{aligned} u_1(t) &= 0.0006814 \cos 20t - 0.26865 \sin 20t \\ u_2(t) &= -0.0639 \cos 20t - 0.137775 \sin 20t \end{aligned} \quad (8.54)$$

which may also be written as

$$\begin{aligned} u_1 &= 0.2686 \sin(20t + 3.144) \text{ in} \\ u_2 &= 0.1516 \sin(20t + 3.571) \text{ in} \end{aligned} \quad (\text{Ans.})$$

When the results are compared with those obtained for the undamped structure in Illustrative Example 8.5, we note only a small change in the amplitude of motion. This is always the case for systems lightly damped and subjected to harmonic excitation of a frequency that is not close to one of the natural frequencies of the system. For this example, the forced frequency $\omega = 20$ rad/sec is relatively far from the natural frequencies, $\omega_1 = 11.83$ rad/sec or $\omega_2 = 32.89$ rad/sec which were calculated in Illustrative Example 5.1.

8.5 Program 10 – Harmonic Response

Program 10 calculates the response to harmonic excitations of a structural system for which the stiffness and mass matrices have been determined by one of the programs modeling the structure (Program 7 to model the structure as a shear building).

Damping in the system is assumed to be proportional to the stiffness and /or mass coefficients, that is, the damping matrix is calculated as

$$[C] = a_0[M] + a_1[K] \quad (8.55)$$

in which a and a_1 are constants specified in the input data. The program calculates the steady-state response for structures subjected to harmonic forces applied at the nodal coordinates or a harmonic acceleration applied at the base of the structure.

Illustrative Example 8.7.

Obtain the response of the damped two-degree-of-freedom shear building of Illustrative Example 8.6 using computer Program 10.

Solution:

Input Data and Output Results

PROGRAM 10: HARMONIC RESPONSE

DATA FILE: D10

INPUT DATA

DAMPING STIFFNESS FACTOR

KFAC = .01

DAMPING MASS FACTOR

MFAC = 0

FORCED FREQUENCY (RAD/ SEC)

W = 20

SYSTEM STIFFNESS MATRIX

7.5000E+04

-4.4300E+04

-4.4300E+04

4.4300E+04

SYSTEM MASS MATRIX

1.3600E+02	0.0000E+00
0.0000E+00	6.6000E+01

FORCE COEFFICIENTS OF [FC* $\cos(W*T)$ + FI* $\sin(W*T)$] FOR EACH COORD.

COORD.	FC COMPONENT	FI COMPONENT
1	0	0
2	0	10000

THE STEADY-STATE SOLUTION IS

[FC* $\cos(W*T)$ + FI* $\sin(W*T)$] FOR EACH COORD.

COORD.	FC COMPONENT	FI COMPONENT
1	6.814E-04	-2.6865E-01
2	-6.309E-02	-1.3777E-01

As given by the output results of computer Program 10, the response for this two-degree-of-freedom system is

$y_1(t) = 0.00068 \cos 20t - 0.2686 \sin 20t$
 $y_2(t) = -0.0631 \cos 20t - 0.1378 \sin 20t$
or

$y_1(t) = 0.2686 \sin (20t + 3.144)$
 $y_2(t) = 0.1516 \sin (20t + 3.571)$

The results given by the computer, as expected, are the same as the values calculated in Example 8.6.

Illustrative Example 8.8.

For the structure modeled as a four-story shear building shown in Fig. 8.6, determine the steady-state response when subjected to the force $F(t) = 10,000 \sin 20t$ (lb) applied at the top floor of the building. Assume damping proportional to stiffness (factor of proportionality $a_1 = 0.01$. Modulus of elasticity $E = 30 \times 10^6$ psi.

Solution:

The solution of this problem is initiated by executing Program 7 to model the structure as a shear building followed by the execution of Program 10 to obtain the response to harmonic forces. Modeling of this structure has been undertaken in the solution of Example 9.1.

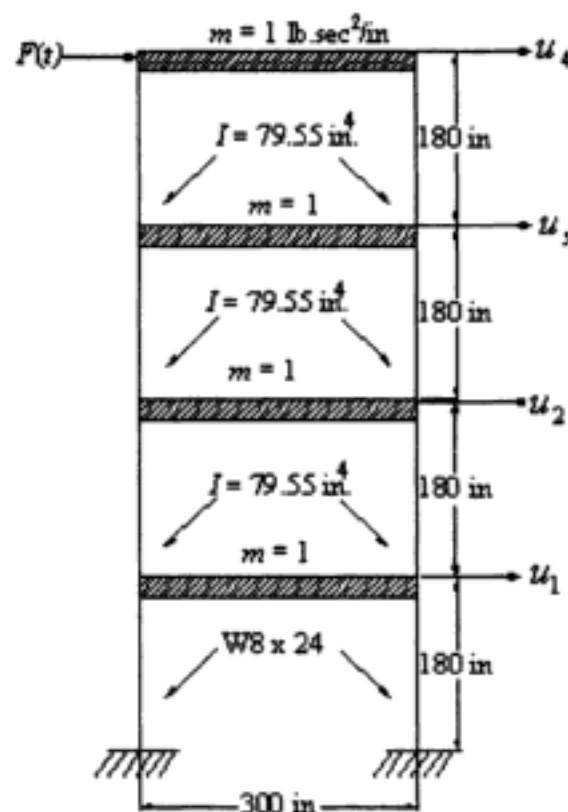


Fig. 8.6 Modeled four-story shear building of Illustrative Example 8.8.

Input Data and Output Results

PROGRAM 10: HARMONIC RESPONSE

DATA FILE: DIO.8

INPUT DATA:

DAMPING STIFFNESS FACTOR

KFA= .01

DAMPING MASS FACTOR

MFA= 0

FORCED FREQUENCY (RAD / SEC)

W =20

SYSTEM STIFFNESS MATRIX

6.5473E+02	-3.2737E+02	0.0000E+00	0.0000E+00
-3.2737E+02	6.5473E+02	-3.2737E+02	0.0000E+00
0.0000E+00	-3.2737E+02	6.5473E+02	-3.2737E+02
0.0000E+00	0.0000E+00	-3.2737E+02	3.2737E+02

SYSTEM MASS MATRIX

1.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
0.0000E+00	0.0000E+00	1.0000E+00	0.0000E+00
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00

FORCE COEFFICIENTS OF $[FC \cdot \cos(W \cdot T) = FI \cdot \sin(W \cdot T)]$ FOR EACH COORD.

COORD.	FC COMPONENT	FI COMPONENT
1	0	0
2	0	0
3	0	0
4	0	10000

THE STEADY-STATE SOLUTION IS

$$[FC \cdot \cos(W \cdot T) = FI \cdot \sin(W \cdot T)] \text{ FOR EACH COORD.}$$

COORD.	FC COMPONENT	FI COMPONENT
1	2.038E+01	2.6492E+01
2	2.304E+01	1.7070E+01
3	2.643E+00	-1.7821E+01
4	-2.505E+01	-3.2396E+01

8.6 Forced Motion Using SAP2000

Illustrative Example 8.9

Use SAP2000 to determine the maximum displacement at the floor levels of the three-story shear building [Fig. 8.7(a)] subjected to impulsive triangular loads shown in Fig. 8.7(b). The modulus of elasticity $E = 30,000$ ksi, the cross-sectional moment of inertia of the columns is $I = 1215$ in⁴, and the total mass at each floor level is $m = 0.3886$ lb.sec²/in. Neglect damping in the structure.

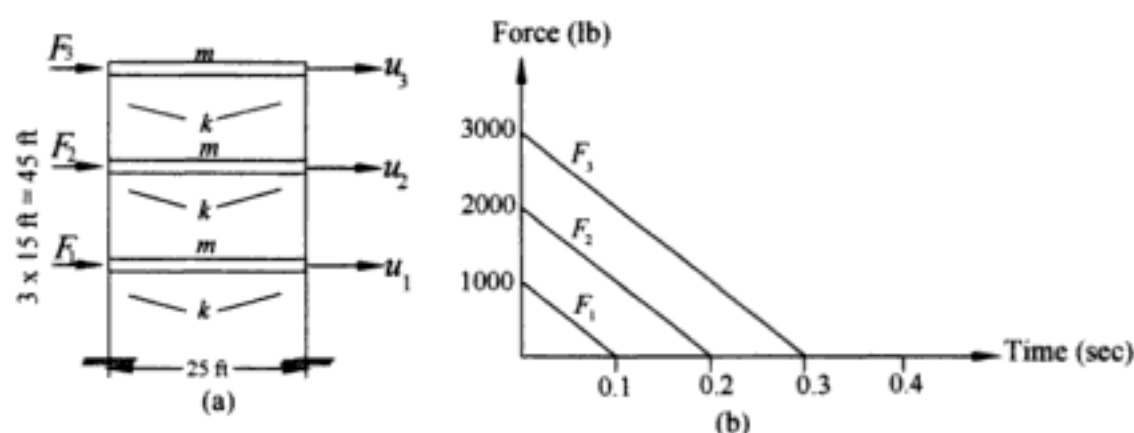


Fig. 8.7 (a) Shear Building model for Illustrative Example 8.9;
(b) Impulsive loading functions.

Solution:

The following commands are implemented in SAP2000:

Begin: Open SAP2000 (already installed)

Hint: Maximize both windows for a full view of the screen.

Select: In the lower right-hand corner of the screen use the drop-down menu to select "lb-in".

Select: From the Main Menu bar:
FILE>NEW MODEL FROM TEMPLATE
Click on Portal Frame

Enter:

Number of Stories = 3
 Number of Bays = 1
 Story Height = 180
 Bay width = 300. Then OK.

Edit: Click on XZ icon in the Menu Bar
 Maximize the XZ window and use the PAN icon in the toolbar to drag the figure to the center of the screen.

Label: From the Main Menu enter:
 VIEW>SET ELEMENTS
 Click on Joint Labels. Then OK.
 Use the PAN icon on the toolbar to center the figure on the screen.

Boundary: Select joints 1 and 5 by clicking on them and enter:
 ASSIGN>JOINT>RESTRAINTS
 Select restraints in all directions. Then OK.

Material: DEFINE > MATERIALS
 Select Material "OTHER"
 Modify/Show Property
 Enter:
 Modulus of Elasticity = 30000000
 Mass per Unit Volume = 0
 Weight per Unit Volume = 0. then OK.

Sections: DEFINE>FRAME SECTIONS
 Click on Add I/Wide Flange
 Then on "Add General" (Section)
 Enter:
 Moment of Inertia about 3 Axis = 1215
 Cross-sectional area=1.0E6
 Shear Area in 2 Direction = 0. Then OK.
 Select: Material "OTHER". Then OK.

Click on "Add General" (Section)
 Enter:
 Moment of Inertia about 3 axis = 1.0E6
 Shear Area in 2 Direction = 0. then OK.
 Select Material "OTHER". Then OK, OK.

Assign: Select all the columns by clicking on them
 Enter: ASSIGN>FRAME>SECTIONS
 Select: FSEC2. Then OK.
 Select all the horizontal members by clicking on them;
 Enter:
 ASSIGN>FRAME>SECTIONS
 Select: FSEC3. Then OK.

Assign Mass: Click on joints 2, 3 and 4 and enter:

ASSIGN>JOINT>MASSES

For direction 1 enter 0.3886. Then OK.

Load: DEFINE>STATIC LOAD CASES

Enter: Load = LOAD1

Type = LIVE

Self Weight Multiplier = 0

Click on Add New Load

Enter: Load = LOAD2

Click on Add New Load

Enter: Load = LOAD 3

Click on Add New Load

Then OK.

Functions: DEFINE>TIME HISTORY FUNCTIONS'

Click on Add New Function and accept default name FUNC1

Enter: Time = 0

Value = 1000 Then press ADD

Enter: Time = 0.1

Value = 0 Then press ADD

Enter: Time = 0.5

Value = 0 then press ADD, and OK.

Click on Add New Function and accept default name FUNC2

Enter: Time = 0

Value = 2000 Then press ADD

Enter: Time = 0.2

Value = 0 Then press ADD

Enter: Time = 0.5

Value = 0 Then press ADD and OK.

Click on Add New Function and accept default name FUNC3

Enter: Time = 0

Value = 3000 Then press ADD

Enter: Time = 0.3

Value = 0 Then press ADD

Enter: Time = 0.5

Value = 0 Then press ADD and OK.

Then OK, OK.

Load Case: DEFINE>TIME HISTORY CASES

Enter: Add New History

Enter: Analysis Type = Linear

Number of Output Time Steps = 50

Time Steps = 0.01.

Click on "Envelopes"

Load Assignments:

Enter: Load = LOAD1

Function = FUNC1. Then press ADD

Enter: Load = LOAD2
 Function = FUNC2. Then press ADD
 Enter: Load = LOAD3
 Function = FUNC3. Then press ADD
 Then OK, OK.

Assign Loads: Click on Joint 2 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES
 Select LOAD1 from the drop down menu
 Enter: Load, Force Global X = 1. Then OK.

Click on joint 3 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES
 Select LOAD2 from the drop-down menu
 Enter: Load, Force Global X = 1. Then OK.

Click on joint 4 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES
 Select LOAD3 from the drop-down menu
 Enter: Load, Force Global X = 1. Then OK.

Analyze: ANALYZE>SET OPTIONS

Check only UX in Available DOF's (Degrees of Freedom)
 Check Dynamic Analysis and Set Dynamic Parameters
 Enter: Number of Modes = 3.
 Then OK, OK.

ANALYZE>RUN

Enter filename: EXD 8.9. Then SAVE.

At the conclusion of the calculations, after checking for errors, press OK.
 Select X-Z view.

Output Table Mode: DISPLAY>SET OUTPUT TABLE MODE

Select LOAD1 Load Case, LOAD2 Load Case, LOAD3 Load Case and
 HIST1 History (hold down CTRL key to select all) then OK.

Print Input Tables: FILE>PRINT INPUT TABLES

CLICK "Print to File" and accept all other default values.
 Then OK.

(The edited Input Tables of Illustrative Example 8.9 are given in Table 8.1)

Table 8.1 Edited Input Data for Illustrative Example 8.9.

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	LIVE	0.0000
LOAD2	LIVE	0.0000
LOAD3	LIVE	0.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	TIME STEP INCREMENT
HIST1	LINEAR	50	0.01000

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-150.00000	0.00000	0.00000	1 1 1 1 1 1
2	-150.00000	0.00000	180.00000	0 0 0 0 0 0
3	150.00000	0.00000	360.00000	0 0 0 0 0 0
4	-150.00000	0.00000	540.00000	0 0 0 0 0 0
5	150.00000	0.00000	0.00000	1 1 1 1 1 1
6	150.00000	0.00000	180.00000	0 0 0 0 0 0
7	150.00000	0.00000	360.00000	0 0 0 0 0 0
8	150.00000	0.00000	540.00000	0 0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
2	0.389	0.000	0.000	0.000	0.000	0.000
3	0.389	0.000	0.000	0.000	0.000	0.000
4	0.389	0.000	0.000	0.000	0.000	0.000

FRAME ELEMENT DATA

FRAME	JNT-1	JNT-2	SECT	ANG	RELEASES	SEG	R1	R2	FACTOR	LENGTH
1	1	2	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
2	2	3	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
3	3	4	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
4	5	6	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
5	6	7	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
6	7	8	FSEC2	0.000	000000	2	0.000	0.000	1.000	180.000
7	2	6	FSEC3	0.000	000000	4	0.000	0.000	1.000	300.000
8	3	7	FSEC3	0.000	000000	4	0.000	0.000	1.000	300.000
9	4	8	FSEC3	0.000	000000	4	0.000	0.000	1.000	300.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
2	1.000	0.000	0.000	0.000	0.000	0.000

JOINT FORCES Load Case LOAD2

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
3	1.000	0.000	0.000	0.000	0.000	0.000

JOINT FORCES Load Case LOAD3

JOINT	GLBL-X	GLBL-Y	GLBL-Z	GLBL-XX	GLBL-YY	GLBL-ZZ
2	1.000	0.000	0.000	0.000	0.000	0.000
4	1.000	0.000	0.000	0.000	0.000	0.000
3	1.000	0.000	0.000	0.000	0.000	0.000

Plot Displacement Function:

DISPLAY>SHOW TIME HISTORY TRACES

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 2

Vector Type = Displacement

Component = UX.

Mode Number = Include All. Then OK.

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 3

Vector Type = Displacement

Component = UX.

Mode Number = Include All. Then OK.

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 4

Vector Type = Displacement

Component = UX.

Mode Number = Include All. Then OK, OK.

Under List of Functions click once on Joint2,

Click on ADD to move Joint2 to Plot Functions column

Repeat these two steps for joints 3 and 4.

(All three functions will then appear on the Time History Traces Plot)

Axis Labels

Horizontal = Time (sec)

Vertical = Displacements Joints 2, 3 and 4 (in)

Click "Display" to view Displacement functions

Enter: FILE>PRINT GRAPHICS to obtain a printed version of these plots.

(Figure 8.8 represents the plot of the Displacement Function Values for Joints 2, 3 and 4 of Illustrative Example 8.9.)

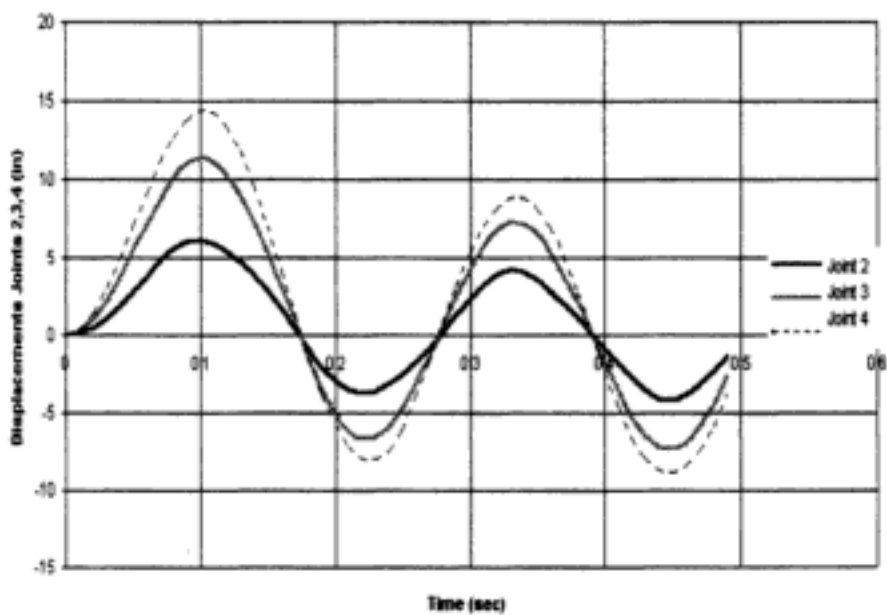


Fig. 8.8 Displacements for Joints 2, 3 and 4 of Illustrative Example 8.9.

Alternatively, a table of the displacement function may be obtained by entering:

FILE>PRINT TABLES TO FILE

The values in this table may be plotted using Excel or similar software.

(Table 8.2 contains the Displacement Function Values for Joints 2, 3 and 4 of Illustrative Example 8.9.)

Table 8.2 Displacement Function Values of Joints 2, 3 and 4 of Illustrative Example 8.9.

Time (sec)	Joint 2	Joint 3	Joint 4
0	0	0	0
0.01	0.12446	0.25299	0.37763
0.02	0.48273	0.99197	1.4467
0.03	1.05737	2.16921	3.05747
0.04	1.8315	3.69292	5.02009
0.05	2.76758	5.41934	7.141
0.48	-2.37705	-4.51742	-5.79848
0.49	-1.40215	-2.7831	-3.69537

Natural Periods and Frequencies:

Using Notepad or other word editor, open the following file:

C:/SAP2000e/EXD 8.9.OUT

The Natural Period and Frequency table may be found in this file.

(Table 8.2 contains the edited Natural Period and Frequency values of Illustrative Example 8.9)

Table 8.3 Natural Periods and Natural Frequencies for Illustrative Example 8.9.

PERIODS AND FREQUENCIES

MODE	PERIOD	FREQUENCY (CYC/SEC)	FREQUENCY (RAD/SEC)	EIGENVALUE (RAD/SEC)**2
1	0.227	4.401	27.650	764.522
2	0.081	12.330	77.474	6002.154
3	0.056	17.818	111.953	12533.375

Mode Shapes: DISPLAY>SHOW MODE SHAPE

Modal Shape 1 : Click "Wire Shadow" and "Cubic Curve"

Scale Factor = 1. Then OK.

(To view Modes 2 and 3 repeat previous entries)

* Numerical values for a mode shape in the Z direction and rotation around the Y axis may be read on the screen by right-clicking on a node in the nodal shape shown on the screen. Values for Modes 1, 2 and 3 are given in Table 8.4.

Table 8.4 Mode Shapes for the shear building of Illustrative Example 8.9.

Joint	Mode		
	1	2	3
	(T1=0.227 sec)	(T2=0.081 sec)	(T3=0.056 sec)
2	0.52614	1.18223	-0.94808
3	0.94808	0.52614	1.18223
4	1.18223	-0.94808	-0.52614

Illustrative Example 8.10

For the structure modeled as a four-story shear building shown in Fig.8.9, determine the steady-state response when subjected to the force $F(t) = 10,000 \sin 20t$ (lb) applied at the top floor of the building. Assume damping proportional to stiffness (Factor of proportionality $\alpha_1 = 0.01$, Modulus of Elasticity = $30E6$ psi).

Solution:

The following commands are implemented in SAP2000:

Begin: Open SAP2000 (already installed)**Hint:** Maximize both windows for a full view of the screen.

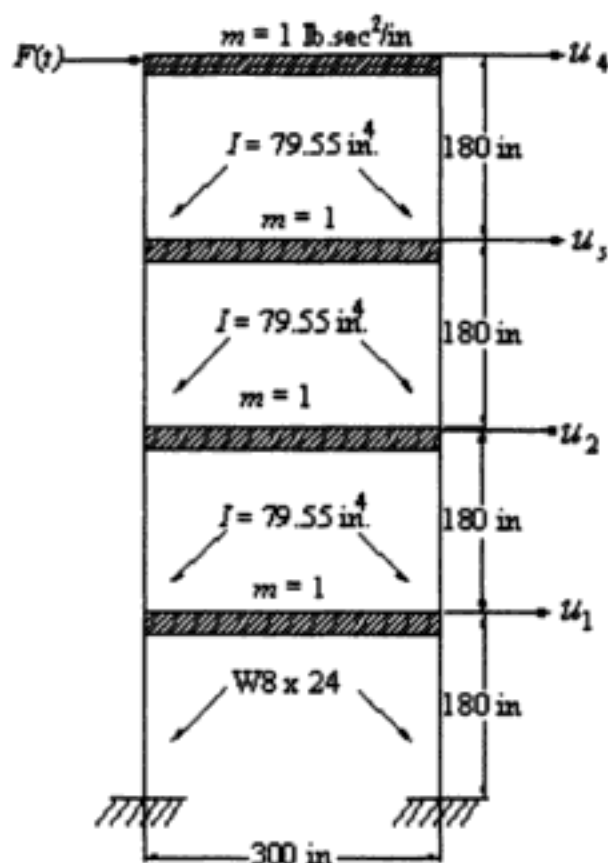


Fig. 8.9 Modeled four-story shear building of Illustrative Example 8.10.

Select: In the lower right-hand corner of the screen use the drop-down menu to select "lb-in".

Select: From the Main Menu bar:
 FILE>NEW MODEL FROM TEMPLATE
 Click on Portal Frame
 Enter:

Number of Stories = 4
 Number of Bays = 1
 Story Height = 180
 Bay width = 300. Then OK.

Edit: Click on XZ icon in the Menu Bar
 Maximize the XZ window and use the PAN icon in the toolbar to drag the figure to the center of the screen.

Label: From the Main Menu enter:
 VIEW>SET ELEMENTS
 Click on Joint Labels. Then OK.
 Use the PAN icon on the toolbar to center the figure on the screen.

Boundary: Select joints 1 and 6 by clicking on them and enter:
 ASSIGN>JOINT>RESTRAINTS
 Select restraints in all directions. Then OK.

Material: DEFINE > MATERIALS

Select Material "STEEL"

Modify/Show Property

Enter:

Modulus of Elasticity = 30000

Mass per Unit Volume = 0

Weight per Unit Volume = 0. then OK.

Sections: DEFINE > FRAME SECTIONS

Click on Import I/Wide Flange

In next window select "Sections" then OPEN

Then on "Add General" (Section)

Enter:

Moment of Inertia about 3 Axis = 1215

Shear Area in 2 Direction = 0. Then OK.

Select: Material "OTHER". Then OK.

Click on "Add General" (Section)

Enter:

Moment of Inertia about 3 axis = 1.0E6

Shear Area in 2 Direction = 0. then OK.

Select Material "OTHER". Then OK, OK.

Assign: Select all the columns by clicking on them

Enter: ASSIGN > FRAME > SECTIONS

Select: FSEC2. Then OK.

Select all the horizontal members by clicking on them;

Enter:

ASSIGN > FRAME > SECTIONS

Select: FSEC3. Then OK.

Assign Mass: Click on joints 2, 3 and 4 and enter:

ASSIGN > JOINT > MASSES

For direction 1 enter 0.3886. Then OK.

Load: DEFINE > STATIC LOAD CASES

Enter: Load = LOAD1

Type = LIVE

Self Weight Multiplier = 0

Click on Add New Load

Enter: Load = LOAD2

Click on Add New Load

Enter: Load = LOAD 3

Click on Add New Load

Then OK.

Functions: DEFINE > TIME HISTORY FUNCTIONS'

Click on Add New Function and accept default name FUNC1

Enter: Time = 0
Value = 1000 Then press ADD
Enter: Time = 0.1
Value = 0 Then press ADD
Enter: Time = 0.5
Value = 0 then press ADD, and OK.

Click on Add New Function and accept default name FUNC2

Enter: Time = 0
Value = 2000 Then press ADD
Enter: Time = 0.2
Value = 0 Then press ADD
Enter: Time = 0.5
Value = 0 Then press ADD and OK.

Click on Add New Function and accept default name FUNC3

Enter: Time = 0
Value = 3000 Then press ADD
Enter: Time = 0.3
Value = 0 Then press ADD
Enter: Time = 0.5
Value = 0 Then press ADD and OK.
Then OK, OK.

Load Case: DEFINE>TIME HISTORY CASES

Enter: Add New History
Enter: Analysis Type = Linear
Number of Output Time Steps = 50
Time Steps = 0.01.
Click on "Envelopes"
Load Assignments:
Enter: Load = LOAD1
Function = FUNC1. Then press ADD

Enter: Load = LOAD2
Function = FUNC2. Then press ADD
Enter: Load = LOAD3
Function = FUNC3. Then press ADD
Then OK, OK.

Assign Loads: Click on Joint 2 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES
Select LOAD1 from the drop down menu
Enter: Load, Force Global X = 1. Then OK.

Click on joint 3 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES
Select LOAD2 from the drop-down menu

Enter: Load, Force Global X = 1. Then OK.

Click on joint 4 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES

Select LOAD3 from the drop-down menu

Enter: Load, Force Global X = 1. Then OK.

Analyze: ANALYZE>SET OPTIONS

Check only UX in Available DOF's (Degrees of Freedom)

Check Dynamic Analysis and Set Dynamic Parameters

Enter: Number of Modes = 3.

Then OK, OK.

ANALYZE>RUN

Enter filename: EXD 8.10. Then SAVE.

At the conclusion of the calculations, after checking for errors, press OK.

Select X-Z view.

Output Table Mode: DISPLAY>SET OUTPUT TABLE MODE

Select LOAD1 Load Case, LOAD2 Load Case, LOAD3 Load Case

8.7 Combining Maximum Values of Modal Response

The square root of the sum of squared contributions (SRSS), to estimate the total response from calculated maximum modal values, may be expressed, in general, from eq. (11.41) or eq. (11.42), as

$$R = \sqrt{\sum_{i=1}^N R_i^2} \quad (8.56)$$

where R is the estimated response (force, displacement, etc.) at a specified coordinate and R_i is the corresponding maximum response of the i th mode at that coordinate.

Application of the SRSS method for combining modal response generally provides an acceptable estimation of the total maximum response. However, when some of the modes are closely spaced, the use of the SRSS method may result in grossly underestimating or overestimating the maximum response. In particular, large errors have been found in the analysis of three-dimensional structures in which torsional effects are significant. The term "closely spaced" referring to modes, may be arbitrarily define the case when the difference between two natural frequencies is within 10% of the smallest of the two frequencies.

A formulation known as the Complete Quadratic Combinations (CQC), which is based on the theory of random vibrations, has been proposed by Kiureghian (1980) and by Wilson, et.al. (1981). The CQC method, which may be considered as an extension of the SRSS method, is given by the following equation:

$$R = \sqrt{\sum_{i=1}^N \sum_{j=1}^N R_i \rho_{ij} R_j} \quad (8.57)$$

in which the cross-modal coefficient ρ_{ij} may be approximated by

$$\rho_{ij} = \frac{8(\xi_i \xi_j)^{1/2} (\xi_i + r \xi_j) r^{3/2}}{(1-r^2)^2 + 4\xi_i \xi_j r(1-r^2) + 4(\xi_i^2 + \xi_j^2) r^2} \quad (8.58)$$

where $r = \omega_j / \omega_i$ is the ratio of the natural frequencies of order i and j and ξ_i and ξ_j the corresponding damping ratios for modes i and j . For constant modal damping ξ , eq. (11.58) reduces to

$$\rho_{ij} = \frac{8\xi^2(1-r)r^{3/2}}{(1-r^2)^2 + 4\xi^2 r(1+r)^2} \quad (8.59)$$

It is important to note that, for $i = j$, eq. (11.58) or eq. (11.59) yields $\rho_{ij} = 1$ for any value of the damping ratio, including $\xi = 0$. Thus, for an undamped structure, the CQC method [eq. (11.57)] is identical to the SRSS method [eq. (11.56)].

8.8 Summary

For the solution of linear equations of motion, we may employ either the modal superposition method of dynamic analysis or a step-by-step numerical integration procedure. The modal superposition method is restricted to the analysis of structures governed by linear systems of equations whereas the step-by-step methods of numerical integration are equally applicable to systems with linear or nonlinear behavior. We have deferred the presentation of the multi-degree-of-freedom systems.

In the present chapter, we have introduced the modal superposition method in obtaining the responses of the shear building subjected to either force excitation or to base motion and have demonstrated that the use of normal modes of free vibration for transforming the coordinates leads to a set of uncoupled differential equations. The solution of these equations may then be obtained by any of the methods presented in Part I for the single-degree-of-freedom system.

When use is made of response spectra to determine maximum values for modal responses, these values are usually combined by the square root of the sum of squares (SRSS) method. However, the SRSS method could seriously overestimate or underestimate the total response when some of the natural frequencies are closely spaced. A more precise method of combining maximum values of the modal response is the Complete Quadratic Combination (CQC). This method has been strongly recommended in lieu of the SRSS method.

In the particular case of harmonic excitation, the response may be obtained in closed form by simply solving a system of algebraic equations in which the unknowns are the amplitudes of the response at the various coordinates.

8.9 Problems

Problem 8.1

Determine the response as a function of time for the two-story shear building of Problem 7.1 when a constant force of 5000 lb is suddenly applied at the level of the second floor as shown in Fig. P8.1. Bays are 15 ft. apart.

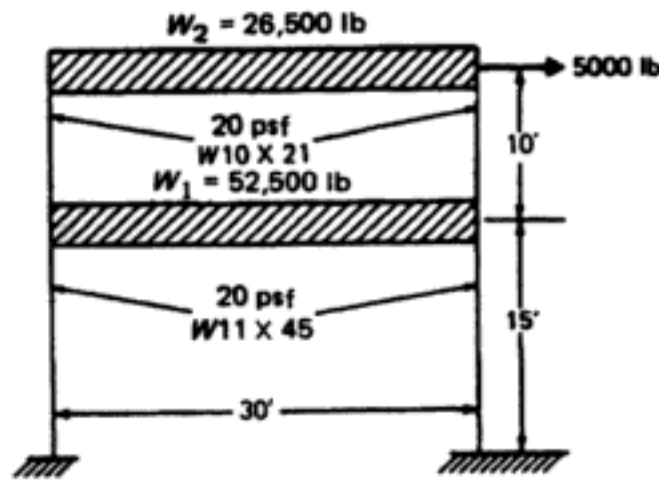


Fig. P8.1.

Problem 8.2

Repeat Problem 8.1 if the excitation is applied to the base of the structure in the form of a suddenly applies acceleration of magnitude 0.5 g.

Problem 8.3

Determine the maximum displacement at the floor levels of the three-story shear building [Fig P8.3(a)] subjected to impulsive triangular loads as shown in Fig. P8.3(b). The total stiffness of the columns of each story is $k=1500$ lb/in and the mass at each floor is $m=0.386$ lb. sec²/in.

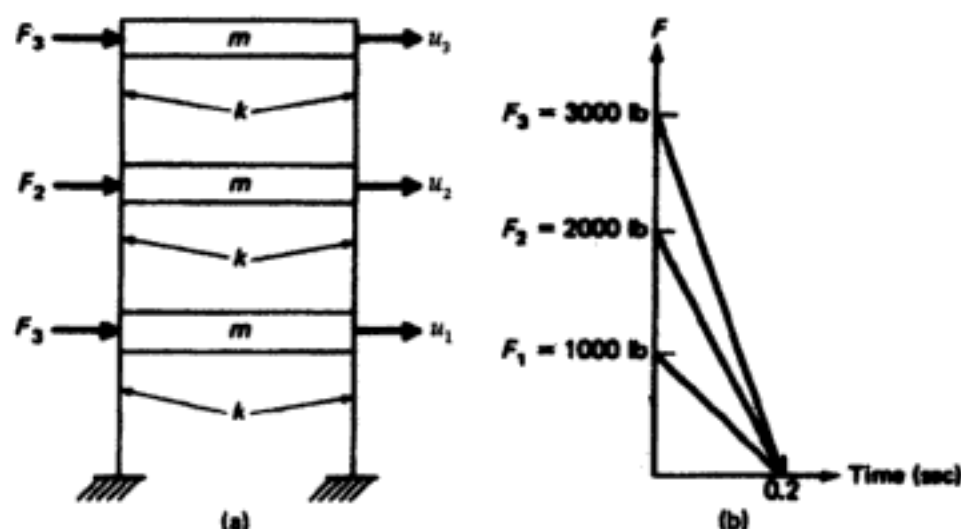


Fig. P8.3.

Problem 8.4

Determine the maximum shear force in the columns of the second story of Problem 8.3. (Hint: Calculate modal shear forces and combine contributions using method of square root sum of squares.)

Problem 8.5

Use SAP 2000 to obtain the time history response of the three-story building in Fig. P8.5(a) subjected to the support acceleration plotted in Fig. P8.5(b). Determine the response for a total time of 1.0 sec using time step $\Delta t = 0.05$ sec, and modal damping coefficient of 10% for all the modes ($E = 30 \times 10^6$ psi).

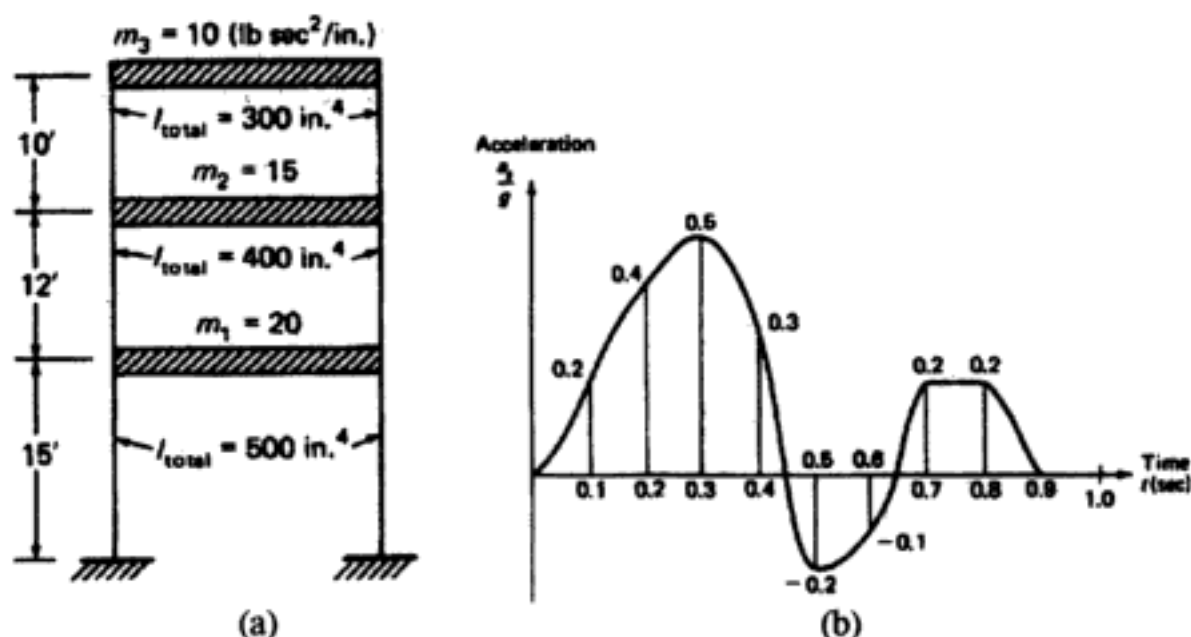


Fig. P8.5.

Problem 8.6

Find the steady-state response of the shear building shown in Fig. P8.6 subjected to the harmonic forces indicated in the figure. Neglect damping.

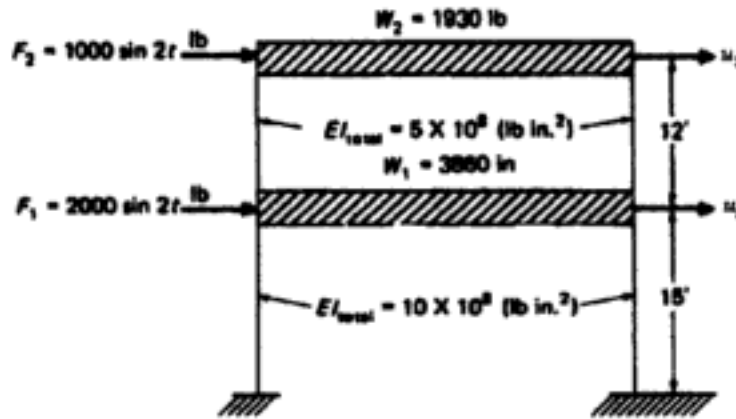


Fig. P8.6.

Problem 8.7

Solve Problem 8.6 assuming damping coefficients proportional to the story stiffness, $c_i = 0.05 K_i$.

Problem 8.8

For the structure (shear building) shown in Fig. P8.8 determine the steady-state motion for the following load systems (loads in pounds):

- (a) $F_1(t) = 1000 \sin t$, $F_2(t) = 2000 \sin t$, $F_3(t) = 1500 \sin t$
 (b) $F_1(t) = 2000 \cos t$, $F_2(t) = 3000 \cos t$, $F_3(t) = 4000 \cos t$

Also load the structure simultaneously with load systems (a) and (b) and verify the superposition of results.

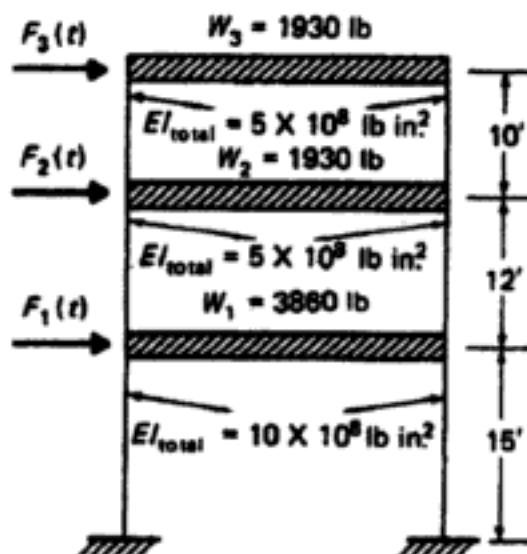


Fig. P8.8

Problem 8.9

For the structure modeled as a four-story shear building shown in Fig. P8.9 determine the steady-state response when it is subjected to a force $F = 10000 \sin 20t$ (lb) applied at the top floor of the building. The modulus of elasticity is $E = 2.0 \times 10^6$ psi. Assume damping in the system is proportional to the stiffness coefficient ($c_0 = 0.01$)

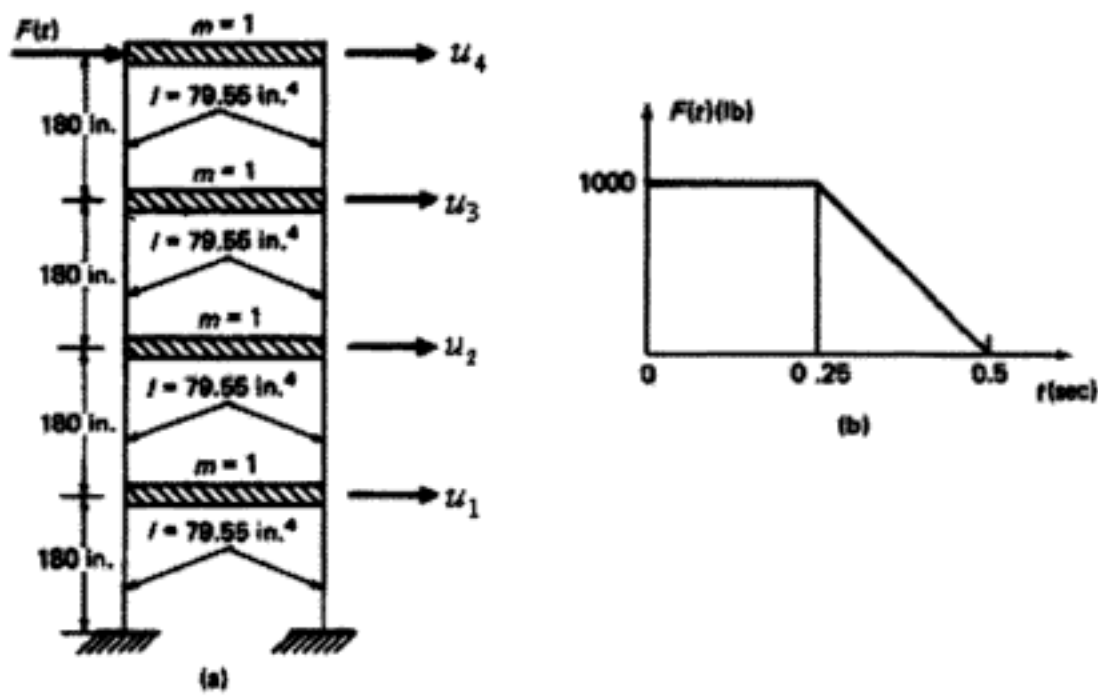


Fig. P8.9

9 Reduction of Dynamic Matrices

In the discretization process it is sometimes necessary to divide a structure into a large number of elements because of changes in geometry, loading, or material properties. When the elements are assembled for the entire structure, the number of unknown displacements, that is, the number of degrees of freedom, may be quite large. As a consequence, the stiffness, mass, and damping matrices will be of large dimensions. The solution of the corresponding eigenproblem to determine natural frequencies and modal shapes will be difficult and, in addition, expensive. In such cases it is desirable to reduce the size of these matrices in order to make the solution of the eigenproblem more manageable and economical. Such reduction is referred to as condensation.

A popular method of reduction is the Static Condensation Method. This method, though simple to apply, is only approximate and may produce relatively large errors in the results when applied to dynamic problems. An improved method for condensation of dynamic problems, which gives virtually exact results, has recently been proposed. This method, called the Dynamic Condensation Method, will be presented in this chapter after the introduction of the Static Condensation Method.

9.1 Static Condensation

A practical method of accomplishing the reduction of the stiffness matrix is to identify those degrees of freedom to be condensed as dependent or secondary degrees of freedom, and express them in the term of the remaining independent or primary degrees of freedom. The relationship between the secondary or primary degrees of freedom is found by establishing the static relation between them, hence the name Static Condensation Method (Guyan, R.J., 1965). This relationship provides the

means to reduce the stiffness matrix. This method is also used in the static problems to eliminate unwanted degrees of freedom such as the internal degrees of freedom of an element used with the Finite Element Method. In order to describe the Static Condensation Method, let us assume that those (secondary) degrees of freedom to be reduced or condensed are arranged as the first s coordinates, and the remaining (primary) degrees of freedom are the last p coordinates. With this arrangement, the stiffness equation for the structure may be written using partition matrices as

$$\begin{bmatrix} [K_{ss}] & [K_{sp}] \\ [K_{ps}] & [K_{pp}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \{u_p\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{F_p\} \end{Bmatrix} \quad (9.1)$$

where $\{u_s\}$ is the displacement vector corresponding to the s degrees of freedom to be reduced and $\{u_p\}$ is the vector corresponding to the remaining p independent degrees of freedom. In eq. (8.1), it is assumed that the external forces were zero at the dependent (i.e., secondary) degrees of freedom; this assumption is not mandatory (Gallagher, R.H., 1975), but serves to simplify explanations without affecting the final results. A simple multiplication of the matrices on the left side of eq. (9.1) expands this equation into two matrix equations, namely,

$$[K_{ss}]\{u_s\} + [K_{sp}]\{u_p\} = \{0\} \quad (9.2)$$

$$[K_{ps}]\{u_s\} + [K_{pp}]\{u_p\} = \{F_p\} \quad (9.3)$$

Equation (9.2) is equivalent to

$$\{u_s\} = [\bar{T}]\{u_p\} \quad (9.4)$$

where $[\bar{T}]$ is the transformation matrix given by

$$[\bar{T}] = -[K_{ss}]^{-1}[K_{sp}] \quad (9.5)$$

Substituting eq. (9.4) and using eq. (9.5) in eq. (9.3) results in the reduced stiffness equation relating forces and displacements at the primary coordinates, that is,

$$[\bar{K}]\{u_p\} = \{F_p\} \quad (9.6)$$

where $[\bar{K}]$ is the reduced stiffness matrix given by

$$[\bar{K}] = [K_{pp}] - [K_{ps}][K_{ss}]^{-1}[K_{sp}] \quad (9.7)$$

Equation (9.4), which expresses that static relationship between the secondary coordinates $\{u_s\}$ and primary coordinates $\{u_p\}$, may also be written using the identity $\{u_p\} = [I]\{u_p\}$ as

$$\begin{Bmatrix} \{u_s\} \\ \{u_p\} \end{Bmatrix} = \begin{bmatrix} [\bar{T}] \\ [I] \end{bmatrix} \{u_p\}$$

or

$$\{u\} = [T]\{u_p\} \quad (9.8)$$

where

$$\{u\} = \begin{Bmatrix} \{u_s\} \\ \{u_p\} \end{Bmatrix} \quad \text{and} \quad [T] = \begin{bmatrix} [\bar{T}] \\ [I] \end{bmatrix} \quad (9.9)$$

Substituting eqs. (9.8) and (9.9) into eq. (9.1) and pre-multiplying by the transpose of $[T]$ results in

$$[T]^T [K] [T] \{u_p\} = [T]^T [F] \begin{Bmatrix} \{0\} \\ \{F_p\} \end{Bmatrix}$$

or

$$[T]^T [K] [T] \{u_p\} = \{F_p\}$$

and using eq. (9.6)

$$[\bar{K}] = [T]^T [K] [T] \quad (9.10)$$

thus showing that the reduced stiffness matrix $[\bar{K}]$ can be expressed as a transformation of the system stiffness matrix $[K]$.

It may appear that the calculation of the reduced stiffness matrix $[\bar{K}]$ given by eq. (9.7) requires the inconvenient calculation of the inverse matrix $[K_{ss}]^{-1}$. However, the practical application of the Static Condensation Method does not require a matrix inversion. Instead, the standard Gauss-Jordan elimination process, as it used in the solution of a system of linear equations, is applied systematically on the system's stiffness matrix $[K]$ up to the elimination of the secondary coordinates $\{u_s\}$. At this stage of the elimination process the stiffness equation (9.1) has been reduced to

$$\begin{bmatrix} [I] & -[\bar{T}] \\ [0] & [\bar{K}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \{u_p\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{F_p\} \end{Bmatrix} \quad (9.11)$$

It may be seen by expanding eq. (9.11) that the partition matrices $[\bar{T}]$ and $[\bar{K}]$ are precisely the transformation matrix and the reduced stiffness matrix defined by eqs. (9.4) and (9.6), respectively. In this way, the Gauss-Jordan elimination process yields both the transformation matrix $[\bar{T}]$ and the reduced stiffness matrix $[\bar{K}]$. There is thus no need to calculate $[K_{ss}]^{-1}$ in order to reduce the secondary coordinates of the system.

Illustrative Example 9.1

Consider the two-degree-of-freedom system represented by the model shown in Fig. 9.1 and use static condensation to reduce the first coordinate.

Solution:

For this system the equations of equilibrium are readily obtained as

$$\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix} \quad (9.12)$$

The reduction of u_1 using Gauss elimination leads to

$$\begin{bmatrix} 1 & -1/2 \\ 0 & k/2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix} \quad (9.13)$$

Comparing eq. (9.13) with eq. (9.11), we identify in this example

$$\begin{aligned} [\bar{T}] &= \frac{1}{2} \\ [\bar{K}] &= k/2 \end{aligned} \quad (9.14)$$

Consequently, from eq. (9.9) the transformation matrix is

$$[\bar{T}] = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \quad (9.15)$$

We can now check eq. (9.10) by simply performing the indicated multiplications, namely

$$[\bar{K}] = [1/2 \quad 1] \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{k}{2} \quad (9.16)$$

which agrees with the result given in eqs. (9.14).

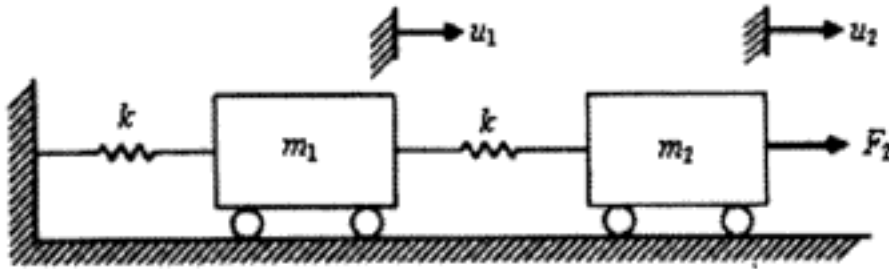


Fig. 9.1 Mathematical model for a two-degree-of-freedom system.

9.2 Static Condensation Applied to Dynamic Problems

In order to reduce the mass and the damping matrices, it is assumed that the same static relationship between the secondary and primary degrees of freedom remains valid in the dynamic problem. Hence the same transformation based on static condensation for the reduction of the stiffness matrix is also used in reducing the mass and damping matrices. In general this method of reducing the dynamic problem is not exact and introduces errors in the results. The magnitude of these errors depends on the relative number of degrees of freedom reduced as well as on the specific selection of these degrees of freedom for a given structure.

We consider first the case in which the discretization of the mass results in a number of massless degrees of freedom selected to be condensed. In this case it is only necessary to carry out the static condensation of the stiffness matrix and to delete from the mass matrix the rows and columns corresponding to the massless degrees of freedom. The Static Condensation Method in this case does not alter the original problem, and thus results in an equivalent eigenproblem without introducing any error.

In the general case, that is, the case involving the condensation of degrees of freedom to which the discretization process has allocated mass, the reduced mass and damping matrices are obtained using transformations analogous to eq. (9.10). Specifically, if $[M]$ is the mass matrix of the system, then the reduced mass matrix is given by

$$[\bar{M}] = [T]^T [M] [T] \quad (9.17)$$

where $[T]$ is the transformation matrix defined by eq. (9.9).

Analogously, for a damped system, the reduced damping matrix is given by

$$[\bar{C}] = [T]^T [C] [T] \quad (9.18)$$

where $[C]$ is the damping matrix of the system.

This method of reducing the mass and damping matrices may be justified as follows: The potential elastic energy V and the kinetic energy KE of the structure may be written, respectively, as

$$V = \frac{1}{2} \{u\}^T [K] \{u\} \quad (9.19)$$

$$KE = \frac{1}{2} \{\dot{u}\}^T [M] \{\dot{u}\} \quad (9.20)$$

Analogously, the virtual work of δW_d done by the damping forces $F_d = [C] \{\dot{u}\}$ corresponding to the virtual displacement $\{\delta u\}$ may be expressed as

$$\delta W_d = \{\delta u\}^T [C] \{\dot{u}\} \quad (9.21)$$

Introduction of the transformation eq. (9.8) in the above equation results in

$$V = \frac{1}{2} \{u_p\}^T [T]^T [K] [T] \{u_p\} \quad (9.22)$$

$$KE = \frac{1}{2} \{\dot{u}_p\}^T [T]^T [M] [T] \{\dot{u}_p\} \quad (9.23)$$

$$\delta W_d = \{\delta u_p\}^T [T]^T [C] [T] \{\dot{u}_p\} \quad (9.24)$$

The respective substitution of $[\bar{K}]$, $[\bar{M}]$ and $[\bar{C}]$ from eqs. (9.10), (9.17), and (9.18) for the product of the three central matrices in the eqs. (9.22), (9.23), and (9.24) yields

$$V = \frac{1}{2} \{u_p\}^T [\bar{K}] [T] \{u_p\} \quad (9.25)$$

$$KE = \frac{1}{2} \{\dot{u}_p\}^T [\bar{M}] \{\dot{u}_p\} \quad (9.26)$$

$$\delta W_d = \{\delta u_p\}^T [\bar{C}] \{\dot{u}_p\} \quad (9.27)$$

These last three equations express the potential energy, the kinetic energy, and the virtual work of the damping forces in terms of the primary coordinates $\{u_p\}$. Hence the matrices $[\bar{K}]$, $[\bar{M}]$ and $[\bar{C}]$ may be interpreted, respectively, as the stiffness, mass, and damping matrices of the structure corresponding to the primary degrees of freedom $\{u_p\}$ resulting in the same potential energy, kinetic energy and virtual work calculated with all the original nodal coordinate.

Illustrative Example 9.2

Determine the natural frequencies and modal shapes for the three-degree-of-freedom shear building shown in Fig. 9.2; then condense the first degree of freedom and compare the resulting values obtained for natural frequencies and modal shapes. The stiffness of each story and the mass at each floor level are indicated in the figure.

Solution:

1. Calculation of Natural Frequencies and Modal Shapes:

The equation of motion in free vibration for this structure is given by eq. (8.3) with the force vector $\{F\} = \{0\}$, namely,

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \quad (a)$$

where the matrices $[M]$ and $[K]$ are given, respectively, by eqs. (7.4) and (7.5). Substituting the corresponding numerical values in eq. (a) yields

$$\begin{bmatrix} 25 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

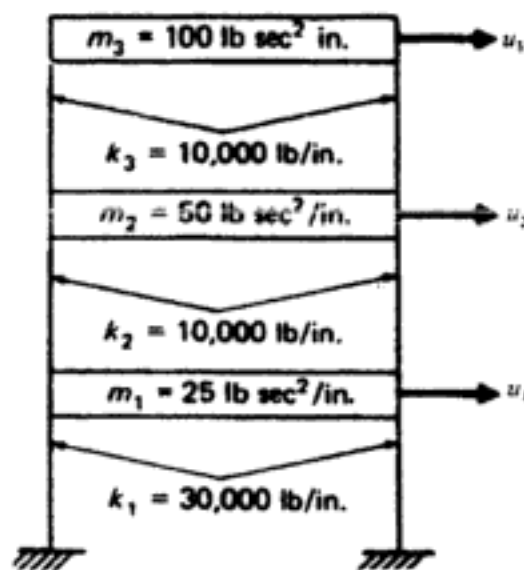


Fig. 9.2 Shear building for Example 9.2.

Upon substitution $u_i = U_i \sin \omega t$ and cancellation of the factor $\sin \omega t$, we obtain the following system of equations:

$$\begin{bmatrix} 40,000 - 25\omega^2 & -10,000 & 0 \\ -10,000 & 20,000 - 50\omega^2 & -10,000 \\ 0 & -10,000 & 10,000 - 100\omega^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (b)$$

which for a nontrivial solution requires that the determinant of the coefficients of the unknowns be equal to zero, that is,

$$\begin{vmatrix} 40,000 - 25\omega^2 & -10,000 & 0 \\ -10,000 & 20,000 - 50\omega^2 & -10,000 \\ 0 & -10,000 & 10,000 - 100\omega^2 \end{vmatrix} = 0$$

The expansion of this determinant results in a third degree equation in terms of ω^2 having the following roots:

$$\omega_1^2 = 36.1 \qquad \omega_2^2 = 400.0 \qquad \omega_3^2 = 1664.0 \qquad (c)$$

The natural frequencies are calculated as $f = \omega / 2\pi$, so that

$$f_1 = 0.96 \text{ cps} \qquad f_2 = 3.18 \text{ cps} \qquad f_3 = 264.8 \text{ cps}$$

The modal shapes are then determined by substituting in turn each of the values for the natural frequencies into eq. (b), deleting a redundant equation, and solving the remaining two equations for two of the unknowns in terms of the third. As we mentioned previously, in solving for these unknowns it is expedient to set the first nonzero unknown equal to one. Performing these operations, we obtain from eqs. (b) and (c) the following values for the modal shapes:

$$\begin{aligned} U_{11} &= 1.00, & U_{12} &= 1.00, & U_{13} &= 1.00 \\ U_{21} &= 3.91, & U_{22} &= 3.00, & U_{23} &= 3.338 \\ U_{31} &= 6.11, & U_{32} &= -1.00, & U_{33} &= -2.025 \end{aligned}$$

in which the second sub-index in U refers to the modal order.

2. Condensation of Coordinate u_1 :

The stiffness matrix for this structure is

$$\begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix}$$

Gauss elimination applied to the first row gives

$$\left[\begin{array}{c|cc} 1 & -0.25 & 0 \\ \hline 0 & 17,500 & -10,000 \\ 0 & -10,000 & 10,000 \end{array} \right] \qquad (d)$$

A comparison of eq. (d) with eq. (9.11) indicates that

$$[\bar{T}] = \begin{bmatrix} 0.25 & 0 \end{bmatrix}$$

$$[\bar{K}] = \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix} \quad (e)$$

Then from eq. (9.9)

$$[T] = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (f)$$

To check results, we use eq. (9.10) to compute $[\bar{K}]$. Hence

$$[\bar{K}] = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\bar{K}] = \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix}$$

which checks with eqs. (e).

The reduced mass matrix is calculated by substituting matrix $[T]$ and its transpose from eq. (f) into eq. (9.17), so that

$$[\bar{M}] = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which results in

$$[\bar{M}] = \begin{bmatrix} 51.6 & 0 \\ 0 & 100 \end{bmatrix}$$

The condensed dynamic problem is then

$$\begin{bmatrix} 51.6 & 0 \\ 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The natural frequencies and modal shapes are determined from the solution of the following reduced eigenproblem:

$$\begin{bmatrix} 17,500 - 51.6\omega^2 & -10,000 \\ -10,000 & 10,000 - 100\omega^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (g)$$

Equating to zero the determinant of the coefficient matrix in eq. (g) and solving the resulting quadratic equation in ω^2 gives

$$\omega_1^2 = 36.1 \qquad \omega_2^2 = 403.3 \qquad (h)$$

from which

$$f_1 = \sqrt{36.1/2\pi} = 0.95 \text{ cps}$$

$$f_2 = \sqrt{403.3/2\pi} = 3.20 \text{ cps}$$

Corresponding modal shapes are obtained from eq. (g) after substituting in turn the numerical values for ω_1^2 or ω_2^2 and solving the first equation for U_1 with $U_2 = 1$. We then obtain

$$U_{21} = 1.00, \quad U_{22} = 1.00$$

$$U_{31} = 1.56, \quad U_{32} = -0.33$$

in which the second subindexes in U serve to indicate the mode 1 or 2.

Application of eq. (9.8) for the first mode gives

$$\begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.56 \end{Bmatrix} = \begin{Bmatrix} 0.25 \\ 1.00 \\ 1.56 \end{Bmatrix}$$

or, after normalizing so that the first component is 1,

$$U_{11} = 1.00, \quad U_{21} = 4.00, \quad U_{31} = 6.24$$

and analogously for the second mode

$$U_{12} = 1.00, \quad U_{22} = 4.00, \quad U_{32} = -1.32$$

For this system of only three degrees of freedom, the reduction of one coordinate gives natural frequencies that compare rather well for the first two modes [eqs. (h) and (c)]. However, experience shows that static condensation may produce large errors in the calculation of eigenvalues and eigenvectors obtained from the reduced system. A general recommendation given by users of this method is to assume that the static condensation process results in an eigenproblem that provides acceptable approximations of only about a third of the calculated eigenvalues (natural frequencies) and eigenvectors (modal shapes).

Illustrative Example 9.3.

Figure 9.3 shows a uniform four-story shear building. For this structure, determine the following: (a) the natural frequencies and corresponding modal shapes as a four-degree-of-freedom system, (b) the natural frequencies and modal shapes after static condensation of coordinates u_1 and u_3 .

Solution:

(a) Natural Frequencies and Modal Shapes as a Four-Degree-of-Freedom System:

The stiffness and the mass matrices for this structure are respectively

$$[K] = 327.35 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad (a)$$

and

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (b)$$

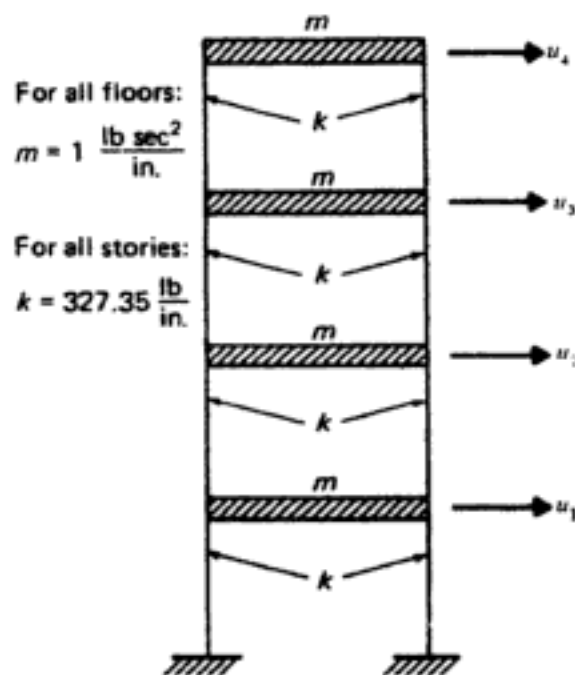


Fig.9.3 Uniform four-story shear building for Illustrative Example 9.3

Substituting eqs. (a) and (b) into eq. (9.3) and solving the corresponding eigenvalue problem (using Program 8) yields

$$\omega_1^2 = 39.48, \quad \omega_2^2 = 327.35, \quad \omega_3^2 = 768.3, \quad \text{and} \quad \omega_4^2 = 1156.00$$

corresponding to the natural frequencies

$$\begin{aligned} f_1 &= \frac{\omega_1}{2\pi} = 1.00 \text{ cps} & f_2 &= \frac{\omega_2}{2\pi} = 2.88 \text{ cps} \\ f_3 &= \frac{\omega_3}{2\pi} = 4.41 \text{ cps} & f_4 &= \frac{\omega_4}{2\pi} = 5.41 \text{ cps} \end{aligned} \quad (c)$$

and the normalized modal matrix (see Section 7.2)

$$[\Phi] = \begin{bmatrix} 0.2280 & 0.5774 & -0.6565 & 0.4285 \\ 0.4285 & 0.5774 & 0.2280 & -0.6565 \\ 0.5774 & 0 & 0.5774 & 0.5774 \\ 0.6565 & -0.5774 & 0.4285 & -0.2280 \end{bmatrix} \quad (d)$$

(b) Natural Frequencies and Modal Shapes after Reduction to Two-Degree-of-Freedom System:

To reduce coordinates u_1 and u_3 , for convenience, we first rearrange the stiffness matrix in eq. (a) to have the coordinates in order u_1, u_3, u_2, u_4

$$[K] = 327.35 \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (e)$$

Applying Gauss-Jordan elimination to the first two rows of the matrix in eq. (e) results in

$$[D] = \left[\begin{array}{cc|cc} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -0.5 & -0.5 \\ \hline 0 & 0 & 327.35 & -163.70 \\ 0 & 0 & -163.70 & 163.70 \end{array} \right] \quad (f)$$

A comparison of eq. (f) with eq. (9.11) reveals that

$$[\bar{T}] = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} \quad (g)$$

and

$$[\bar{K}] = \begin{bmatrix} 327.35 & -163.70 \\ -163.70 & 163.70 \end{bmatrix} \quad (\text{h})$$

Use of eq. (9.9) gives

$$[T] = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reduced mass matrix can now be calculated by eq. (9.17) as

$$[\bar{M}] = [T]^T [M] [T] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix} \quad (\text{i})$$

The condensed eigenproblem is then

$$\begin{bmatrix} 327.35 - 1.5\omega^2 & -163.70 - 0.25\omega^2 \\ -163.70 - 0.25\omega^2 & 163.70 - 1.25\omega^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{j})$$

and its solution is

$$\omega_1^2 = 40.39, \quad \omega_2^2 = 365.98 \quad (\text{k})$$

$$[U]_p = \begin{bmatrix} 0.4380 & 0.7056 \\ 0.6723 & -0.6128 \end{bmatrix} \quad (\text{l})$$

where $[U]_p$ is the modal matrix corresponding to the primary degrees of freedom.

The eigenvectors for the four-degree-of-freedom system are calculated for the first mode using eq. (9.8) as

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_1 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.4380 \\ 0.6723 \end{Bmatrix} = \begin{Bmatrix} 0.2190 \\ 0.5552 \\ 0.4380 \\ 0.6723 \end{Bmatrix}_1$$

or rearranging the component of the first modal shape:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.2190 \\ 0.4380 \\ 0.5552 \\ 0.6723 \end{Bmatrix}_1 \quad (\text{m})$$

and for the second mode

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_2 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.7056 \\ -0.6128 \end{Bmatrix} = \begin{Bmatrix} 0.3528 \\ 0.0464 \\ 0.7056 \\ -0.6128 \end{Bmatrix}_2$$

or

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.3528 \\ 0.7056 \\ 0.0464 \\ -0.6128 \end{Bmatrix}_2$$

Illustrative Example 9.4.

The shear building of Illustrative Example 9.3 is subjected to an earthquake motion at its foundation. For design purposes, use the response spectrum of Fig. 5.10 (Section 5.4) and determine the maximum horizontal displacements of the structure at the level of the floors.

Solution:

The participation factor for the i th mode of a shear building with N stories is given by eq. (8.40) as

$$\Gamma_i = \sum_{j=1}^N (m_j \phi_{ji}) \quad (\text{a})$$

where m_j is the mass at the j th floor and the ϕ_{ji} the j th element of the normalized i^{th} eigenvector of the mode.

(a) Response Considering Four Degrees of Freedom:

The substitution into eq. (a) of the corresponding numerical results from Illustrative Example 9.3 gives

$$\Gamma_1 = -1.890, \quad \Gamma_2 = -0.5775, \quad \Gamma_3 = -0.2797, \quad \text{and} \quad \Gamma_4 = -0.1213 \quad (\text{b})$$

The spectral displacements corresponding to the values of the natural frequencies of this building [eq. (c) of Illustrative Example 9.3] are obtained from the response spectrum, Fig 5.10, as

$$S_{D1} = 14.32, \quad S_{D2} = 3.240, \quad S_{D3} = 1.433, \quad \text{and} \quad S_{D4} = 0.969 \quad (\text{c})$$

The maximum displacements at the floor levels relative to the displacement at the base of the building are calculated using eq. (8.41), namely,

$$u_{i \max} = \sqrt{\sum_{j=1}^n (T_j S_{Dj} \phi_{ij})^2} \quad (d)$$

to obtain

$$u_{1 \max} = 6.274 \text{ in, } u_{2 \max} = 11.65 \text{ in, } u_{3 \max} = 15.64 \text{ in, and } u_{4 \max} = 17.81 \text{ in}$$

(b) Response Considering the System Reduce to Two Degrees of Freedom:

The natural frequencies, calculated from eq. (k) in Illustrative Example 9.3, are

$$\begin{aligned} f_1 &= \sqrt{40.39} / 2\pi = 1.011 \text{ cps} \\ f_2 &= \sqrt{365.98} / 2\pi = 3.044 \text{ cps} \end{aligned} \quad (e)$$

Upon introducing, into eq. (a), the corresponding eigenvectors give in eqs. (m) and (n) of Illustrative Example 9.3, we obtain the participation factors

$$\Gamma_1 = -1.884 \quad \Gamma_2 = -0.492$$

The values of spectral displacements corresponding to the frequencies in eq. (e) can be read from Fig. 5.10:

$$S_{D1} = 14.16 \quad S_{D4} = 2.913$$

Use of eq. (d) gives the relative maximum displacements at the level of the floors as

$$\begin{aligned} u_{1 \max} &= \sqrt{(1.884 \times 14.16 \times 0.2190)^2 + (0.4920 \times 2.913 \times 0.3528)^2} = 5.864 \text{ in} \\ u_{2 \max} &= \sqrt{(1.884 \times 14.16 \times 0.4380)^2 + (0.4920 \times 2.913 \times 0.7056)^2} = 11.73 \text{ in} \\ u_{3 \max} &= \sqrt{(1.884 \times 14.16 \times 0.5552)^2 + (0.4920 \times 2.913 \times 0.0464)^2} = 14.81 \text{ in} \\ u_{4 \max} &= \sqrt{(1.884 \times 14.16 \times 0.6723)^2 + (0.4920 \times 2.913 \times 0.6128)^2} = 17.97 \text{ in} \end{aligned}$$

9.3 Dynamic Condensation

A method of reduction that may be considered an extension of the Static Condensation Method has been proposed (Paz 1984). The algorithm for this method starts by assigning an approximate value (e.g., zero) to the first eigenvalues ω_1^2 , applying dynamic condensation to the dynamic matrix of the system $[D_1] = [K] - \omega_1^2[M]$, and then solving the reduced eigenproblem to determine the first and second eigenvalues, ω_1^2 and ω_2^2 . The process continues in this manner, with one virtually

exact eigenvalue and an approximate value for the next order eigenvalue calculated at each step.

The Dynamic Condensation Method requires neither matrix inversion nor series expansion. To demonstrate this fact, consider the eigenvalues problem of a discrete structural system for which it is desired to reduce the secondary degrees of freedom $\{u_o\}$ and retain the primary degrees of freedom $\{u_p\}$. In this case, the equations of free motion may be written in partitioned matrix form as

$$\begin{bmatrix} [M_{ss}] & [M_{sp}] \\ [M_{ps}] & [M_{pp}] \end{bmatrix} \begin{Bmatrix} \ddot{u}_s \\ \ddot{u}_p \end{Bmatrix} + \begin{bmatrix} [K_{ss}] & [K_{sp}] \\ [K_{ps}] & [K_{pp}] \end{bmatrix} \begin{Bmatrix} u_s \\ u_p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (9.28)$$

The substitution of $\{u\} = \{U\} \sin \omega_i t$ in eq. (9.28) results in the generalized eigenproblem

$$\begin{bmatrix} [K_{ss}] - \omega_i^2 [M_{ss}] & [K_{sp}] - \omega_i^2 [M_{sp}] \\ [K_{ps}] - \omega_i^2 [M_{ps}] & [K_{pp}] - \omega_i^2 [M_{pp}] \end{bmatrix} \begin{Bmatrix} u_s \\ u_p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (9.29)$$

where ω_i^2 is the approximation of the i th eigenvalues which was calculated in the preceding step of the process. To start the process one takes an approximate or zero value for the first eigenvalue ω_1^2 .

The following three steps are executed to calculate the i th eigenvalue ω_i^2 and the corresponding eigenvector $\{U\}_i$ as well as an approximation of the eigenvalue of the next order ω_{i+1}^2 :

Step 1: The approximation of ω_i^2 is introduced in eq. (9.29). Gauss-Jordan elimination of the secondary coordinates $\{U_s\}$ is then used to reduce eq. (9.29) to

$$\begin{bmatrix} [I] & -[\bar{T}_i] \\ [0] & [D_i] \end{bmatrix} \begin{Bmatrix} \{U_s\} \\ \{U_p\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \quad (9.30)$$

The first equation in eq. (9.30) can be written as

$$\{U_s\} = [\bar{T}_i] \{U_p\} \quad (9.31)$$

Consequently, the i th modal shape $\{U\}_i$ can be expressed as

$$\{U\}_i = [T] \{U_p\} \quad (9.32)$$

where

$$[T] = \begin{bmatrix} [\bar{T}_i] \\ [I] \end{bmatrix} \quad \text{and} \quad \{U\}_i = \begin{Bmatrix} \{U_s\} \\ \{U_p\} \end{Bmatrix} \quad (9.33)$$

Step 2: The reduced mass matrix $[\bar{M}_i]$ and the reduced stiffness matrix $[\bar{K}_i]$ are calculated as

$$[\bar{M}_i] = [T_i]^T [M] [T_i] \quad (9.34)$$

and

$$[\bar{K}_i] = [\bar{D}_i] + \omega_i^2 [\bar{M}_i] \quad (9.35)$$

where the transformation matrix $[T_i]$ is given by eq. (9.33) and the reduced dynamic matrix $[D_i]$ is defined eq. (9.30).

Step 3: The reduced eigenproblem

$$[[\bar{K}_i] - \omega^2 [\bar{M}_i]] \{U_p\} = \{0\} \quad (9.36)$$

is then solved to obtain improved eigenvalues ω_i^2 , its corresponding eigenvector $\{U_p\}_i$, and also an approximation for the next order eigenvalue ω_{i+1}^2 .

These three-step process may be applied iteratively; that is, the value of ω_i^2 obtained in step 3 may be used as an improved approximate value in step 1 to obtain a further improved value of ω_i^2 in step 3. Experience has shown that one or two such iterations will produce virtually exact eigensolutions

Illustrative Example 9.5.

Repeat Example 9.3 of Section 9.2 using the Dynamic Condensation Method.

Solution:

The stiffness matrix and the mass matrix with the coordinates in the order u_1, u_3, u_2, u_4 are given, respectively, by eqs. (e) and (b) of Illustrative Example 9.3. Substitution of these matrices into eq. (9.29) results in the dynamic matrix for the system:

$$[D] = \begin{bmatrix} 654.70 - \omega_i^2 & 0 & -327.35 & 0 \\ 0 & 654.70 - \omega_i^2 & -327.35 & -327.35 \\ -327.35 & -327.35 & 654.70 - \omega_i^2 & 0 \\ 0 & -327.35 & 0 & 327.35 - \omega_i^2 \end{bmatrix} \quad (a)$$

Step 1: Assuming we have no initial approximation of ω_1^2 , we start step 1 by setting $\omega_1^2 = 0$ and substituting this value into eq. (a):

$$[D] = \begin{bmatrix} 654.70 & 0 & -327.35 & 0 \\ 0 & 654.70 & -327.35 & -327.35 \\ -327.35 & -327.35 & 654.70 & 0 \\ 0 & -327.35 & 0 & 327.35 \end{bmatrix} \quad (b)$$

Application of the Gauss-Jordan elimination process to the first two rows gives

$$\begin{bmatrix} 1 & 0 & -0.5 & 0.0 \\ 0 & 1 & -0.5 & -0.5 \\ 0 & 0 & 327.35 & -163.67 \\ 0 & 0 & -163.67 & 163.67 \end{bmatrix}$$

from which, by eqs. (9.30) and (9.33)

$$[T_1] = \begin{bmatrix} 0.5 & 0.0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix}$$

Step 2: The reduced mass and stiffness matrices by eqs. (9.34) and (9.35), are

$$[\bar{M}_1] = [T_1]^T [M] [T_1] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix}$$

and

$$[\bar{K}_1] = [\bar{D}_1] + \omega_1^2 [\bar{M}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix}$$

Step 3: The solution of the reduced eigenproblem $[[\bar{K}_1] - \omega^2 [\bar{M}_1]] \{U_p\}_1 = \{0\}$ yields

$$\omega_1^2 = 40.39 \quad \text{and} \quad \omega_2^2 = 365.98$$

These values for ω_1^2 and ω_2^2 may be improved by iterating the calculations, that is, by introducing $\omega_1^2 = 40.39$ into eq. (9.29). This substitution results in

$$[D_1] = \begin{bmatrix} 614.31 & 0 & -327.35 & 0 \\ 0 & 614.31 & -327.35 & -327.35 \\ -327.35 & -327.35 & 614.31 & 0 \\ 0 & -327.35 & 0 & 286.96 \end{bmatrix}$$

Applications of Gauss-Jordan elimination to the first two rows gives

$$\left[\begin{array}{cc|cc} 1 & 0 & -0.533 & 0.0 \\ 0 & 1 & -0.533 & -0.533 \\ \hline 0 & 0 & 265.44 & -174.44 \\ 0 & 0 & -174.44 & 112.53 \end{array} \right]$$

from which

$$[T_1] = \begin{bmatrix} 0.533 & 0.0 \\ 0.533 & 0.533 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_1] = \begin{bmatrix} 265.44 & -174.44 \\ -174.44 & 112.53 \end{bmatrix}$$

The reduced mass and stiffness matrices are then

$$[\bar{M}_1] = [T_1]^T [M] [T_1] = \begin{bmatrix} 1.568 & 0.284 \\ 0.284 & 1.284 \end{bmatrix}$$

and

$$[\bar{K}_1] = \bar{D}_1 + \omega_1^2 [\bar{M}_1] = \begin{bmatrix} 328.76 & -162.97 \\ -162.67 & 164.39 \end{bmatrix}$$

The solution of the reduced eigenproblem,

$$([\bar{K}_1] - \omega^2 [\bar{M}_1]) \{U_p\} = \{0\}$$

yields the eigenvalues

$$\omega_1^2 = 39.48, \quad \omega_2^2 = 360.21 \quad (c)$$

and corresponding eigenvectors

$$\{U_p\}_1 = \begin{bmatrix} 0.4283 \\ 0.6562 \end{bmatrix} \quad \{U_p\}_2 = \begin{bmatrix} 0.6935 \\ -0.6171 \end{bmatrix} \quad (d)$$

The same process is now applied to the second mode, starting by substituting into eq. (9.29) the approximate eigenvalues $\omega_2^2 = 360.21$ calculated for the second mode in eq. (c). In this case we obtain

$$[D_2] = \left[\begin{array}{cc|cc} 294.49 & 0 & -327.35 & 0 \\ 0 & 294.49 & -327.35 & -327.35 \\ \hline -327.35 & -327.35 & 294.49 & 0 \\ 0 & -327.35 & 0 & -32.86 \end{array} \right]$$

Gauss-Jordan elimination of the first two rows yields

$$\left[\begin{array}{cc|cc} 1 & 0 & -1.112 & 0.0 \\ 0 & 1 & -1.112 & -1.112 \\ \hline 0 & 0 & -433.27 & -363.88 \\ 0 & 0 & -363.88 & -396.74 \end{array} \right]$$

from which

$$[T_2] = \begin{bmatrix} 1.112 & 0.0 \\ 1.112 & 1.112 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [\bar{D}_2] = \begin{bmatrix} -433.27 & -363.88 \\ -363.88 & -396.74 \end{bmatrix}$$

The reduced mass and stiffness matrices are

$$[\bar{M}_2] = [T_2]^T [M] [T_2] = \begin{bmatrix} 3.471 & 1.236 \\ 1.236 & 2.236 \end{bmatrix}$$

$$[\bar{K}_2] = [\bar{D}_2] + \omega_1^2 [\bar{M}_2] = \begin{bmatrix} 817.12 & 81.21 \\ 81.21 & 408.56 \end{bmatrix}$$

The solution of the reduced eigenproblem $[[\bar{K}_2] - \omega^2 [\bar{M}_2]] \{U_p\}_2 = \{0\}$ yields for the second mode

$$\omega_2^2 = 328.61$$

An iteration is performed by introducing $\omega_2^2 = 328.61$ into eq. (9.29) to obtain the following:

$$[D_2] = \left[\begin{array}{cc|cc} 326.09 & 0 & -327.35 & 0 \\ 0 & 326.09 & -327.35 & -327.35 \\ \hline -327.35 & -327.35 & 326.09 & 0 \\ 0 & -327.35 & 0 & -1.26 \end{array} \right]$$

Applying Gauss-Jordan elimination to the first two rows yields

$$\left[\begin{array}{cc|cc} 1 & 0 & -1.004 & 0.0 \\ 0 & 1 & -1.004 & -1.004 \\ \hline 0 & 0 & -331.14 & -328.62 \\ 0 & 0 & -328.62 & -329.88 \end{array} \right]$$

from which

$$[T_2] = \begin{bmatrix} 1.004 & 0.0 \\ 1.004 & 1.004 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_2] = \begin{bmatrix} -331.14 & -328.62 \\ -328.62 & -329.88 \end{bmatrix}$$

The reduced mass and stiffness matrices are:

$$[\bar{M}_2] = [T_2]^T [M] [T_2] = \begin{bmatrix} 3.015 & 1.008 \\ 1.008 & 2.008 \end{bmatrix}$$

and

$$[\bar{K}_2] = [\bar{D}_2] + \omega_1^2 [\bar{M}_2] = \begin{bmatrix} 659.78 & 2.54 \\ 2.54 & 329.89 \end{bmatrix}$$

The solution of the reduced eigenproblem $[[\bar{K}_2] - \omega^2 [\bar{M}_2]] \{U_p\}_2 = \{0\}$ now gives for the second mode

$$\{U_p\}_2 = \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix} \quad \omega_2^2 = 327.35 \quad (e)$$

Therefore, from eqs. (c), (d), and (e) we have obtained for the first two eigenvalues

$$\omega_1^2 = 39.48 \quad \text{and} \quad \omega_2^2 = 327.35 \quad (f)$$

and corresponding eigenvectors

$$\{U_p\}_1 = \begin{Bmatrix} 0.4283 \\ 0.6562 \end{Bmatrix}, \quad \{U_p\}_2 = \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix}$$

The eigenvectors of the system are then computed using eq. (9.32) as follows:

$$\begin{bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{bmatrix}_1 = \begin{bmatrix} 0.533 & 0.0 \\ 0.533 & 0.533 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.4283 \\ 0.6562 \end{Bmatrix} = \begin{Bmatrix} 0.2283 \\ 0.5780 \\ 0.4283 \\ 0.6562 \end{Bmatrix}_1$$

Hence ordering the elements of the modal vector:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.2283 \\ 0.4283 \\ 0.5780 \\ 0.6562 \end{Bmatrix}_1 \quad (h)$$

and

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_2 = \begin{bmatrix} 1.004 & 0.0 \\ 1.004 & 1.004 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix} = \begin{Bmatrix} 0.5789 \\ 0.0 \\ 0.5766 \\ -0.5766 \end{Bmatrix}_2$$

Hence ordering the modal vector:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.5789 \\ 0.5766 \\ 0.0 \\ -0.5766 \end{Bmatrix}_2 \quad (i)$$

The eigenvalues and eigenvectors [eqs. (f), (h), and (i)] calculated for the first two modes in this example using dynamic condensation are virtually identical to the exact solution determined in eqs. (c) and (d) of Illustrative Example 9.3.

It should be noted that normalization of the eigenvectors is not needed in eqs. (h) and (i) if the reduced vectors are normalized with respect to the reduced mass of the system, that is, if a reduced eigenvector $\{U_p\}$ satisfies the normalizing equation

$$\{U_p\}^T [\bar{M}] \{U_p\} = 1$$

then by eq. (9.34)

$$\{U_p\}^T [T]^T [M] [T] \{U_p\} = 1 \quad (9.37)$$

and since by eq. (9.32)

$$\{U\}_i = [T] \{U_p\}$$

and

$$\{U_i\}^T = \{U_p\}_i^T [T]^T$$

Upon substitution of $\{U\}_i$ and $\{U_i\}^T$ into eq. (9.37) results in

$$\{U\}_i^T [M] \{U\}_i = 1$$

thus demonstrating that the eigenvector $\{U\}_i$ is normalized with respect to the mass matrix $[M]$ of the system, if $\{U_p\}$ has been normalized with respect to the reduced mass matrix $[\bar{M}]$.

9.4 Modified Dynamic Condensation

The dynamic condensation method requires the application of elementary operations, as it is routinely done to solve a linear system of algebraic equations, using the Gauss-Jordan elimination process. The elementary operations are required to transform eq. (9.29) to the form given by eq. (9.30). However, the method also requires the calculation of the reduced mass matrix by eq. (9.34). This last calculation involves the multiplication of three matrices of dimensions equal to the total number of coordinates in the system. Thus, for a system defined with many degrees of freedom, the calculation of the reduced mass matrix $[M]$ requires a large number of numerical operations. A modification (Paz 1989), recently proposed, obviates such large number of numerical operations. This modification consists of calculating the reduced stiffness matrix $[\bar{K}]$ only once by simple elimination of s displacements in eq. (9.29) after setting $\omega^2 = 0$, thus making unnecessary the repeated calculation of $[\bar{K}]$ for each mode using eq. (9.35). Furthermore, it also eliminates the time consumed in calculating the reduced mass matrix $[\bar{M}]$ using eq. (9.34). In the modified method, the reduced mass matrix for any mode i is calculated from eq. (9.35) as

$$[\bar{M}_i] = \frac{1}{\omega_i^2} [[\bar{K}] - [\bar{D}_i]] \quad (9.38)$$

where $[\bar{K}]$ is the reduced stiffness matrix, already calculated, and $[\bar{D}_i]$ is the dynamic matrix given in the partitioned matrix of eq. (9.30).

As can be seen, the modified algorithm essentially requires, for each eigenvalues calculated, only the application of the Gauss-Jordan process to eliminate s unknowns in a linear system of equations such as the system in eq. (9.29).

Illustrative Example 9.6.

Repeat Illustrative Example 9.5 using the modified dynamic condensation method.

Solution:

The initial calculations of the modified method are the same as those in Illustrative Example 9.5. Thus, from Illustrative Example 9.5 we have

$$[\bar{K}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix} \quad (a)$$

$$[\bar{M}_1] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix} \quad (b)$$

$$\omega_1^2 = 40.39 \qquad \omega_2^2 = 365.98, \qquad (c)$$

$$[\bar{D}_1] = \begin{bmatrix} 265.44 & -174.44 \\ -174.44 & 112.53 \end{bmatrix} \qquad (d)$$

and

$$[\bar{T}_1] = \begin{bmatrix} 0.533 & 0 \\ 0.533 & 0.533 \end{bmatrix} \qquad (e)$$

Now, the reduced mass matrix $[\bar{M}_1]$ is calculated from eq. (9.38), after substitution in this equation of $[\bar{K}_1]$ from eq. (a) and $[\bar{D}_1]$ from eq. (d), as

$$[\bar{M}_1] = \frac{1}{\omega_1^2} [[\bar{K}_1] - [\bar{D}_1]] = \begin{bmatrix} 1.530 & 0.267 \\ 0.267 & 1.266 \end{bmatrix} \qquad (f)$$

Then the reduced stiffness and mass matrix given by eqs. (a) and (f) are used to solve the reduced eigenproblem:

$$[[\bar{K}_1] - \omega^2 [\bar{M}_1]] \{U_p\}_1 = \{0\}$$

to obtain the eigenvalues

$$\omega_1^2 = 39.46 \qquad \omega_2^2 = 363.67 \qquad (g)$$

and the eigenvector for the first mode

$$\begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.43359 \\ 0.66424 \end{Bmatrix}_1 \qquad (h)$$

The eigenvector for the first mode, in terms of the original four coordinate, is then obtained from eq. (9.31) as

$$\begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix} = \begin{bmatrix} 0.533 & 0 \\ 0.533 & 0.533 \end{bmatrix} \begin{Bmatrix} 0.43359 \\ 0.66424 \end{Bmatrix} = \begin{Bmatrix} 0.23110 \\ 0.58514 \end{Bmatrix} \qquad (i)$$

The combination of eqs. (h) and (i) give the eigenvector for the first mode as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.23110 \\ 0.43359 \\ 0.58514 \\ 0.66424 \end{Bmatrix}_1 \qquad (j)$$

Analogously, for the second mode, we substitute $\omega_2^2 = 363.67$ from eqs. (g) into eq. (a) of Illustrative Example 9.5, to obtain the following matrices after reducing the first two coordinates:

$$[\bar{T}_2] = \begin{bmatrix} 1.1248 & 0 \\ 1.1248 & 1.1248 \end{bmatrix}$$

$$[\bar{D}_2] = \begin{bmatrix} -445.43 & -368.20 \\ -368.20 & -404.54 \end{bmatrix}$$

The reduced mass matrix $[\bar{M}_2]$ is calculated from eq. (9.37) as

$$[\bar{M}_2] = \frac{1}{363.67} \left\{ \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix} - \begin{bmatrix} -445.43 & -368.20 \\ -368.20 & -404.54 \end{bmatrix} \right\}$$

$$[\bar{M}_2] = \begin{bmatrix} 2.1250 & 0.5624 \\ 0.5624 & 1.5624 \end{bmatrix}$$

Then, for the second mode, the solution of the corresponding reduced eigenproblem gives

$$\omega_2^2 = 319.41, \quad \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.61894 \\ -0.63352 \end{Bmatrix}$$

$$\begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix}_2 = \begin{bmatrix} 1.1248 & 0 \\ 1.1248 & 1.1248 \end{bmatrix} \begin{Bmatrix} 0.61894 \\ -0.63352 \end{Bmatrix} = \begin{Bmatrix} 0.69618 \\ -0.01640 \end{Bmatrix}$$

and

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.69618 \\ 0.61894 \\ -0.01640 \\ -0.63352 \end{Bmatrix}_2$$

The results obtained for this example using the modified method, although sufficiently approximate, are not as close to the exact solution as those obtained in Example 9.5 by the direct application of the dynamic condensation method. Table 9.1 shows and compares eigenvalues calculated in Illustrative Examples 9.5 and 9.6 with the exact solution obtained previously in Illustrative Example 9.3.

TABLE 9.1 Comparison of Results in Illustrative Examples 9.5 and 9.6 using Dynamic Condensation and Modified Dynamic Condensation

Mode	Eigenvalue				
	Exact Solution	Dynamic Condensation	Error %	Modified Method	Error%
1	39.48	39.48	0.00	39.46	0.05
2	327.35	327.35	0.00	319.41	2.42

9.5 Program 12---Reduction Of The Dynamic Problem

The computer program described in this section reduces by static condensation or by dynamic condensation the stiffness and mass matrices of a structural system and solves the reduced eigenproblem. The user has the option of selecting either of these two methods.

Illustrative Example 9.7.

For the four-story shear building shown in Fig. 9.3, use Program 12 to determine the natural frequencies and modal shapes after reducing the system to coordinates u_2 and u_4 using the following methods:

- Static condensation,
- Dynamic condensation,
- Exact solution as a system with four degrees of freedom.

Solution:

The execution of "Program 12: Condensation," requires the previous preparation of a file modeling the structure, that is, a file storing the stiffness and mass matrix of the system. For this example, this file can be prepared by either executing Program 7 to Model the structure as a shear building or executing the Auxiliary Program X1, which directly stores in a file the stiffness and mass matrices from input data.

Input Data and Output Results

SYSTEM STIFFNESS MATRIX

+ .65470E+03	-.32735E+03	+ .00000E+00	+ .00000E+00
-.32735E+03	+ .65470E+03	-.32735E+03	+ .00000E+00
+ .00000E+00	-.32735E+03	+ .65470E+03	-.32735E+03
+ .00000E+00	+ .00000E+00	-.32735E+03	+ .32735E+03

SYSTEM MASS MATRIX

+ .10000E+01	+ .00000E+00	+ .00000E+00	+ .0000E+00
+ .00000E+00	+ .10000E+01	+ .00000E+00	+ .00000E+00
+ .00000E+00	+ .00000E+00	+ .10000E+01	+ .00000E+00
+ .00000E+00	+ .00000E+00	+ .00000E+00	+ .10000E+01

(a) STATIC CONDENSATION

PROGRAM 12: CONDENSATION

DATA FILE: DI2

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM
 NUMBER OF PRIMARY COORDINATES
 CONDENSATION INDEX

ND= 4
 NL= 2
 INDSX=-1

PRIMARIES COORDINATES:

2 4

OUTPUT RESULTS:

FREQ. #	EIGENVALUE	FREQUENCY (C.P.S.)
---------	------------	----------------------

1	4.039E+00I	1.011
---	------------	-------

EIGENVECTOR # 1

0.218979	0.437958	0.555152	0.672345
----------	----------	----------	----------

FREQ. #	EIGENVALUE	FREQUENCY (C.P.S.)
---------	------------	----------------------

2	3.660E+02	3.045
---	-----------	-------

EIGENVECTOR # 2

0.352792	0.705584	0.046386	-.612812
----------	----------	----------	----------

(b) DYNAMIC CONDENSATION

PROGRAM 12: CONDENSATION

DATA FILE: DI2

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM
 NUMBER OF PRIMARY COORDINATES
 CONDENSATION INDEX

ND= 4
 NL= 2
 INDEX= 1

PRIMARIES COORDINATES:

2 4

OUTPUT RESULTS:

FREQ. #	EIGENVALUE	FREQUENCY (C.P.S.)
---------	------------	---------------------

1	3.948E+001	1.000
---	------------	-------

EIGENVECTOR # 1

0.228219 0.428282 0.577871 0.656167

FREQ. # EIGENVALUE FREQUENCY (C.P.S.)

2 3.289E+0.2 2.885

EIGENVECTOR # 2

0.618702 0.556594 0.002393 -.554441

(c) EXACT SOLUTION

PROGRAM 8: NATURAL FREQUENCIES
INPUT DATA:

DATA FILE: SK

STIFFNESS MATRIX

0.65473E+03	-.32737E+03	0.00000E+00	0.00000E+00
-.32737E+03	0.65473E+03	-.32737E+03	0.00000E+00
0.00000E+00	-.32737E+03	0.65473E+03	-.32737E+03
0.00000E+00	0.00000E+00	-.32737E+03	0.32737E+03

MASS MATRIX

0.10000E+01	0.00000E+00	0.00000E+00	0.00000E+00
0.00000E+00	0.10000E+01	0.00000E+00	0.00000E+00
0.00000E+00	0.00000E+00	0.10000E+01	0.00000E+00
0.00000E+00	0.00000E+00	0.00000E+00	0.10000E+01

OUTPUT RESULTS:

EIGENVALUES:

3.948E+01 3.274E+02 7.684E+02 1.156E+03

NATURAL FREQUENCIES (C.P.S.):

1.000 2.880 4.411 5.412

EIGENVECTORS BY ROWS:

0.22801	0.42853	0.57735	0.65654
-0.57735	-0.57735	0.00000	0.57735
-0.65654	0.22801	0.57735	-0.42853
0.42853	-0.65654	0.57735	-0.22801

As expected, results given by the computer for Illustrative Example 9.7 agree with calculation for the same structure obtained previously using static condensation in Example 9.3 and dynamic condensation in Illustrative Example 9.5.

9.6 Summary

The reduction of secondary or dependent degrees of freedom is usually accomplished in practice by the Static Condensation Method. This method consists of determining, by a partial Gauss-Jordan elimination, the reduced stiffness matrix corresponding to the primary degrees of freedom and the transformation matrix relating the secondary and primary degrees of freedom. The same transformation matrix is used in an orthogonal transformation to reduce the mass and damping matrices of the system. Static condensation introduces errors when applied to the solution structural dynamics problems. However, as is shown in this chapter, the application of the Dynamic Condensation Method substantially reduces or eliminates these errors. Furthermore, the Dynamic Condensation Method converges rapidly to the exact solution when iteration is applied.

9.7 Problems

Problem 9.1

The stiffness and mass matrices of a certain structure are given by

$$[K] = \begin{bmatrix} 10 & -2 & -1 & 0 \\ -2 & 6 & -3 & -2 \\ -1 & -3 & 12 & -1 \\ 0 & -2 & -1 & 8 \end{bmatrix}, \quad [M] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) Use the Static Condensation Method to determine the transformation matrix and the reduced stiffness and mass matrices corresponding to the elimination of the first two degrees of freedom (the massless degrees of freedom).

(b) Determine the natural frequencies and corresponding normal modes for reduced system.

Problem 9.2

Repeat (a) and (b) of Problem 9.1 for a structure having stiffness matrix as indicated in that problem, but mass matrix given by

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 9.3

Determine the natural frequencies and modal shape of the system in Problem 9.2 in terms of its four original coordinates; find the errors in the two modes obtained in Part (b) of Problem 9.2.

Problem 9.4

Consider the shear building shown in Fig. P9.4. 1.0

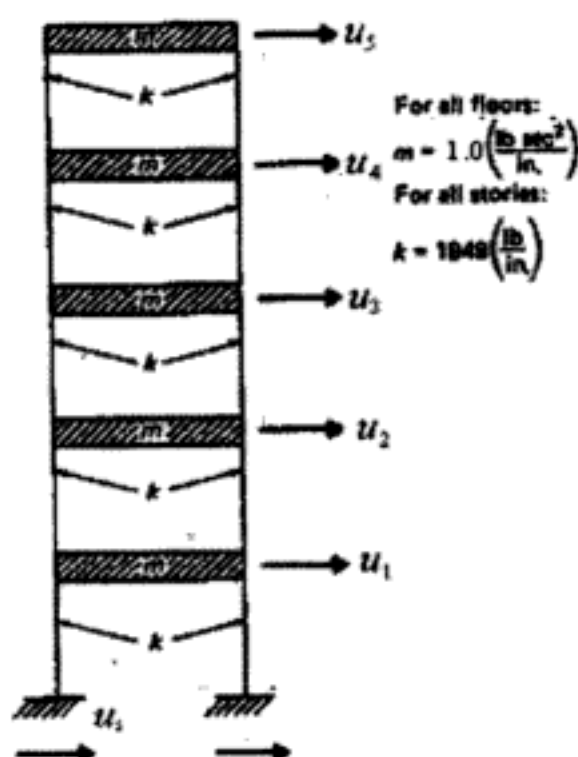


Fig. P9.4.

- Determine the transformation matrix and the reduced stiffness and mass matrices corresponding to the static condensation of the coordinates u_1 , u_3 , and u_4 as indicated in the figure.
- Determine the natural frequencies and normal modes for the reduced system obtained in (a).
- Use the results of (b) to determine the modal shapes, described by the five original coordinates, corresponding to the two lowest frequencies.

Problem 9.5

Use the results obtained in Problem 9.4 to determine the maximum shear forces in the stories of the building in Fig. P9.4 when subjected to an earthquake for which the response spectrum is given in Fig. 5.10 of Section 5.4.

Problem 9.6

Use the results in Problem 9.4 to determine the maximum shear forces in the stories of the building in Fig. 9.4 when subjected to an earthquake the response spectrum is given in Fig. 5.10 of Section 5.4.

Problem 9.7

Repeat Problem 9.2 using the Dynamic Condensation Method.

Problem 9.8

Repeat Problem 9.4 using the Dynamic Condensation Method and compare results with the exact solution.

Problem 9.9

Consider the five-story shear building of Fig. P9.4 subjected at its foundation to the time-acceleration excitation depicted in Fig. P9.9. Use static condensation of the coordinates u_1 , u_3 , and u_4 and determine:

- The two natural frequencies and corresponding modal shapes of the reduced system.
- The displacements at the floor levels considering two modes.
- The shear forces in the columns of the structure also considering two modes.

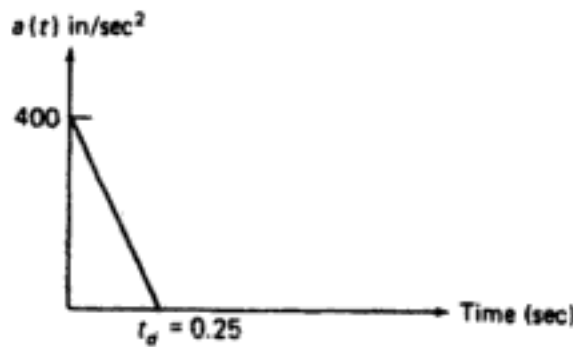


Fig. P9.9.

Problem 9.10

Solve Problem 9.9 using the Dynamic Condensation Method.

Problem 9.11

The stiffness and mass matrices for a certain structure are

$$[K] = 10^6 \begin{bmatrix} 0.906 & 0.294 & 0.424 \\ 0.294 & 0.318 & 0.176 \\ 0.424 & 0.176 & 80.000 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 288 & -8 & 1566 \\ -8 & 304 & 644 \\ 1566 & 644 & 80,000 \end{bmatrix}$$

Calculate the fundamental natural frequency of the system after reduction of the first coordinate by the following methods: (a) Static condensation; and (b) Dynamic condensation. Also obtain the natural frequencies as a three-degrees-of-freedom system and compare results for the fundamental frequency.

Problem 9.12

Repeat Problem 9.11 using the Modified Dynamic Condensation Method and compare results with the solution obtained with no condensation.

PART III

Framed Structures Modeled as Discrete Multi-Degree-of-Freedom Systems

10 Dynamic Analysis of Beams

In this chapter, we shall study the dynamic behavior of structures designated as beams, that is, structures that carry loads that are mainly transverse to the longitudinal direction, thus producing flexural stresses and lateral displacements. We begin by establishing the static characteristics for a beam segment; and then introduce the dynamic effects produced by the inertial forces. Two approximate methods are presented to take into account the inertial effect in the structure: 1) the lumped mass method in which the distributed mass is assigned to point masses, and 2) the consistent mass method in which the assignment to point masses includes rotational effects. The latter method is consistent with the static deflections of the beam. In Part IV on Special Topics the exact theory for dynamics of beams considering the elastic and inertial distributed properties will be presented.

10.1 Shape Functions for a Beam Segment

Consider a uniform beam element of cross-sectional moment of inertia I , length L , and material modulus of elasticity E as shown in Fig.10.1. We shall establish the relation between static forces and moments designated as P_1 , P_2 , P_3 and P_4 and the corresponding linear and angular displacements δ_1 , δ_2 , δ_3 and δ_4 at the ends of the beam element as indicated in Fig 10.1. The relationship thus obtained is known as the stiffness matrix equation for a beam element. The forces P_i and the displacements δ_i are said to be at the nodal coordinates defined at the ends for the beam element.

The differential equation for small transverse displacements of a beam, which is well known from elementary studies of strength and materials, is given by

$$EI \frac{d^2 u}{dx^2} = M(x) \quad (10.1)$$

in which $M(x)$ is the bending moment at a section x of the beam and u is the transverse displacement.

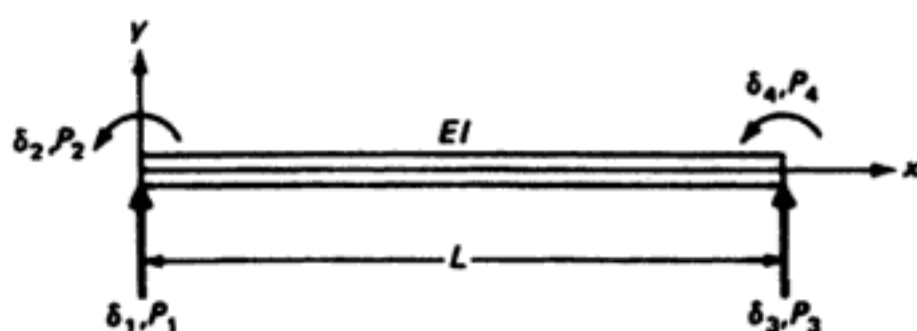


Fig.10.1 Beam segment showing forces and displacement at the nodal coordinates.

The differential equation (10.1) for a uniform beam element is equivalent to

$$EI \frac{d^4 u}{dx^4} = p(x) \quad (10.2)$$

since

$$\frac{dM(x)}{dx} = V(x) \quad (10.3)$$

and

$$\frac{dV(x)}{dx} = p(x)$$

Where $p(x)$ is the beam load per unit length and $V(x)$ is the shear force.

We state first the general definition of the stiffness coefficient which is designated by k_{ij} , that is, k_{ij} is the force at nodal coordinate i due to a unit displacement at nodal coordinate j while all other nodal coordinates are maintained at zero displacement. Figure 10.2 shows the displacement curves and the corresponding stiffness coefficients due to a unit displacement at each one of the four nodal coordinates of the beam element. To determine the expressions for the stiffness coefficients k_{ij} , we begin by finding the equations for displaced curves shown in Fig. 10.2. We first consider the beam element in Fig 10.1 free of loads [$p(x) = 0$], except for the forces P_1 , P_2 , P_3 and P_4 applied at the nodal coordinates. In this case, eq. (10.2) is reduced to

$$\frac{d^4 u}{dx^4} = 0 \quad (10.4)$$

Successive integrations of eq. (10.4) yields

$$\begin{aligned} \frac{d^3 u}{dx^3} &= C_1 \\ \frac{d^2 u}{dx^2} &= C_1 x + C_2 \end{aligned}$$

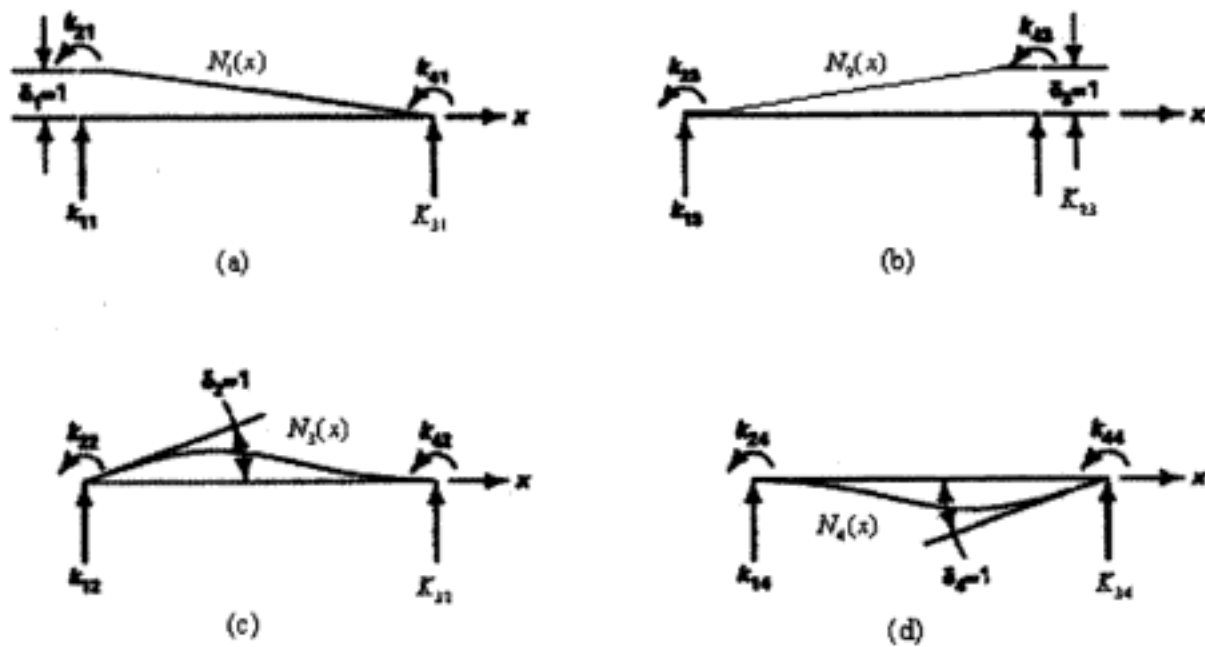


Fig.10.2 Beam element showing static deflection curves due to a unit displacement at one of the nodal coordinates.

$$\frac{du}{dx} = \frac{1}{2}C_1x^2 + C_2x + C_3 \quad (10.5)$$

$$u = \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4 \quad (10.6)$$

in which C_1 , C_2 , C_3 and C_4 are constants of integration to be evaluated using boundary conditions. For example, to determine the function $N_1(x)$ for the curve shown in Fig. 10.2 (a), we make use of the following boundary conditions:

$$\text{at } x = 0 \quad u(0) = 1 \quad \text{and} \quad \frac{du(0)}{dx} = 0 \quad (10.7)$$

$$\text{at } x = L \quad u(L) = 0 \quad \text{and} \quad \frac{du(L)}{dx} = 0 \quad (10.8)$$

Use of these conditions in eqs. (10.5) and (10.6), results in an algebraic system of four equations to determine the constants C_1 , C_2 , C_3 and C_4 .

Then the subsequent substitution of these constants into eq. (10.6) results in the equation of the deflected curve for the beam element in Fig. (10.2(a) as

$$N_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \quad (10.9a)$$

in which $N_1(x)$ is used instead of $u(x)$ to correspond to the condition $\delta_1 = 1$ imposed on the beam element. Proceeding in analogous fashion, we obtain for the equations of the deflected curves in the other cases depicted in Fig. 10.2 the following equations:

$$N_2(x) = x \left(1 - \frac{x}{L} \right)^2 \quad (10.9b)$$

$$N_3(x) = 3 \left(\frac{x}{L} \right)^2 - 2 \left(\frac{x}{L} \right)^3 \quad (10.9c)$$

$$N_4(x) = \frac{x^2}{L} \left(\frac{x}{L} - 1 \right) \quad (10.9d)$$

Since $N_1(x)$ is the deflection corresponding to a unit displacement $\delta_1 = 1$, the displacement resulting from an arbitrary displacement δ_1 , is $N_1(x) \delta_1$. Analogously, the deflection resulting from nodal displacements δ_2 , δ_3 and δ_4 are, respectively $N_2(x) \delta_2$, $N_3(x) \delta_3$ and $N_4(x) \delta_4$. Therefore, the total deflection $u(x)$ at coordinate x due to arbitrary displacements at the nodal coordinates of the beam element is given by superposition as

$$u(x) = N_1(x) \delta_1 + N_2(x) \delta_2 + N_3(x) \delta_3 + N_4(x) \delta_4 \quad (10.10)$$

The shape equations which are given by eqs.(10.9) and which correspond to unit displacements at the nodal coordinates of a beam element may be used to determine expressions for the stiffness coefficients. For example, consider the beam in Fig 10.2(b) which is in equilibrium with the forces producing the displacement $\delta_2 = 1.0$. For this beam in the equilibrium position, we assume that a virtual displacement equal to the deflection curve shown in Fig 10.2(a) takes place. We then apply the principle of virtual work, which states that, for an elastic system in equilibrium, the work done by the external forces is equal to the work of the internal forces during the virtual displacement. In order to apply this principle, we note that the external work W_E is just equal to the product of the force k_{12} displaced by $\delta_1 = 1$, that is

$$W_E = k_{12} \delta_1 \quad (10.11)$$

This work, as stated above, is equal to the work performed by the elastic forces during the virtual displacement. Considering the work performed by the bending moment, we obtain for the internal work

$$W_I = \int_0^L M(x) d\theta \quad (10.12)$$

in which $M(x)$ is the bending moment at section x of the beam and $d\theta$ is the incremental angular displacement of this section of the element.

For the virtual displacement under consideration, the transverse deflection of the beam is given by eq. (10.9b), which is related to the bending moment through the differential equation (10.1). Substitution of the second derivative $N_2''(x)$ of eq. (10.9b) into eq. (10.1) results in

$$EI N_2''(x) = M(x) \quad (10.13)$$

The angular deflections $d\theta$ produced during this virtual displacement is related to the resulting transverse deflection of the beam $N_1(x)$ by

$$\frac{d\theta}{dx} = \frac{d^2 N_1(x)}{dx^2} = N_1''(x)$$

or

$$d\theta = N_1''(x) dx \quad (10.14)$$

Equating the external virtual work W_E from eq. (10.11) with the internal virtual work W_I from eq. (10.12) after using $M(x)$ and $d\theta$, respectively, from eqs. (10.13) and (10.14) finally gives the stiffness coefficient as

$$k_{12} = \int_0^L EI N_1''(x) N_2''(x) dx \quad (10.15)$$

In general, any stiffness coefficient k_{ij} associated with beam flexure, therefore, may be expressed as

$$k_{ij} = \int_0^L EI N_i''(x) N_j''(x) dx \quad (10.16)$$

It may be seen from eq. (10.16) that $k_{ij} = k_{ji}$ since the interchange of indices requires only an interchange of the two factors, $N_i''(x)$ and $N_j''(x)$ in eq. (10.16). This equivalence of $k_{ij} = k_{ji}$ is a particular case of Betti's theorem, but it is better known as *Maxwell's reciprocal theorem*.

It should be pointed out that although the shape functions, eqs. (10.9), were obtained for a uniform beam, in practice they are nevertheless also used in determining the stiffness coefficients for non-uniform beams.

Considering the case of a uniform beam element of length L and cross-sectional moment of inertia I , we may calculate any stiffness coefficient from eqs. (10.16) and the use of eqs. (10.9). In particular, the stiffness coefficient k_{12} is calculated as follows:

From eq. (10.9a), we obtain

$$N_1''(x) = -\frac{6}{L^2} + \frac{12x}{L^3}$$

and from eq. (10.9b)

$$N_2^*(x) = -\frac{4}{L} + \frac{6x}{L^2}$$

Substitution in eq. (10.15) gives

$$k_{12} = EI \int_0^L \left(\frac{-6}{L^2} + \frac{12x}{L^3} \right) \left(\frac{-4}{L} + \frac{6x}{L^2} \right) dx$$

and performing the integration results in

$$k_{12} = \frac{6EI}{L^2}$$

Since the stiffness coefficient k_{ij} is defined as the force at the nodal coordinate 1 due to unit displacement at the coordinate j , the forces at coordinate 1 due to successive displacement δ_1 , δ_2 , δ_3 and δ_4 at the four nodal coordinates of the beam element are given, respectively, by $k_{11} \delta_1$, $k_{12} \delta_2$, $k_{13} \delta_3$ and $k_{14} \delta_4$. Therefore, the total force P_1 at coordinate 1 resulting from these nodal displacements is obtained by the superposition of the resulting forces, that is,

$$P_1 = k_{11} \delta_1 + k_{12} \delta_2 + k_{13} \delta_3 + k_{14} \delta_4$$

Analogously, the forces at the other nodal coordinates resulting from the nodal displacement δ_1 , δ_2 , δ_3 , δ_4 are

$$\begin{aligned} P_2 &= k_{21} \delta_1 + k_{22} \delta_2 + k_{23} \delta_3 + k_{24} \delta_4 \\ P_3 &= k_{31} \delta_1 + k_{32} \delta_2 + k_{33} \delta_3 + k_{34} \delta_4 \\ P_4 &= k_{41} \delta_1 + k_{42} \delta_2 + k_{43} \delta_3 + k_{44} \delta_4 \end{aligned} \quad (10.17)$$

The above equations are written conveniently in matrix notation as

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \quad (10.18)$$

or in condensed notation as

$$\{P\} = [k] \{\delta\} \quad (10.19)$$

in which $\{P\}$ and $\{\delta\}$ are, respectively, the force and the displacement vectors at the four nodal coordinates of the beam element and $[k]$ is the beam element stiffness matrix.

The use of eq. (10.16) in the manner shown above to determine the coefficient k_{12} will result in the evaluation of all the coefficients of the stiffness matrix. This result for a uniform beam element is

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \quad (10.20)$$

or in condensed notation

$$\{P\} = [k]\{\delta\} \quad (10.21)$$

10.2 System Stiffness Matrix

Thus far we have established in eq. (10.20) the stiffness equation for a uniform beam element, that is, we have obtained the relationship between nodal displacements (linear and angular) and nodal forces (forces and moments). Our next objective is to obtain the same type of relationship between the nodal displacements and the nodal forces, but now for the entire structure (system stiffness equation). Furthermore, our aim is to obtain the system stiffness matrix from the stiffness matrix of each element of the system. The procedure is perhaps better explained through a specific example such as the cantilever beam shown in Fig. 10.3.

The first step in obtaining the system stiffness matrix is to divide the structure into elements. The beam in Fig. 10.3 has been divided into three elements which are numbered sequentially for identification. The second step is to identify the nodes or joints between elements and to number consecutively those nodal coordinates that are not constrained.

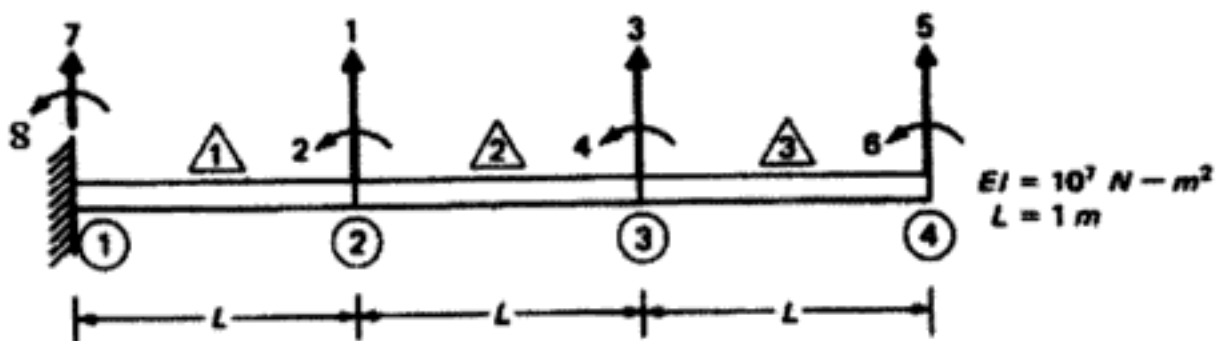


Figure 10.3 Cantilever beam divided into three beam segments with numbered system nodal coordinates.

The constrained or fixed nodal coordinates are the last to be labeled. For beams, we consider only two possible displacements at each node, a vertical deflection and an angular displacement. The cantilever beam in Fig. 10.3 with its three elements results in a total of six free nodal coordinates and two fixed nodal coordinates. The third step

is to obtain systematically the stiffness matrix for each element in the system and to add the element stiffness coefficients appropriately to obtain the system stiffness matrix. This method of assembling the system stiffness matrix is called the direct method. In effect, any stiffness coefficient k_{ij} of the system may be obtained by adding together the corresponding stiffness coefficients associated with those nodal coordinates. Thus, for example, to obtain the system stiffness coefficient k_{33} , it is necessary to add the stiffness coefficients of beam elements 2 and 3 corresponding to node three. These coefficients are designated $k_{33}^{(2)}$ and $k_{11}^{(3)}$, respectively. The upper indices serve to identify the beam element, and the lower indices to locate the appropriate stiffness coefficients in the corresponding element stiffness matrices.

Proceeding with the example in Fig. 10.3 and using eq. (10.20), we obtain the following expression for the stiffness matrix of beam element 2, namely

$$[k^{(2)}] = 10^7 \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{matrix}$$

(10.22)

For the beam element 2, the nodal coordinates numbered one to four coincide with the assignment of system nodal coordinates also numbered 1 to 4, as may be seen in Fig. 10.3. However, for the beam elements 1 and 3 of this beam, the assignment of element nodal coordinates numbered 1 to 4 does not coincide with the assigned system coordinates. For example, for element 1 the assigned system coordinates as seen in Fig. 10.3 are 7, 8, 1, 2; for element \triangle_3 , 3, 4, 5, 6. In the process of assembling the system stiffness, coefficients for element \triangle_2 will be correctly allocated to coordinates 1, 2, 3, 4; for element \triangle_1 to coordinates 7, 8, 1, 2; and for element \triangle_3 to coordinates 3, 4, 5, 6. A simple way to indicate this allocation of coordinates, when working by hand, is to write at the top and on the right of the element stiffness matrix the coordinate numbers corresponding to the system nodal coordinates for the element as it is indicated in eq.(10.22) for element \triangle_2 . The stiffness matrices for elements \triangle_1 and \triangle_3 with the corresponding indication of system nodal coordinates are, respectively,

$$[k^{(1)}] = 10^7 \begin{matrix} & \begin{matrix} 7 & 8 & 1 & 2 \end{matrix} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} & \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix} \end{matrix} \quad (10.23)$$

and

$$[k^{(3)}] = 10^7 \begin{matrix} & \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{matrix} \quad (10.24)$$

Proceeding systematically to assemble the system stiffness matrix, we transfer each entry in the element stiffness matrices, eqs. (10.22), (10.23), and (10.24), to the appropriate location in the system stiffness matrix. For instance, the stiffness coefficient for element $\triangle 3$, $k^{(3)}_{13} = -12 \times 10^7$ should be transferred to location at row 3 and column 5 since these are the nodal coordinates indicated at right and top of matrix eq. (10.24) for this entry. Every element stiffness coefficient transferred to its appropriate location in the system stiffness matrix is added to the other coefficients accumulated at that location. The stiffness coefficients corresponding to columns or rows carrying a label of a fixed system nodal coordinate (seven and eight in the present example) are simply disregarded since the constrained nodal coordinates are not unknown quantities. The assemblage of the system matrix in the manner described results for this example in a 6×6 matrix, namely

$$[k] = 10^7 \begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \quad (10.25)$$

Equation (10.25) is thus the system stiffness matrix for the cantilever beam shown in Fig. 10.3 which has been segmented into three elements. As such, the system stiffness matrix relates the forces and the displacements at the nodal system coordinates in the same manner as the element stiffness matrix relates forces and displacements at the element nodal coordinates.

10.3 Inertial Properties---Lumped Mass

The simplest method for considering the inertial properties for a dynamic system is to assume that the mass of the structure is lumped at the nodal coordinates where translational displacements are defined, hence the name lumped mass method. The usual procedure is to distribute the mass of each element to the nodes of the element. This distribution of the mass is determined by statics. Figure 10.4 shows, for beam segments of length L and distributed mass $\bar{m}(x)$ per unit of length, the nodal allocation for uniform, triangular, and general mass distribution along the beam segment. The assemblage of the mass matrix for the entire structure will be a simple matter of adding the contributions of lumped masses at the nodal coordinates defined as translations.

In this method, the inertial effect associated with any rotational degree of freedom is usually assumed to be zero, although a finite value may be associated with rotational degrees of freedom by calculating the mass moment of inertia of a fraction of the beam segment about the nodal points

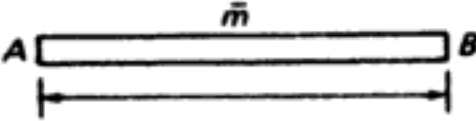
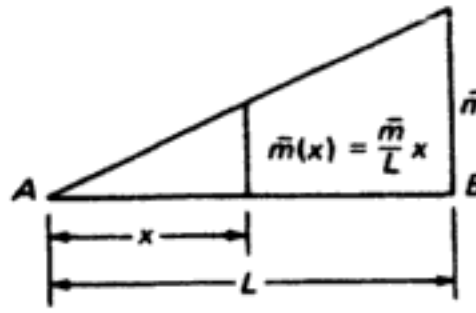
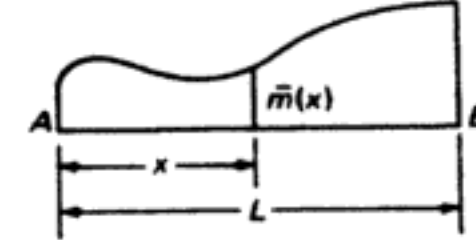
Mass distribution	Lumped mass
 Uniform	$m_A = \frac{\bar{m}L}{2}$ $m_B = \frac{\bar{m}L}{2}$
 Triangular	$m_A = \frac{\bar{m}L}{6}$ $m_B = \frac{\bar{m}L}{3}$
 General	$m_A = \frac{\int_0^L (L - x) \bar{m}(x) dx}{L}$ $m_B = \frac{\int_0^L x \bar{m}(x) dx}{L}$

Fig.10.4 Lumped masses for beam segments with distributed mass.

For example, for a uniform beam, this calculation would result in determining the mass moment of inertia of half of the beam segment about each node, that is

$$I_A = I_B = \frac{1}{3} \left(\frac{\bar{m}L}{2} \right) \left(\frac{L}{2} \right)^2$$

where \bar{m} is the mass per unit length along the beam. For the cantilever beam shown in Fig. 10.3 in which only translation mass effects are considered, the mass matrix of the system would be the diagonal matrix, namely

$$[M] = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} m_1 & & & & & \\ & 0 & & & & \\ & & m_3 & & & \\ & & & 0 & & \\ & & & & m_5 & \\ & & & & & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{matrix} \quad (10.26)$$

in which

$$\begin{aligned} m_1 &= \frac{\bar{m}L_1}{2} + \frac{\bar{m}L_2}{2} \\ m_3 &= \frac{\bar{m}L_2}{2} + \frac{\bar{m}L_3}{2} \\ m_5 &= \frac{\bar{m}L_3}{2} \end{aligned}$$

Using a special symbol $\lceil \rceil$ for diagonal matrices, we may write eq. (10.26) as

$$[M] = \lceil m_1 \quad 0 \quad m_3 \quad 0 \quad m_5 \rceil \quad (10.27)$$

10.4 Inertial Properties---Consistent Mass

It is possible to evaluate the mass coefficients corresponding to the nodal coordinates of a beam element by a procedure similar to the determination of element stiffness coefficients. First, we define the mass coefficient m_{ij} as the force at nodal coordinate i due to a unit acceleration at nodal coordinate j while all other nodal coordinates are maintained at zero acceleration.

Consider the beam segment shown in Fig. 10.5(a) which has distributed mass $\bar{m}(x)$ per unit of length. In the consistent mass method, it is assumed that the deflections resulting from unit dynamic displacements at the nodal coordinates of the beam element are given by the same functions $N_1(x)$, $N_2(x)$, $N_3(x)$, and $N_4(x)$ of eqs. (10.9)

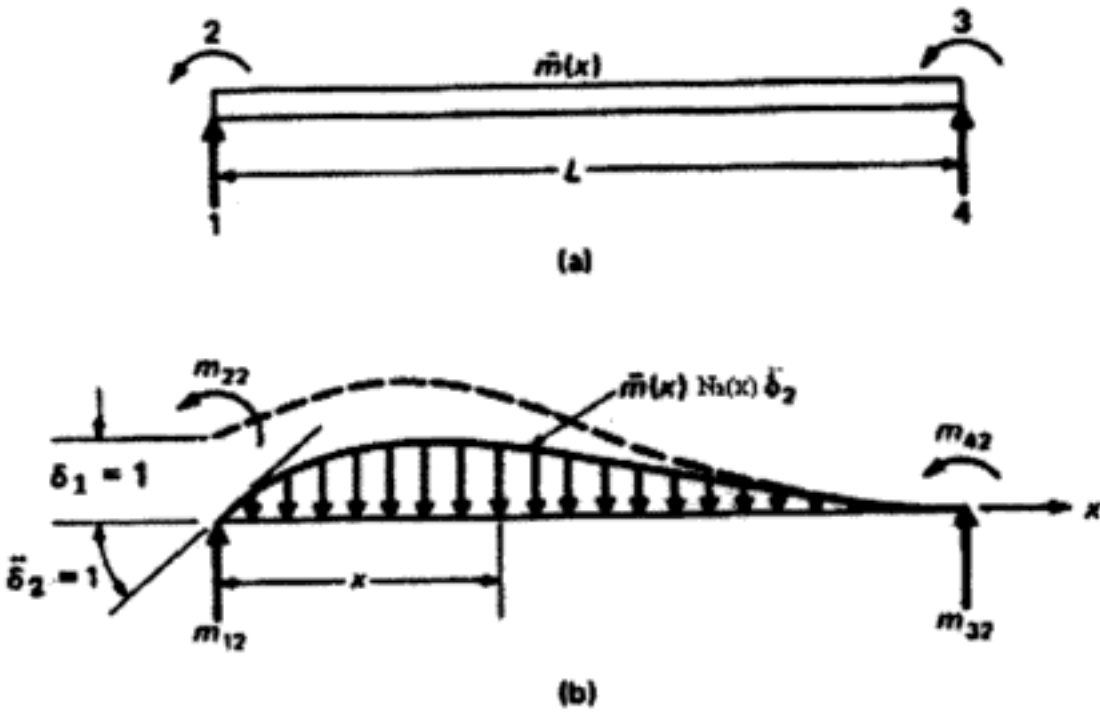


Fig.10.5 (a) Beam element with distributed mass showing nodal coordinates.(b) element supporting inertial load due to acceleration $\ddot{\delta}_2 = 1$, undergoing virtual displacement $\delta_1 = 1$.

which were obtained from static considerations. If the beam segment is subjected to a unit nodal acceleration at one of the nodal coordinates, say $\ddot{\delta}_2$, the transverse acceleration developed along the length of the beam is given by the second derivative with respect to time of eq. (10.10). In this case, with $\ddot{\delta}_1 = \ddot{\delta}_3 = \ddot{\delta}_4 = 0$, we obtain

$$\ddot{u}_2(x) = N_2(x) \ddot{\delta}_2 \tag{10.28}$$

The inertial force $f_I(x)$ per unit of length along the beam due to this acceleration is then given by

$$f_I(x) = \bar{m}(x) \ddot{u}_2(x)$$

or using eq. (10.28) by

$$f_I(x) = \bar{m}(x) N_2(x) \ddot{\delta}_2$$

or, since $\ddot{\delta}_2 = 1$,gb

$$f_I(x) = \bar{m}(x) N_2(x) \tag{10.29}$$

Now to determine the mass coefficient m_{12} , we give to the beam in Fig.10.5(b), which is in equilibrium with the forces resulting from a unit acceleration $\ddot{\delta}_2 = 1$, a virtual displacement corresponding to a unit displacement at coordinate 1, $\delta_1 = 1$, and proceed to apply the Principle of Virtual Work for an elastic system (external work equal to internal virtual work). The virtual work of the external force is simply

$$W_E = m_{12}\delta_1 = m_{12} \quad (10.30)$$

since the only external force undergoing virtual displacement is the inertial force reaction $m_{12}\delta_1$ with $\delta_1 = 1$. The virtual work of the internal forces per unit of length along the beam segment is

$$\delta W_I = f_I(x)N_1(x)$$

or by eq.(10.29)

$$\delta W_I = \bar{m}(x)N_2(x)N_1(x)$$

and for the entire beam

$$W_I = \int_0^L \bar{m}(x)N_2(x)N_1(x)dx \quad (10.31)$$

Equating the external and internal virtual work given, respectively, by eqs.(10.30) and (10.31) results in

$$m_{12} = \int_0^L \bar{m}(x)N_2(x)N_1(x)dx \quad (10.32)$$

which is the expression for the consistent mass coefficient m_{12} . In general, a consistent mass coefficient may be calculated from

$$m_{ij} = \int_0^L \bar{m}(x)N_i(x)N_j(x)dx \quad (10.33)$$

It may be seen from eq. (10.33) that $m_{ij} = m_{ji}$ since the interchange of the subindices only results in an interchange of the order of the factors $N_i(x)$ and $N_j(x)$ under the integral.

In practice, the cubic equations (10.9) are used in calculating the mass coefficients of any straight beam element. For the special case of the beam with uniformly distributed mass, the use of eq. (10.33) gives the following relation between inertial forces and acceleration at the nodal coordinates:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \end{Bmatrix} \quad (10.34)$$

When the mass matrix, eq. (10.34), has been evaluated for each beam element of the structure, the mass matrix for the entire system is assembled by exactly the same procedure (direct method) as described in developing the stiffness matrix for the system. The resulting mass matrix will in general have the same arrangement of nonzero terms as the stiffness matrix.

The dynamic analysis using the lumped mass matrix requires considerably less computational effort than the analysis using the consistent mass method for the following reasons. The lumped mass matrix for the system results in a diagonal mass matrix whereas the consistent mass matrix has many off diagonal terms which are called mass coupling. Also, the lumped mass matrix contains zeros in its main diagonal due to assumed zero rotational inertial forces. This fact permits the elimination by static condensation (Chapter 9) of the rotational degrees of freedom, thus reducing the dimension of the dynamic problem. Nevertheless, the dynamic analysis using the consistent mass matrix gives results that are better approximations to the exact solution compared to the lumped mass method for the same element discretization.

Illustrative Example 10.1.

Determine the lumped mass and the consistent mass matrices for the cantilever beam in Fig. 10.6. Assume uniform mass, $\bar{m} = 420\text{kg/m}$.

Solution:

(a) Lumped Mass Matrix.

The lumped mass at each node of any of the three beam segments, into which the cantilever beam has been divided, is simply half of the mass of the segment. In the present case, the lumped mass at each node is 210 kg as shown in Fig. 10.6. The lumped mass matrix $[M_L]$ for this structure is a diagonal matrix of dimension 6×6 , namely,

$$[M_L] = \begin{bmatrix} 420 & 0 & 420 & 0 & 210 & 0 \\ 0 & 420 & 0 & 0 & 0 & 0 \\ 0 & 0 & 420 & 0 & 0 & 0 \\ 0 & 0 & 0 & 420 & 0 & 0 \\ 0 & 0 & 0 & 0 & 210 & 0 \\ 0 & 0 & 0 & 0 & 0 & 210 \end{bmatrix}$$

(b) Consistent Mass Matrix.

The consistent mass matrix for a uniform beam segment is given by eq. (10.34). The substitution of numerical values for this example $L = 1\text{m}$,

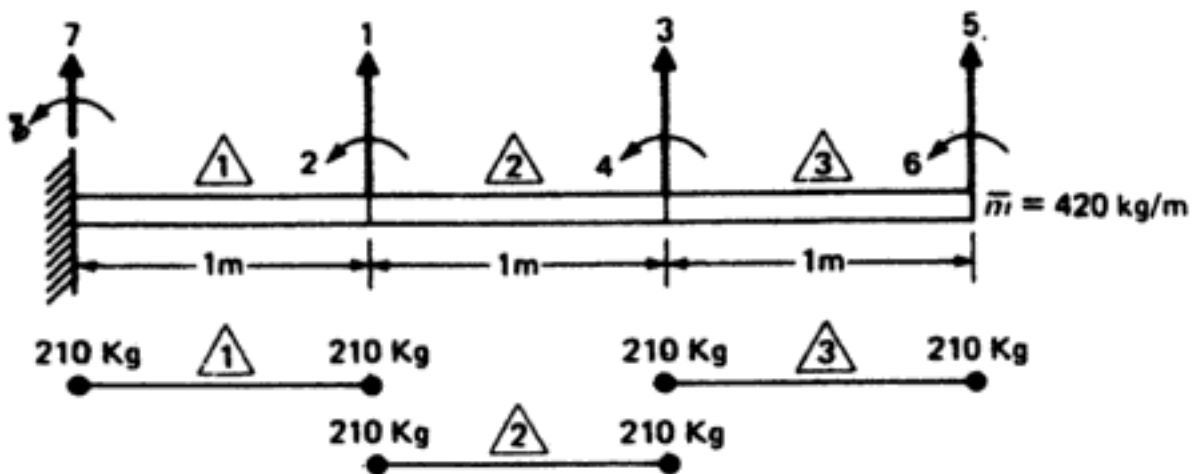


Figure 10.6 Lumped masses for Illustrative Example 10.1

$\bar{m} = 420 \text{ kg/m}$ into eq. (10.34) gives the consistent mass matrix $[m]_e$ for any of the three beam segments as

$$[m]_e = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \quad (a)$$

The assemblage of the system mass matrix from the element mass matrices is carried out in exactly the same manner as the assemblage of the system stiffness matrix from the element stiffness matrices, that is, the element mass matrices are allocated to appropriate entries in the system mass matrix. For the second beam segment, this allocation corresponds to the first four coordinates as indicated above and on the right of eq. (a). For the beam segment $\triangle 3$, the appropriate allocation is 3, 4, 5, 6 and for the beam segment $\triangle 1$, 7, 8, 1, 2 since these are the system nodal coordinates for these beam segments as indicated in Fig. 10.6. The consistent mass matrix $[M]_e$ for this example obtained in this manner is given by

$$[M]_e = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \quad (b)$$

We note that the mass matrix $[M]_e$ is symmetric and also banded as in the case of the stiffness matrix for this system. These facts are of great importance in developing

computer programs for structural analysis, since it is possible to perform the necessary calculations storing in the computer only the diagonal elements and the elements to one of the sides of the main diagonal. The maximum number of nonzero elements in any row which are required to be stored is referred to as the bandwidth of the matrix. For the matrix eq. (b) the bandwidth is equal to four ($NBW = 4$). In this case, it is necessary to store a total of $6 \times 4 = 24$ coefficients, whereas if the square matrix were to be stored, it would require $6 \times 6 = 36$ storage spaces. This economy in storing spaces becomes more dramatic for structures with a large number of nodal coordinates. The dimension of the bandwidth is directly related to the largest difference of the nodal coordinate labels assigned to any of the elements of the structure. Therefore, it is important to number the system nodal coordinates so as to minimize this difference.

10.5 Damping properties

Damping coefficients are defined in a manner entirely parallel to the definition of the stiffness coefficient or the mass coefficient. Specifically, the damping coefficient c_{ij} is defined as the force developed at coordinate i due to a unit velocity at j . If the damping forces distributed in the structure could be determined, the damping coefficients of the various structural elements would then be used in obtaining the damping coefficient corresponding to the system. For example, the damping coefficient c_{ij} for an element might be of the form

$$c_{ij} = \int_0^L c(x) N_i(x) N_j(x) dx \quad (10.35)$$

where $c(x)$ represents the distributed damping coefficient per unit length. If the element damping matrix could be calculated, the damping matrix for the entire structure could be assembled by a superposition process equivalent to the direct stiffness matrix. In practice, the evaluation of the damping property $c(x)$ is impracticable. For this reason, the damping is generally expressed in terms of damping ratios obtained experimentally rather than by a direct evaluation of the damping matrix using eq. (10.35). These damping ratios are evaluated or estimated for each natural mode of vibration. If the explicit expression of the damping matrix $[C]$ is needed, it may be computed from the specified relative damping coefficients by any of the methods described in Chapter 22.

10.6 External loads

When the dynamic loads acting on the structure consist of concentrated forces and moments applied at defined nodal coordinates, the load vector can be written directly.

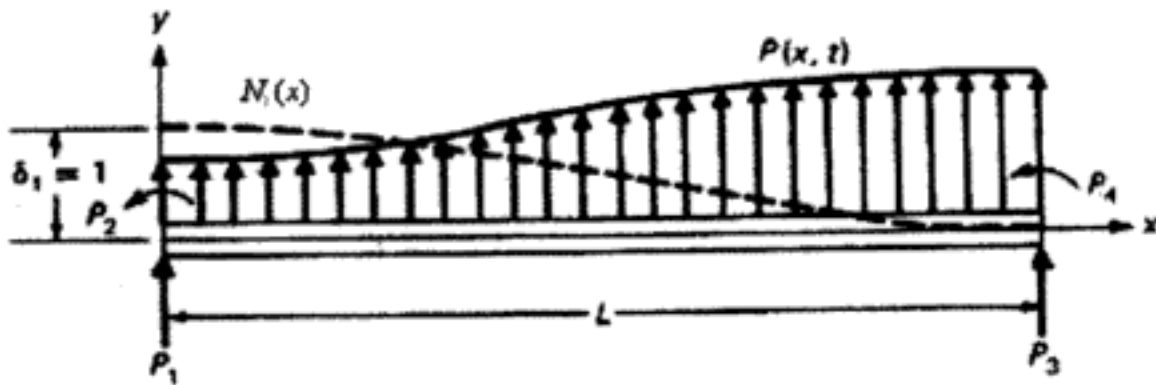


Fig.10.7 Beam element supporting arbitrary distributed load undergoing virtual displacement $\delta_1=1$.

In general, however, loads are applied at points other than nodal coordinates. In addition, the external load may include the action of distributed forces. In this case, the load vector corresponding to the nodal coordinates consists of the equivalent nodal forces. The procedure to determine the equivalent nodal forces which is consistent with the derivation of the stiffness matrix and the consistent mass matrix is to assume the validity of the static deflection functions, eqs. (10.9), for the dynamic problem and to use the principle of virtual work.

Consider the beam element in Fig. 10.7 when subjected to an arbitrary distributed force $p(x, t)$ which is a function of position along the beam as well as a function of time. The equivalent force P_1 at the nodal coordinate 1 may be found by giving a virtual displacement $\delta_1 = 1$ at this coordinate and equating the resulting external work and internal work during this virtual displacement. In this case, the external work is

$$W_E = P_1 \delta_1 = P_1 \quad (10.36)$$

since $\delta_1 = 1$. The internal work per unit of length along the beam is $p(x, t)$ times $N_1(x)$. Then the total internal work is

$$W_I = \int_0^L p(x, t) N_1(x) dx \quad (10.37)$$

Equating external work, eq. (10.36), and internal work, eq. (10.37), gives the equivalent nodal force as

$$P_1(t) = \int_0^L p(x, t) N_1(x) dx \quad (10.38)$$

Thus the element equivalent nodal forces can be expressed in general as

$$P_i(t) = \int_0^L p(x, t) N_i(x) dx \quad (10.39)$$

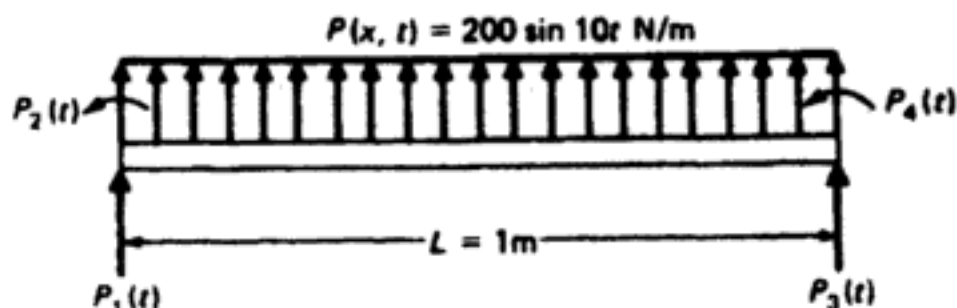


Fig. 10.8 Beam segment with uniformly distributed load, showing the equivalent nodal forces.

Illustrative Example 10.2.

Consider the beam segment in Fig. 10.8 and determine the equivalent nodal forces for a uniform distributed force along the length of the beam given by

$$p(x, t) = 200 \sin 10t \text{ N/m}$$

Solution:

Introduction of numerical values into the displacements functions, eqs. (10.9), and substitution in eq. (10.39) yield

$$P_1(t) = 200 \int_0^1 (1 - 3x^2 + 2x^3) \sin 10t dx = 100 \sin 10t$$

$$P_2(t) = 200 \int_0^1 x(1 - x)^2 \sin 10t dx = 16.67 \sin 10t$$

$$P_3(t) = 200 \int_0^1 (3x^2 - 2x^3) \sin 10t dx = 100 \sin 10t$$

$$P_4(t) = 200 \int_0^1 x^2(x - 1) \sin 10t dx = -16.67 \sin 10t$$

10.7 Geometric stiffness

When a beam element is subjected to an axial force in addition to flexural loading, the stiffness coefficients are modified by the presence of the axial force. The modification corresponding to the stiffness coefficient k_{ij} is known as the geometric stiffness coefficient k_{Gij} , which is defined as the force corresponding to the nodal coordinate i due to a unit displacement at coordinate j and resulting from the axial forces in the structure. These coefficients may be evaluated by application of the principle of virtual work. Consider a beam element as used previously but now subjected to a distributed axial force per unit of length $N(x)$, as depicted in Fig. 10.9(a). In the sketch in Fig. 10.9(b), the beam segment is subjected to a unit rotation of the left end, $\delta_2 = 1$. By definition, the nodal forces due to this displacement are the corresponding geometric stiffness coefficients; for example, k_{G12} is the vertical force at the left end. If we now give to this deformed beam a unit displacement $\delta_1 = 1$, the resulting external work is

$$W_e = k_{G12} \delta_1$$

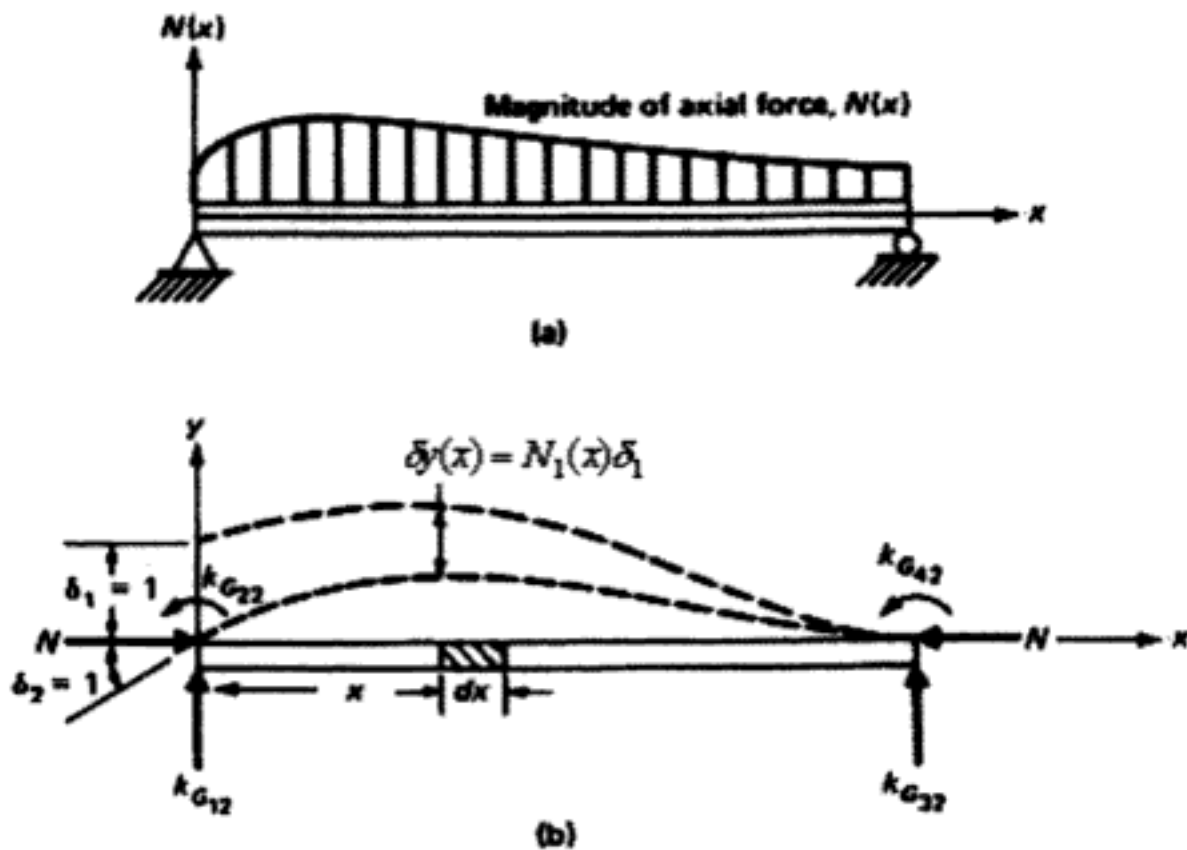


Fig.10.9 (a) Beam element loaded with arbitrary distributed axial force. (b) Beam element acted on by nodal forces resulting from displacement $\delta_2=1$, undergoing a virtual displacement $\delta_1=1$.

or

$$W_e = k_{G12} \quad (10.40)$$

since $\delta_1 = 1$.

The internal work during this virtual displacement is found by considering a differential element of length dx taken from the beam in Fig.10.9.(b) and shown enlarged in Fig.10.10. The work done by the axial force $N(x)$ during the virtual displacement is

$$dW_I = N(x)\delta_e \quad (10.41)$$

where δ_e represents the relative displacement experienced by the normal force $N(x)$ acting on the differential element during the virtual displacement. From Fig. 10.10, by similar triangles (triangles I and II), we have

$$\frac{\delta_e}{dN_1(x)} = \frac{dN_2(x)}{dx}$$

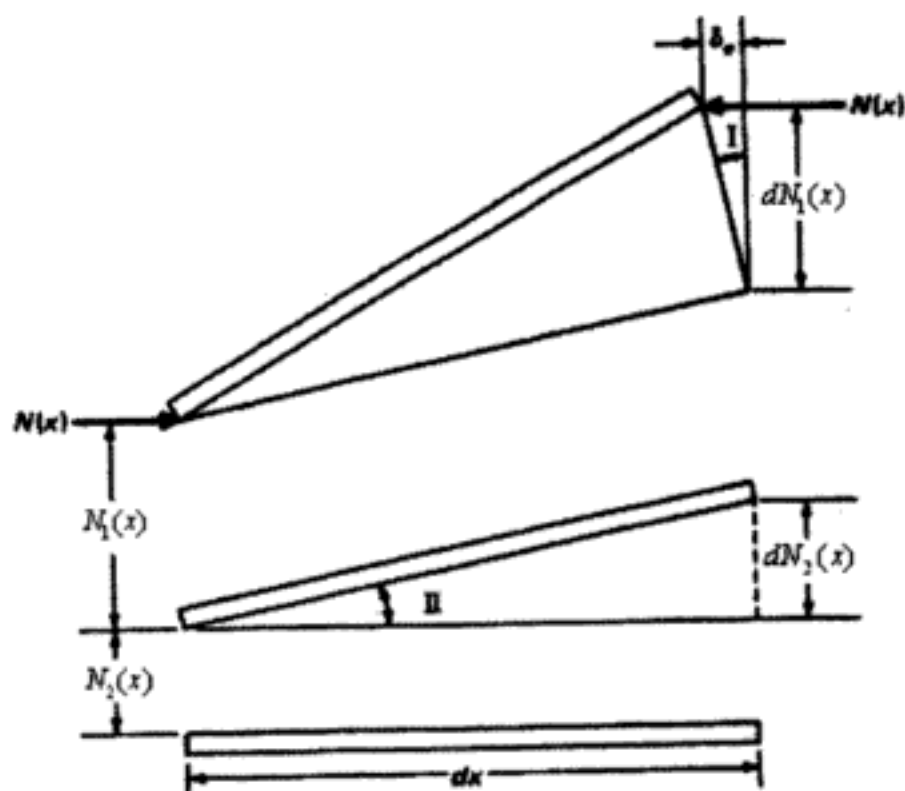


Fig.10.10 Differential segment of deflected beam in Fig.10.9.

or

$$\delta_e = \frac{dN_1(x)}{dx} \cdot \frac{dN_2(x)}{dx} dx$$

$$\delta_e = N'_1(x)N'_2(x)dx$$

in which $N'_1(x)$ and $N'_2(x)$ are the derivatives with respect to x of the corresponding displacement functions defined in eqs. (10.9).

Now, substituting δ_e into eq. (10.41), results in

$$dW_I = N(x)N'_1(x)N'_2(x)dx \quad (10.42)$$

Then integrating this expression and equating the result to the external work, eq. (10.40), finally give

$$k_{G12} = \int_0^L N(x)N'_1(x)N'_2(x)dx \quad (10.43)$$

In general, any geometric stiffness coefficient may be expressed as

$$k_{Gij} = \int_0^L N(x)N'_i(x)N'_j(x)dx \quad (10.44)$$

In the derivation of eq. (10.44), it is assumed that the normal force $N(x)$ is independent of time. When the displacement functions, eqs. (10.9), are used in eq. (10.44) to calculate the geometric stiffness coefficients, the result is called the consistent geometric stiffness matrix. In the special case where the axial force is constant along the length of the beam, use of eqs. (10.44) and (10.9) gives the geometric stiffness matrix equation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{N}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \quad (10.45)$$

The assemblage of the system geometric stiffness matrix can be carried out exactly in the same manner as for the assemblage of the elastic stiffness matrix. The resulting geometric stiffness matrix will have the same configuration as the elastic stiffness matrix. It is customary to define the geometric stiffness matrix for a compressive axial force. In this case, the combined stiffness matrix $[K_c]$ for the structure is given by

$$[K_c] = [K] - [K_G] \quad (10.46)$$

in which $[K]$ is the assembled elastic stiffness matrix for the structure and $[K_G]$ the corresponding geometric stiffness matrix.

Illustrative Example 10.3.

For the cantilever beam in Fig. 10.11, determine the system geometric matrix when an axial force of magnitude 30 newtons is applied at the free end as shown in this figure.

Solution:

The substitution of numerical values into eq. (10.45) for any of the three beam segments in which the cantilever beam has been divided gives the element geometric matrix

$$[K_G] = \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}$$

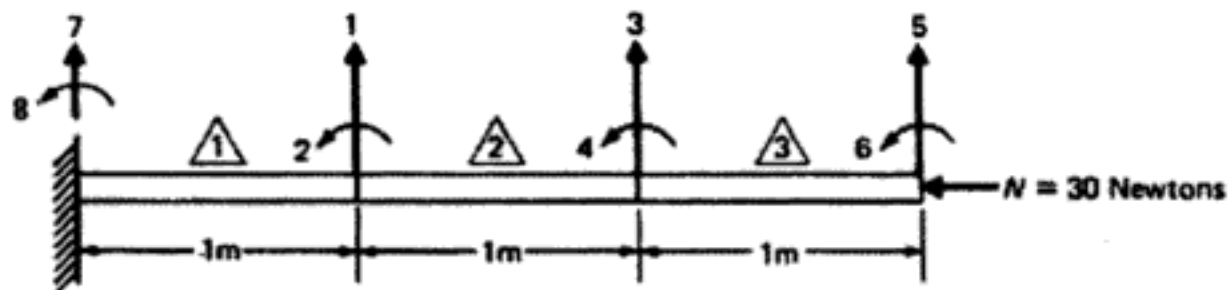


Fig.10.11 Cantilever beam subjected to constant axial force (Illustrative Example 10.3).

since in this example $L = 1\text{m}$ and $N = 30$ newtons. Use of the direct method gives the assembled system geometric matrix as

$$[K_G] = \begin{bmatrix} 72 & 0 & -36 & 3 & 0 & 0 \\ 0 & 8 & -3 & -1 & 0 & 0 \\ -36 & -3 & 72 & 0 & -36 & 3 \\ 3 & -1 & 0 & 8 & -3 & -1 \\ 0 & 0 & -36 & -3 & 36 & -3 \\ 0 & 0 & 3 & -1 & -3 & 4 \end{bmatrix}$$

10.8 Equations of Motion

In the previous sections of this chapter, the distributed properties of a beam and its load were expressed in terms of discrete quantities at the nodal coordinates. The equations of motion as functions of these coordinates may then be established by imposing conditions of dynamic equilibrium between the inertial forces $\{F_I(t)\}$, damping forces $\{F_D(t)\}$, elastic forces $\{F_s(t)\}$, and the external forces $\{F(t)\}$, that is,

$$\{F_I(t)\} + \{F_d(t)\} + \{F_s(t)\} = \{F(t)\} \quad (10.47)$$

For a linear system the forces on the left-hand side of eq. (10.47) are expressed in terms of the system mass matrix, the system damping matrix, and the system stiffness matrix as

$$\{F_I(t)\} = [M]\{\ddot{u}\} \quad (10.48)$$

$$\{F_d(t)\} = [C]\{\dot{u}\} \quad (10.49)$$

$$\{F_s(t)\} = [K]\{u\} \quad (10.50)$$

Substitution of these equations into eq. (10.47) gives the differential equation of motion for a linear system as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\} \quad (10.51)$$

In addition, if the effect of axial forces is considered in the analysis, eq.(10.51) is modified so that

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K_c]\{u\} = \{F(t)\} \quad (10.52)$$

in which

$$[K_c] = [K] - [K_G] \quad (10.53)$$

In practice, the solution of eq. (10.51) or eq. (10.52) is accomplished by standard methods of analysis and the assistance of appropriate computer programs as those described in this and the following chapters. We illustrate these methods by presenting here some simple problems for hand calculation.

Illustrative Example 10.4.

Consider in Fig. 10.12 a uniform beam with the ends fixed against translation or rotation. In preparation for analysis, the beam has been divided into four equal segments. Determine the first three natural frequencies and corresponding modal shapes. Use the lumped mass method in order to simplify the numerical calculations.

Solution:

We begin by numbering sequentially the nodal coordinates starting with the rotational coordinates which have to be condensed in the lumped mass method (no inertial effect associated with rotational coordinates), continuing to number the coordinates associated with translation, and assigning the last numbers 7 through 10 to the fixed nodal coordinates as shown in Fig. 10.12. The stiffness matrix for any of the beam segments for this example is obtained from eq. (10.20) as

$$\begin{array}{cccc}
 4 & 1 & 5 & 2 \\
 7 & 8 & 4 & 1
 \end{array}
 \quad
 [K] = \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{array}{l} 74 \\ 81 \\ 45 \\ 12 \end{array} \quad (10.54)$$

With the aid of the system nodal coordinates for each beam segment written at the top and on the right of the stiffness matrix, eq. (10.54), we proceed to assemble the system stiffness matrix using the direct method. For the beam segment $\triangle 1$, the corresponding labels are 7, 8, 4, 1, we need to transfer only the lowest 2×2 sub-matrix that correspond to labels of rows and columns with indices 4, 1. For the beam segment $\triangle 2$, we translate the 4×4 elements of matrix eq. (10.54) to the system stiffness matrix to rows and columns designated by combination of indices 4, 1, 5, 2 as labeled for this element and so forth for the other two beam segments. The assembled system stiffness matrix obtained in this manner results in the following system stiffness matrix:

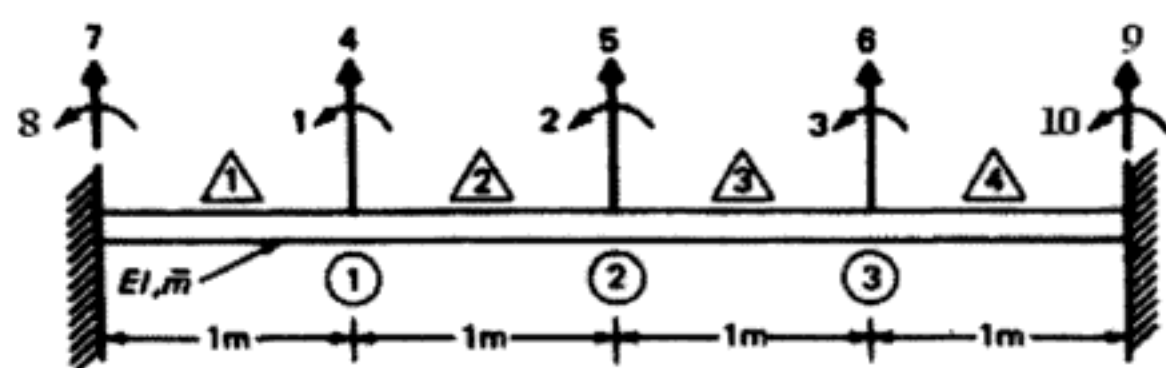


Fig.10.12 Fixed beam divided in four element with indication of system nodal coordinates (Illustrative Example 10.4)

$$[K] = EI \begin{bmatrix} 8 & 2 & 0 & 0 & -6 & 0 \\ 2 & 8 & 2 & 6 & 0 & -6 \\ 0 & 2 & 8 & 0 & 6 & 0 \\ 0 & 6 & 0 & 24 & -12 & 0 \\ -6 & 0 & 6 & -12 & 24 & -12 \\ 0 & -6 & 0 & 0 & -12 & 24 \end{bmatrix} \quad (10.55)$$

The reduction or condensation of eq. (10.55) is accomplished as explained in Chapter 9 by simply performing the Gauss-Jordan elimination of the first three rows since, in this case, we should condense these first three coordinates. This elimination reduces eq. (10.55) to the following matrix:

$$[A] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.214 & -0.750 & 0.214 \\ 0 & 1 & 0 & 0.858 & 0 & -0.858 \\ 0 & 0 & 1 & -0.214 & 0.750 & 0.214 \\ \hline 0 & 0 & 0 & 18.86EI & -12.00EI & 5.14EI \\ 0 & 0 & 0 & -12.00EI & 15.00EI & -12.00EI \\ 0 & 0 & 0 & 5.14EI & -12.00EI & 18.86EI \end{array} \right] \quad (10.56)$$

Comparison of eq. (10.56) in partition form with eq. (9.11) permits the identification of the reduced stiffness matrix $[\bar{k}]$ and the transformation matrix $[\bar{T}]$, so that

$$[\bar{K}] = EI \begin{bmatrix} 18.86 & -12.00 & 5.14 \\ -12.00 & 15.00 & -12.00 \\ 5.14 & -12.00 & 18.86 \end{bmatrix} \quad (10.57)$$

and

$$[\bar{T}] = EI \begin{bmatrix} 0.214 & 0.750 & -0.214 \\ -0.858 & 0 & 0.858 \\ 0.214 & -0.750 & -0.214 \end{bmatrix} \quad (10.58)$$

The general transformation matrix, eq. (9.9), is then

$$[T] = \begin{bmatrix} 0.214 & 0.750 & -0.214 \\ -0.858 & 0 & 0.858 \\ 0.214 & -0.750 & -0.214 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (10.59)$$

As an exercise, the reader may check eq. (9.10) for this example by simply performing the matrix multiplications

$$[\bar{K}] = [T]^T [K] [T]$$

The lumped mass method applied to this example gives three equal masses of magnitude m at each of the three translatory coordinates as indicated in Fig. 10.13. Therefore, the reduced lumped mass matrix is

$$[\bar{M}] = \bar{m} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (10.60)$$

The natural frequencies and modal shapes are found by solving the undamped free vibration problem, that is,

$$[\bar{M}]\{\ddot{u}\} + [\bar{K}]\{u\} = \{0\} \quad (10.61)$$

Assuming the harmonic solution ($\{u\} = \{a\} \sin \omega t$), we obtain

$$([\bar{K}] - \omega^2 [\bar{M}])\{a\} = \{0\} \quad (10.62)$$

requiring for a nontrivial solution that the determinant

$$|[\bar{K}] - \omega^2 [\bar{M}]| = 0 \quad (10.63)$$

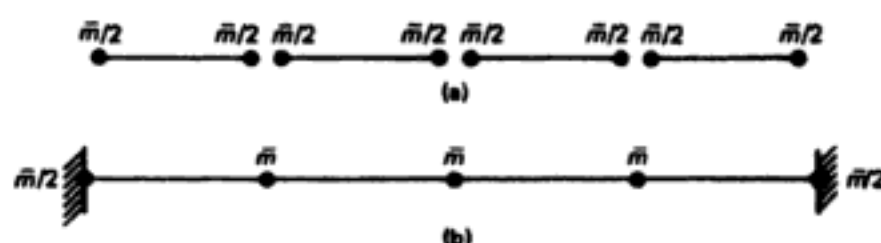


Fig.10.13 (a) Lumped masses for uniform beam segment.
(b) Lumped masses at system nodal coordinates.

Substitution in this last equation $[\bar{k}]$ and $[\bar{M}]$, respectively, from eqs. (10.57) and (10.60) yields

$$\begin{vmatrix} 18.86 - \lambda & -12.00 & 5.14 \\ -12.00 & 15.00 - \lambda & -12.00 \\ 5.14 & -12.00 & 18.86 - \lambda \end{vmatrix} = 0 \quad (10.64)$$

in which

$$\lambda = \frac{\bar{m} \omega^2}{EI} \quad (10.65)$$

The roots of the cubic equation resulting from the expansion of the determinant in eq.(10.64) are found to be

$$\lambda_1 = 1.943, \lambda_2 = 13.720, \lambda_3 = 37.057 \quad (10.66)$$

Then from eq. (10.65)

$$\begin{aligned} \omega_1 &= 1.393 \sqrt{EI / \bar{m}} \\ \omega_2 &= 3.704 \sqrt{EI / \bar{m}} \\ \omega_3 &= 6.087 \sqrt{EI / \bar{m}} \end{aligned} \quad (10.67)$$

The first three natural frequencies for a uniform fixed beam of length $L = 4m$ determined by the exact analysis (Chapter 17) are

$$\begin{aligned} \omega_1(\text{exact}) &= 1.398 \sqrt{EI / \bar{m}} \\ \omega_2(\text{exact}) &= 3.854 \sqrt{EI / \bar{m}} \\ \omega_3(\text{exact}) &= 7.556 \sqrt{EI / \bar{m}} \end{aligned} \quad (10.68)$$

The first two natural frequencies determined using the three-degrees-of-freedom reduced system compare very well with the exact values. A practical rule in condensing degrees of freedom is to condense those nodal coordinates which have the least inertial effect; in this problem these are the rotational coordinates.

The modal shapes are determined by solving two of the equations in eq.(10.62) after substituting successively values of $\omega_1, \omega_2, \omega_3$ from eqs. (10.67) and conveniently setting the first coordinate for each modal shape be equal to one. The resulting modal shapes are

$$\{a\}_1 = \begin{bmatrix} 1.00 \\ 1.84 \\ 1.00 \end{bmatrix}, \{a\}_2 = \begin{bmatrix} 1.00 \\ 0 \\ -1.00 \end{bmatrix}, \{a\}_3 = \begin{bmatrix} 1.00 \\ -1.08 \\ 1.00 \end{bmatrix} \quad (10.69)$$

The normalized modal shapes which are obtained by division of the elements of eqs. (10.69) by corresponding values of $\sqrt{\sum m_i a_{ij}^2}$ ($m_i = 100$ kg for this example) are arranged in the columns of the modal matrix are

$$[\Phi]_p = \begin{bmatrix} 0.0431 & 0.0707 & 0.0562 \\ 0.0793 & 0 & -0.0607 \\ 0.0431 & -0.0707 & 0.0562 \end{bmatrix} \quad (10.70a)$$

The modal shapes in terms of the six original coordinates are then obtained by eq. (9.8) as

$$[\Phi] = [T] [\Phi]_p$$

$$[\Phi] = \begin{bmatrix} 0.0594 & 0.0301 & -0.0457 \\ 0 & -0.1212 & 0 \\ -0.0594 & 0.0301 & 0.0457 \\ 0.0431 & 0.0707 & 0.0562 \\ 0.0793 & 0 & -0.0607 \\ 0.0431 & 0.0707 & 0.0562 \end{bmatrix} \quad (10.70b)$$

Illustrative Example 10.5.

Determine the steady-state response for the beam of Illustrative Example 10.4 when subjected to the harmonic forces

$$F_1 = F_{01} \sin \omega t$$

$$F_2 = F_{02} \sin \omega t$$

$$F_3 = F_{03} \sin \omega t$$

applied, respectively, at joints, 1, 2 and 3 of the beam in Fig. 10.12. Neglect damping and let $EI = 10^8$ ($N \cdot m^2$), $\bar{m} = 100$ kg/m, $\omega = 3000$ rad/sec, $F_{01} = 2000$ N, $F_{02} = 3000$ N, and $F_{03} = 1000$ N.

Solution:

The modal equations (uncoupled equations) can readily be written using the results of Illustrative Example 10.4. In general the n th normal equation is given by

$$\ddot{z}_n + \omega_n^2 z_n = P_n \sin \omega t \quad (10.71)$$

in which

$$P_n = \sum_{i=1}^N \phi_{in} F_{0i}$$

The steady-state solution of eq. (10.71) is given by eq. (3.4) as

$$z_n = Z_n \sin \omega t = \frac{P_n \sin \omega t}{\omega_n^2 - \omega^2} \quad (10.72)$$

The calculations required in eq. (10.72) are conveniently arranged in Table 10.1. The deflections at the modal coordinates are found from the transformation

$$\begin{aligned} \{u\} &= [\Phi]\{z\} \\ \{U\} &= [\Phi]\{Z\} \end{aligned} \quad (10.73)$$

in which $[\Phi]$ is the modal matrix and

$$\begin{aligned} \{u\} &= \{U\} \sin \omega t \\ \{z\} &= \{Z\} \sin \omega t \end{aligned}$$

The substitution into eq. (10.73) of the modal matrix from eq. (10.70b) and the values of $\{Z\}$ from the last column of Table 10.1 gives the amplitudes at nodal coordinates as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{bmatrix} 0.0594 & 0.0301 & -0.0457 \\ 0 & -0.1212 & 0 \\ -0.0594 & 0.0301 & 0.0457 \\ 0.0431 & 0.0707 & 0.0562 \\ 0.0793 & 0 & -0.0607 \\ 0.0431 & -0.0707 & 0.0562 \end{bmatrix} \begin{Bmatrix} -5.200 \\ 1.500 \\ -0.048 \end{Bmatrix} 10^{-5}$$

or

$$U_1 = -2.616 \times 10^{-6} \text{ rad}$$

$$U_2 = -1.818 \times 10^{-6} \text{ rad}$$

$$U_3 = 3.524 \times 10^{-6} \text{ rad}$$

$$U_4 = -1.207 \times 10^{-6} \text{ m}$$

$$U_5 = -4.094 \times 10^{-6} \text{ m}$$

$$U_6 = -3.329 \times 10^{-6} \text{ m}$$

TABLE 10.1 Modal Response for Illustrative Example 10.5

Mode (n)	ω_n^2	$P_n = \sum_i \phi_{ni} F_{oi}$	$Z_n = \frac{P_n}{\omega_n^2 - \omega^2}$
1	1.943×10^6	367.2	-5.200×10^{-5}
2	13.720×10^6	70.7	1.5000×10^{-5}
3	37.057×10^6	-13.5	-0.048×10^{-5}

Therefore, the motion at these nodal coordinates is given by

$$\begin{aligned}
 u_1(t) &= -2.616 \times 10^{-6} \sin 3000t \text{ rad} \\
 u_2(t) &= -1.818 \times 10^{-6} \sin 3000t \text{ rad} \\
 u_3(t) &= -3.524 \times 10^{-6} \sin 3000t \text{ rad} \\
 u_4(t) &= -1.207 \times 10^{-6} \sin 3000t \text{ m} \\
 u_5(t) &= -4.094 \times 10^{-6} \sin 3000t \text{ m} \\
 u_6(t) &= -3.329 \times 10^{-6} \sin 3000t \text{ m}
 \end{aligned} \tag{10.74}$$

The minus sign in the resulting amplitudes of motion simply indicates that the motion is 180° out of phase with the applied harmonic forces.

10.9 Element Forces At Nodal Coordinates

The central problem to be solved using the dynamic stiffness method is to determine the displacements at the nodal coordinates. Once these displacements have been determined, it is a simple matter of substituting the appropriate displacements in the condition of dynamic equilibrium for each element to calculate the forces at the nodal coordinates. The nodal element forces $\{P\}$ may be obtained by adding the inertial force $\{P_I\}$, the damping force $\{P_D\}$, the elastic force $\{P_S\}$, and subtracting the nodal equivalent forces $\{P_E\}$.

Therefore, we may write:

$$\{P\} = \{P_I\} + \{P_D\} + \{P_S\} - \{P_E\}$$

or

$$\{P\} = [m]\{\ddot{\delta}\} + [c]\{\dot{\delta}\} + [k]\{\delta\} - \{P_E\} \tag{10.75}$$

In eq. (10.75) the inertial force, the damping force, and the elastic force are, respectively,

$$\begin{aligned}
 \{P_I\} &= [m]\{\ddot{\delta}\} \\
 \{P_D\} &= [c]\{\dot{\delta}\} \\
 \{P_S\} &= [k]\{\delta\}
 \end{aligned} \tag{10.76}$$

where $[m]$ is the element mass matrix; $[c]$ the element damping matrix; $[k]$ the element stiffness matrix; and $\{\ddot{\delta}\}$, $\{\dot{\delta}\}$, $\{\delta\}$ represents, respectively, the displacement, velocity, and acceleration vectors at the nodal coordinates of the element.

Although, the determination of element end-forces at nodal coordinates is given by eq. (10.75), commercial computer programs only use the forces due to the elastic displacements, that is, the element end forces are calculated as

$$\{P\} = [k]\{\delta\}$$

This simplified approach may be justified by the mathematical model in which the inertial forces, damping forces, and external forces are assumed to be acting directly at the nodes. Therefore, it is important in the process of discretization to select beam elements relatively short as to approximate the distributed properties with discrete concentrated values at the nodes between the elements. The determination of the element nodal forces is illustrated in the following example.

Illustrative Example 10.6.

Determine the element nodal forces and moments for the four beam elements of Illustrative Example 10.5.

Solution:

Since in this example damping is neglected and there are no external forces applied on the beam elements except those at the nodal coordinates, eq. (10.75) reduces to

$$\{P\} = [m]\{\ddot{\delta}\} + [k]\{\delta\} \quad (10.77)$$

The displacement functions for the six nodal coordinates of the beam in Fig. 10.12 are given by eq. (10.74). These displacements are certainly also the displacements of the element nodal coordinates. The identification for this example of corresponding nodal coordinates between beam segments and system nodal coordinates is

$$\{\delta\}_1 = \begin{bmatrix} 0 \\ 0 \\ u_4 \\ u_1 \end{bmatrix}, \{\delta\}_2 = \begin{bmatrix} u_4 \\ u_1 \\ u_5 \\ u_2 \end{bmatrix}, \{\delta\}_3 = \begin{bmatrix} u_5 \\ u_2 \\ u_6 \\ u_3 \end{bmatrix}, \{\delta\}_4 = \begin{bmatrix} u_6 \\ u_3 \\ 0 \\ 0 \end{bmatrix} \quad (10.78)$$

where $\{\delta\}_i$ is the vector of nodal displacement for i beam segment. The substitution of appropriate quantities into eq. (10.77) for the first beam element results in

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{\bar{m}}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_4 \\ u_1 \end{Bmatrix} (-\omega^2) \sin \omega t + EI \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_4 \\ u_1 \end{Bmatrix} \sin \omega t$$

To complete this example, we substitute the numerical values of $\omega = 3000$ rad/sec, $m = 100$ kg/m, and $EI = 10^8$ (N. m²) and obtain

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} 7.5 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 7.5 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -120.7 \\ -261.6 \end{Bmatrix} \sin 3000t$$

which then gives

$$\begin{aligned} P_1 &= -121.2 \sin 3000t \text{ N} \\ P_2 &= 201.0 \sin 3000t \text{ N.m} \\ P_3 &= 664.3 \sin 3000t \text{ N} \\ P_4 &= -322.2 \sin 3000t \text{ N.m} \end{aligned}$$

The nodal element forces found in this manner for all of the four beam segments in this example are given in Table 10.2.

The results in Table 10.2 may be used to check that the dynamic conditions of equilibrium are satisfied in each beam segment. The free body diagrams of the four elements of this beam are shown in Fig. 10.14 with inclusion of nodal inertial forces. These forces are computed by multiplying the nodal mass by the corresponding nodal acceleration.

TABLE 10.2 Element Nodal Forces (Amplitudes) for Illustrative Example 10.6

Force	Beam Segment				Units
	1	2	3	4	
P_1	-121.2	1347.2	1947.9	-382.3	N
P_2	201.0	322.2	-481.4	-587.0	N.m
P_3	664.3	1038.3	1392.4	1880.4	N
P_4	-322.2	481.4	587.0	-1292.6	N.m

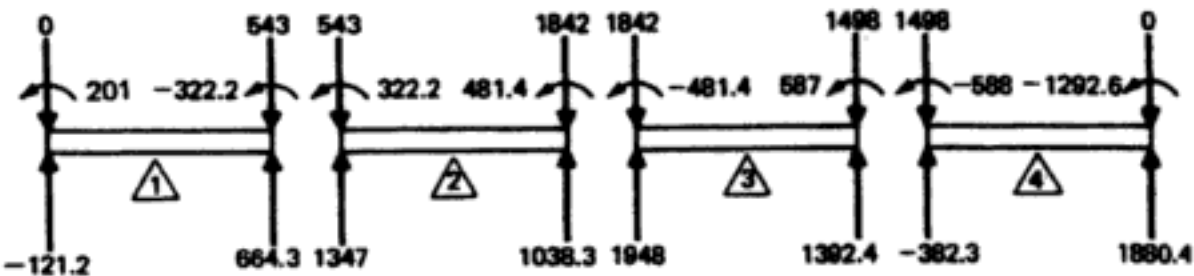


Fig.10.14 Dynamic equilibrium for beam element of Illustrative Example 10.6.

10.10 Program 13—Modeling Structures as Beams

The computer program presented in this section calculates the stiffness and mass matrices for a beam and stores the coefficients of these matrices in a file, for future use. Since the stiffness and mass matrices are symmetric, only the upper triangular portion of these matrices needs to be stored. The program also stores in another file, named by the user, the general information on the beam. The information stored in these files is needed by programs which perform dynamic analysis such as calculation of natural frequencies or determination of the response of the structure subjected to external excitation.

Illustrative Example 10.7.

Determine the stiffness and mass matrices for the uniform fixed-ended beam shown in Fig. 10.15. The following are the properties of the beam:

- Length: $L = 200$ in
- Cross-section moment of inertia: $I = 100$ in⁴
- Modulus of elasticity: $E = 6.58 E6$ lb/in²
- Mass per unit length: $m = 0.10$ (lb. sec²/in/in)

Solution:

The beam is divided in four elements of equal length as shown in Fig. 10.15. This division results in a total of five joints of which two are fixed, thus giving a total of four fixed nodal coordinates (the two ends of the beam are fixed for translation and for rotation).

Input Data and Output Results

PROGRAM 13: MODELING BEAMS	DATA FILE: D13
GENERAL DATA:	
NUMBER OF JOINTS	NJ=5
NUMBER OF BEAM ELEMENTS	NE=4
NUMBER OF FIXED COORDINATES	NC=4
MODULUS OF ELASTICITY	E=6580000

JOINT DATA:

JOINT #	X-COORDINATE	CONCENTRATED JOINT MASS
1	0.00	0.00
2	50.00	0.00
3	100.00	0.00
4	150.00	0.00
5	200.00	0.00

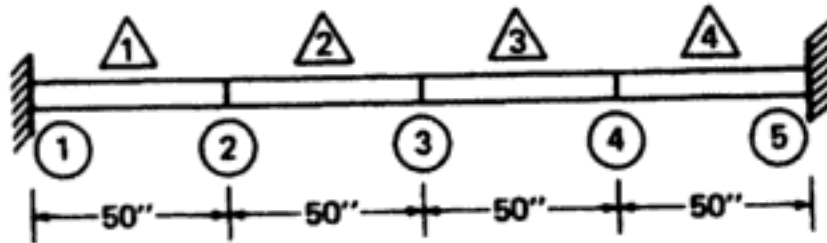


Fig.10.15 Fixed beam modeled for Illustrative Example 10.7.

BEAM ELEMENT DATA:

(NEGATIVE JOINT NUMBERS TO RELEASED JOINTS)

BEAM-ELEMENT#, LEFT JOINT#, RIGHT JOINT#, MASS/LENGTH, MOMENT OF INERTIA

1	1	2	0.10	100.00
2	2	3	0.10	100.00
3	3	4	0.10	100.00
4	4	5	0.10	100.00

FIXED COORDINATE DATA:

JOINT #, DIRECTION # (VERTICAL=1, ROTATION=2)

1	1
1	2
5	1
5	2

OUTPUT RESULTS:

SYSTEM STIFFNESS MATRIX

1.2634E+05	0.0000E+00	-6.3168E+04	1.5792E+06	0.0000E+00	0.0000E+00
0.0000E+00	1.0528E+08	-1.5792E+06	2.6320E+07	0.0000E+00	0.0000E+00
-6.3168E+04	-1.5792E+06	1.2634E+05	0.0000E+00	-6.3168E+04	1.5792E+06
-1.5792E+06	2.6320E+07	0.0000E+00	1.0528E+08	-1.5792E+06	2.6320E+07
0.0000E+00	0.0000E+00	-6.3168E+04	-1.5792E+06	1.2634E+05	0.0000E+00
0.0000E+00	0.0000E+00	1.5792E+06	2.6320E+07	0.0000E+00	1.0528E+08

SYSTEM MASS MATRIX

3.7143E+00	0.0000E+00	6.4296E+04	-7.7381E+00	0.0000E+00	0.0000E+00
0.0000E+00	2.3810E+02	7.7381E+00	-8.9286E+01	0.0000E+00	0.0000E+00
6.4286E+01	7.7381E+00	3.7143E+00	0.0000E+00	6.4286E-01	7.7381E+00
-7.7381E+00	-8.9286E+01	0.0000E+00	2.3810E+02	7.7381E+00	-8.9286E+01
0.0000E+00	0.0000E+00	6.4286E-01	7.7381E+00	3.7143E+00	0.0000E+00
0.0000E+00	0.0000E+00	-7.7381E+00	-8.9286E+01	0.0000E+00	2.3810E+02

Illustrative Example 10.8.

For the fixed beam modeled in Illustrative Example 10.7, determine: (a) the natural frequencies and modal shapes, (b) the response to concentrated force of 10000 lb suddenly applied at the center of the beam for 0.1 sec and removed linearly as shown in Fig. 10.16. Use time step of integration $\Delta t = 0.01$ sec.

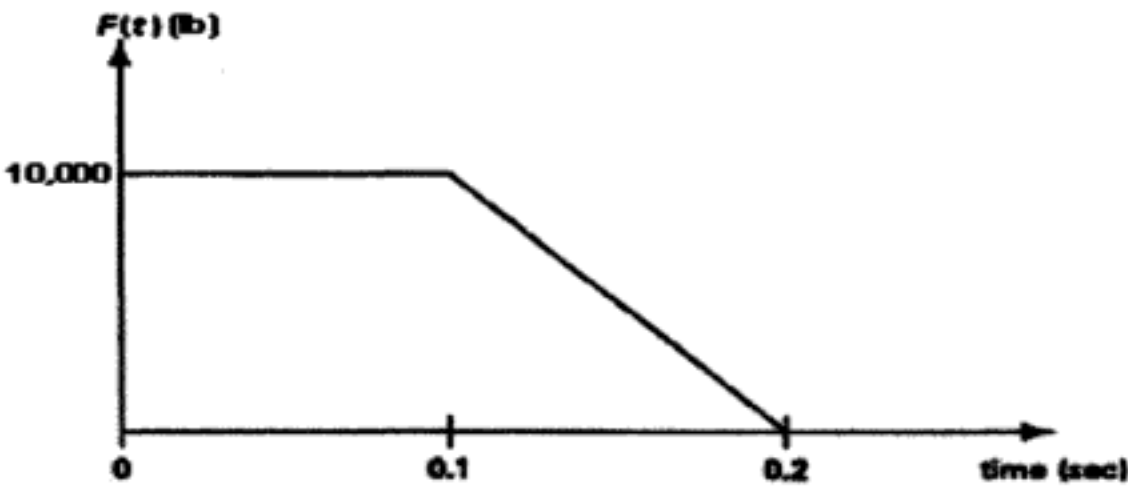


Fig.10.16 Force applied to the beam of Illustrative Example 10.8.

Solution:

(a) Natural Frequencies

During the execution of Program 13, to model the beam in Problem 10.7, a file named "SK" has been created to store the data necessary to calculate natural frequencies and corresponding modal shapes (Program 8.) The execution of Program 8 gives the following output:

Output Results

PROGRAM 8: NATURAL FREQUENCIES

DATA FILE: SK

OUTPUT RESULTS

EIGENVALUES:

2.064E+03 1.593E+04 6.271E+04 2.245E+05 6.104E+05 1.594E+06

NATURAL FREQUENCIES (C.P.S.):

7.23 20.09 39.86 75.40 124.71 200.93

EIGENVECTORS BY ROWS:

0.19352	0.00542	0.35606	- 0.00000	0.19352	- 0.00542
- 0.32920	- 0.00355	0.00000	0.01299	0.32920	- 0.00355
0.32051	- 0.00754	- 0.33013	- 0.00000	0.32051	0.00754
- 0.15157	0.02892	- 0.00000	- 0.03233	0.06682	0.04865
0.18579	0.04758	0.00000	0.07737	- 0.18579	0.04758

(b) Calculation Of The Response.

The execution of Program 8 "Jacobi" creates a file named "EA" in which the eigenvalues ω_i^2 and the eigenvectors $\{\Phi\}_i$ are stored for subsequent use by other programs such as Program 9 which uses the modal superposition method to calculate the response of the structure to externally applied forces. The following is the computer output for execution of Program 9:

Input Data and Output Results

PROGRAM 9: MODAL

DATA FILE: D10.88

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM	ND=6
NUMBER OF EXTERNAL FORCES	NF= 1
TIME STEP OF INTEGRATION	H=.01
GRAVITATIONAL INDEX	G=0
PRINT TIME HISTORY NPRT=1; ONLY MAX. VALUES NPRT=0	NPRT=0

FORCE #,	COORD. # OF APPLIED FORCE	NUM. OF POINTS DEFINING THE FORCE
1	3	3

EXCITATION FUNCTION FOR FORCE #1:

TIME	EXCITATION
0.0000	10000.00
0.1000	10000.00
0.2000	0.00

MODAL DAMPING RATIOS:

0	0	0	0	0	0
---	---	---	---	---	---

MAXIMUM RESPONSE:

COORD.	MAX. DISPL.	MAX. VELOC.	MAX. ACC.
1	0.654	17.13	1740.37
2	0.019	0.70	177.90
3	1.243	30.15	3345.84
4	0.000	0.00	0.00
5	0.654	17.13	1740.37
6	0.019	0.70	177.90

10.11 Dynamic Analysis Of Beams Using SAP2000**Illustrative Example 10.9**

For the fixed beam shown in Fig. 10.17 use SAP2000 to determine: (a) Natural frequencies and modal shapes, (b) the response to the concentrated force of 10,000 lb suddenly applied at the center of the beam for 0.1 sec and removed linearly as shown in Fig. 10.18. Use time step of integration $\Delta t = 0.01$ sec.

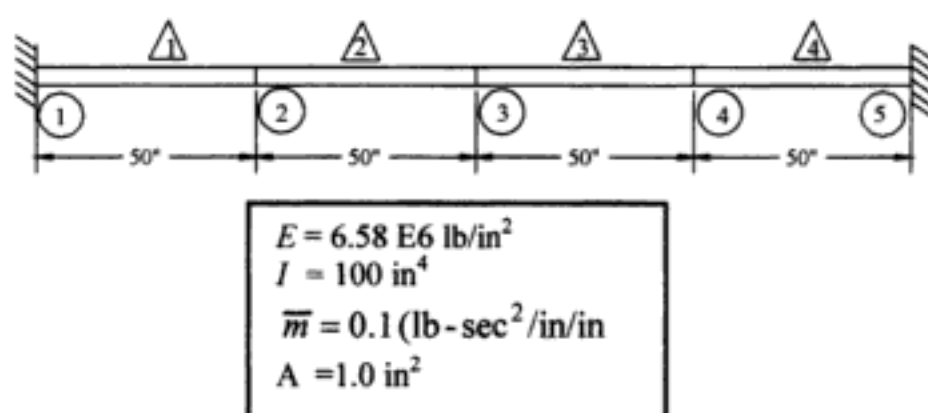


Fig. 10.17 Fixed beam modeled for Illustrative Example 10.9.

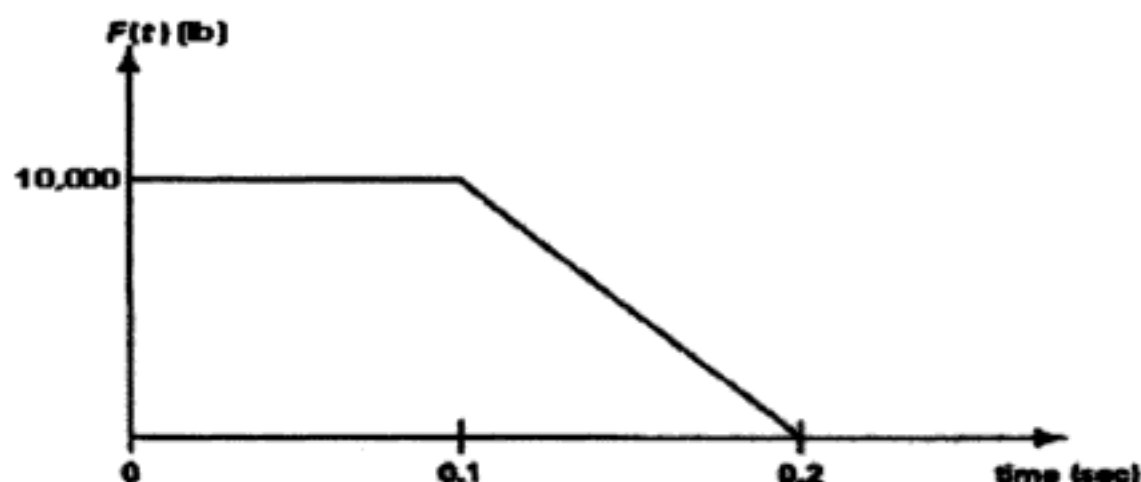


Fig. 10.18 Force applied to the beam of Illustrative Example 10.9.

Solution:

Analytical Model: This beam is divided into four segments of 50 in length as shown in Fig. 10.17. The following commands are implemented in SAP2000.

Begin: Open SAP2000e and select "lb-in" from the drop down menu in the lower right hand corner of the screen.

FILE>NEW MODEL FROM TEMPLATE

Under "New Model Templates" click on the button marked with a beam and enter:

Number of Spans = 4

Span length = 50

Check Restraints and Gridlines. Then OK.

Maximize the X-Z Plane window and center the drawing in the screen using the PAN icon.

Labels: VIEW>SET ELEMENTS

Under Joints check Labels and Restraints

Under Frames check Labels.

Accept all other default selections. Then OK.

Use PAN icon to center drawing once again.

Restraints: Select Joints ① and ⑤ by clicking on them.
 ASSIGN>JOINT>RESTRAINTS
 Restrain in all directions. Then OK.

Select joints ②, ③ and ④ by clicking on them.
 ASSIGN>JOINT>RESTRAINTS
 No restraints in any direction (free).

Materials: DEFINE>MATERIALS
 Under Materials select OTHER
 Click to Modify/Show Material
 Material Name = OTHER
 Mass per Unit Volume = 0.1
 Weight per Unit Volume = 0
 Modulus of Elasticity = 6.58E6
 Accept all other default entries. Then OK, OK.

Define: DEFINE>FRAME SECTIONS
 Click on the second drop-down menu, Add/Wide Flange and
 select ADD GENERAL
 Enter: Moment of Inertia about 3 Axis = 100
 Shear Area in 2 Direction = 0
 Area = 1.0
 Accept all other default entries. Then OK.

In General Section window change Section Name to GENERAL
 Under Material use drop down menu to select OTHER. Then OK, OK.

Assign: Select frame elements 1, 2, 3 and 4 by clicking on them
 ASSIGN>FRAME>SECTIONS
 In Frame Sections select GENERAL. Then OK.

Define: DEFINE>STATIC LOAD CASES
 Load = LOAD1
 Type = LIVE
 Self-Weight Multiplier = 0
 Click on "Change Load". Then OK.

Time History Function: DEFINE>TIME HISTORY FUNCTIONS
 Click on Add New Function
 Add the following values and click "add" after each row is entered.

Time	Value
0	-10000
0.1	-10000
0.2	0
0.5	0

Then OK, OK.

Time History Cases:

DEFINE>TIME HISTORY CASES
Click to Add New History
Accept HIST1 as history case name
Enter: Analysis Type = Linear
Number of Time Steps = 50
Output Time Step Size = 0.01
Click on envelopes
Under Load Assignments
Load = LOAD 1
Function = FUNC1
Scale Factor = 1
Arrival Time = 0
Then Click "ADD". Then OK, OK.

Joint Load:

Mark Joint ① by clicking on it.
ASSIGN>JOINT STATIC LOADS>FORCES
Load Case Name = LOAD1
Force Global Z = 1, all others 0. Click Add to existing loads.
Then OK.

Set Option:

ANALYZE>SET OPTIONS
Under Available DOF's (Degrees of Freedom) check only UZ and RY.
Click on "Dynamic Analysis" and on "Set Dynamic Parameters"
Enter: Number of Modes = 6
Then OK, OK.

Analyze:

ANALYZE>RUN
Enter filename EXD 10.9. Then SAVE.
When analysis is complete click on OK.
Select X-Z plane to view deformed shape of beam.

Output Table Mode: DISPLAY>OUTPUT TABLE MODE

Select LOAD1 Load Case and HIST1 History (hold down CTRL key to select both) then OK.

Print Input Tables: FILE>PRINT INPUT TABLES

CLICK "Print to File" and accept all other default values.
Then OK.

(The edited Output Tables of Illustrative Example 10.9 are in Table 10.3)

Table 10.3 Edited Input Data for Illustrative Example 10.9.**-STATIC LOAD CASES**

STATIC	CASE	SELF-WT
CASE	TYPE	FACTOR
LOAD1	LIVE	0.0000

TIME HISTORY CASES

HISTORY	HISTORY	NUMBER OF	TIME STEP
CASE	TYPE	TIME STEPS	INCREMENT
HIST1	LINEAR	50	0.01000

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-100.00000	0.00000	0.00000	1 1 1 1 1
2	50.00000	0.00000	0.00000	0 0 0 0 0
3	0.00000	0.00000	0.00000	0 0 0 0 0
4	50.00000	0.00000	0.00000	0 0 0 0 0
5	100.00000	0.00000	0.00000	1 1 1 1 1

FRAME ELEMENT DATA

FRAME	JNT-1	JNT-2	SECTION	ANG	RELS	SEGMT	R1	R2	LENGTH
1	1	2	GENERAL	0.000	000000	4	0.000	0.000	50.000
2	2	3	GENERAL	0.000	000000	4	0.000	0.000	50.000
3	3	4	GENERAL	0.000	000000	4	0.000	0.000	50.000
4	4	5	GENERAL	0.000	000000	4	0.000	0.000	50.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
3	0.000	0.000	1.000	0.000	0.000	0.000

Plot Displacement Function: DISPLAY>SHOW TIME HISTORY TRACES

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 3

Vector Type = Displacement

Component = UZ.

Mode Number = Include All. Then OK, OK.

Under List of Functions click on Joint3,

Click on ADD to move Joint3 to the Plot Functions column

Axis Labels

Horizontal = Time (sec)

Vertical = Displacement Joint 3 (in)

Click on "Display" to view the Displacement function

Max = 1.241(in)

Min = -0.45 (in)

Enter: FILE>PRINT GRAPHICS to obtain a printed version of the plot.

Alternatively, a table of the displacement function may be obtained by entering:

FILE>PRINT TABLES TO FILE

The values in this table may be plotted using Excel or similar software (Fig. 10.19 contains the plot of the displacement function of Joint ③)

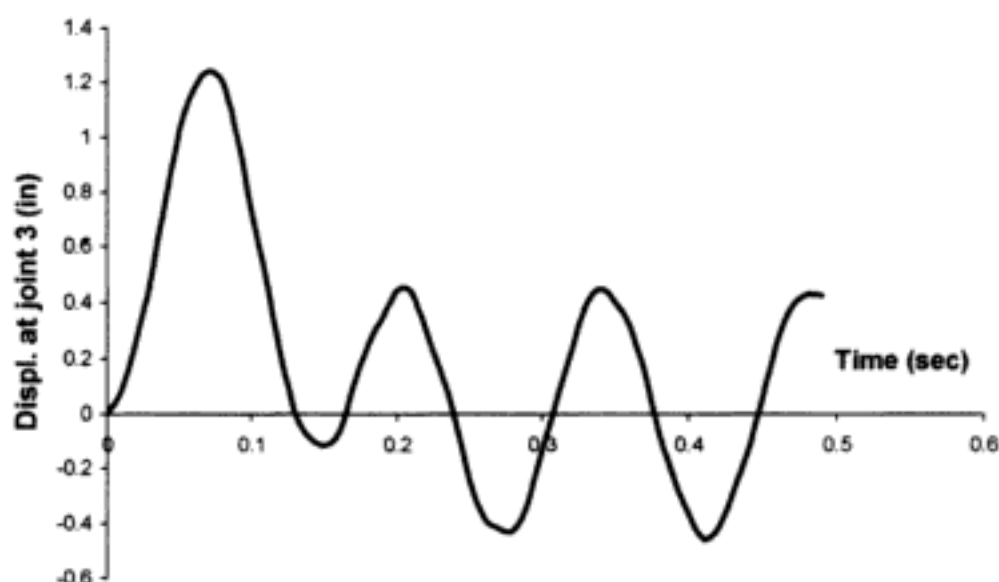


Fig. 10.19 Displacement function at joint ③ of Illustrative Example 10.9.

Note: Displacement values may be viewed on the screen by moving the cursor along the curve shown on the screen.

Plot Frame Element Force Function:

Enter: DISPLAY>SHOW TIME HISTORY TRACES

Click on "Define Functions"

Click to "Add Frame Element Forces"

Element ID = 2

Station = 1

Component = Moment 3-3

Accept all other default settings. Then OK, OK.

Click once on "Frame2" and then ADD to move "Frame2" to the Plot Function column

Highlight "Joint3" and click remove "Joint3" back to List of Functions column

Axis Labels

Horizontal = Time (sec)

Vertical = Bending Moment at Element 2 (lb-in)

Click Display to view the function plot.

Max = 7.284E04

Min = -7.115E04

FILE>PRINT GRAPHICS

To obtain a table of function values in the Time History Function window:

Enter:

FILE>PRINT TABLES TO FILE

Alternatively, The values of this function may be plotted using Excel or similar software.

(Fig. 10.20 contains a plot of the bending moment time function for element 3 of Illustrative Example 10.7))

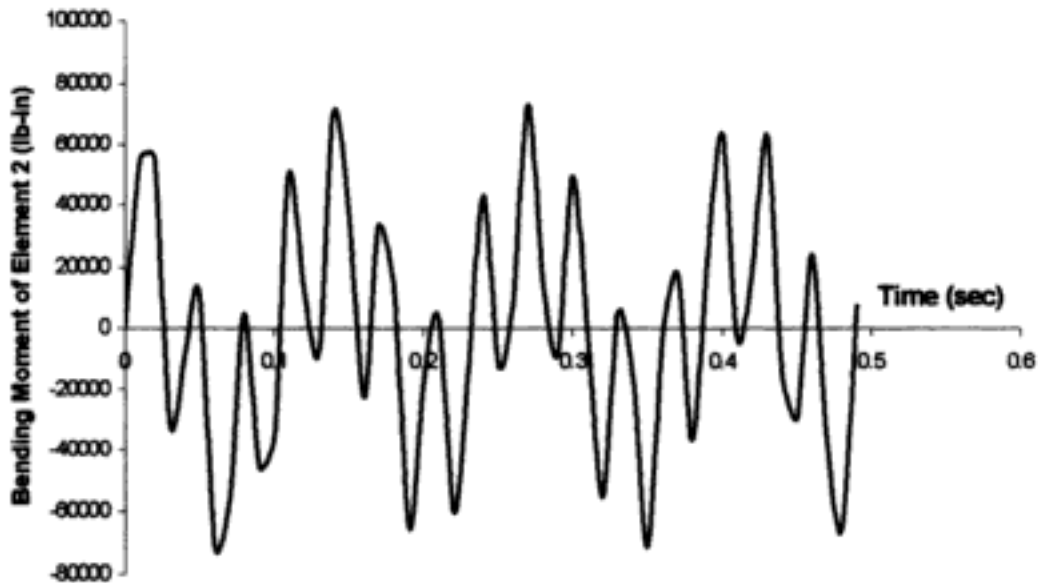


Fig. 10.20 Bending moment time function for element 2 of Illustrative Example 10.9.

Note: Numerical values at any time of the bending moment time function may be read on the screen by moving the cursor along the plot line shown on the screen.

Natural Periods and Frequencies:

Using Notepad or other word editor and open the file:

C:/SAP2000e/EXD 10.9.OUT

The Modal Period and Frequency table may be found in this file.

(Table 10.4 contains the table of the edited Natural Period and Frequency values of Illustrative Example 10.9)

Table 10.4 Natural Periods and Frequencies for Illustrative Example 10.7.

MODAL PERIODS AND FREQUENCIES

MODE	PERIOD (SEC)	FREQUENCY (CYC/SEC)	FREQUENCY (RAD/SEC)	EIGENVALUE (RAD/SEC)**2
1	0.138924	7.198200	45.227623	2045.538
2	0.052290	19.124041	120.159893	14438.400
3	0.031811	31.436088	197.518764	39013.662

Mode Shapes: DISPLAY>SHOW MODE SHAPE

Modal Shape 1 : Click "Wire Shadow" and "Cubic Curve"

Scale Factor = 1. Then OK.

FILE>PRINT GRAPHICS

(Figure 10.21 reproduces plot of Modal Shape 1 for Illustrative Example 10.9)

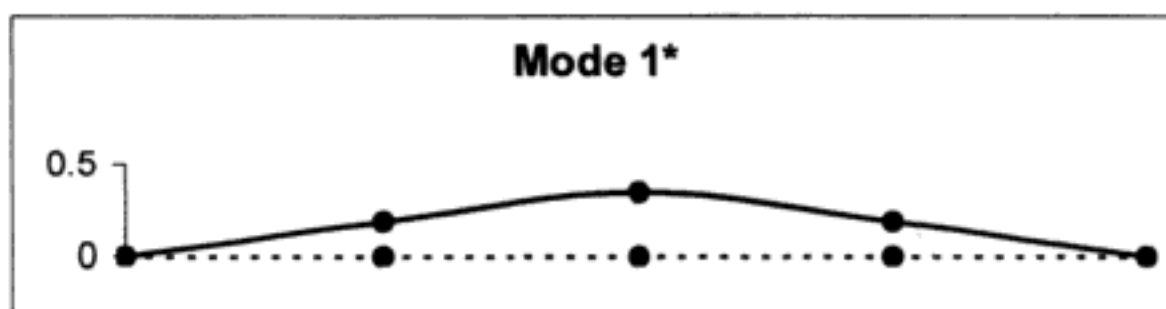


Fig. 10.21 Modal shape 1 of Illustrative Example 10.9.

Mode shape 2: Click on the directional arrow in the lower right-hand corner of the screen to select Mode 2.

Scale Factor = 1. Then OK.

(Figure 10.22 reproduces plot of Modal Shape 2 for Illustrative Example 10.9)

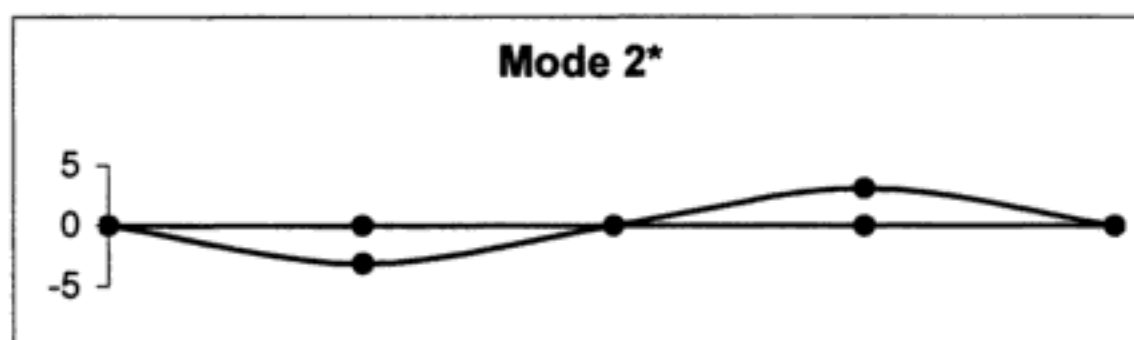


Fig. 10.22 Modal Shape 2 of Illustrative Example 10.9

Modal shape 3: Click on the directional arrow in the lower right-hand corner of the screen to select Mode 3

Scale Factor = 1. Then OK.

(Figure 10.23 reproduces plot of Modal Shape 3 for Illustrative Example 10.9)

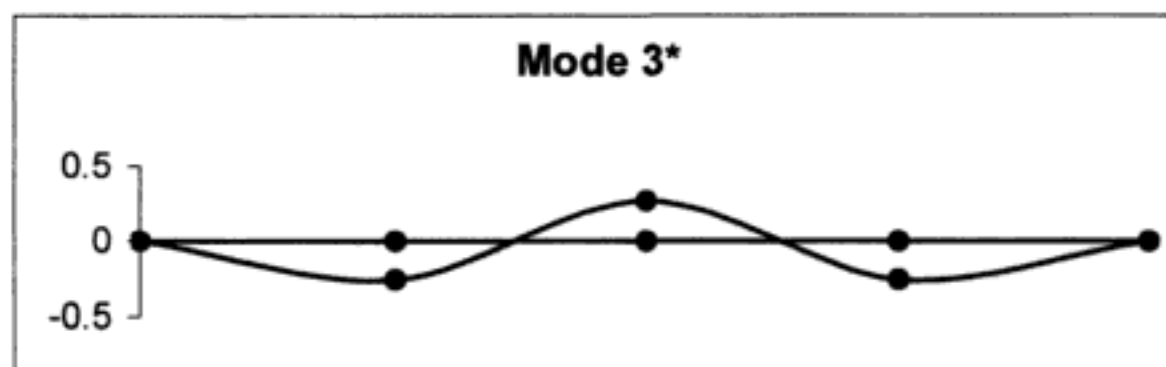


Fig. 10.23 Modal Shape 2 of Illustrative Example 10.9.

* Numerical values for a mode shape in the Z direction and rotation around the Y axis may be read on the screen by right-clicking on a node in the modal shape shown on the screen.

10.12 Summary

In this chapter, we have formulated the dynamic equations for beams in reference to a discrete number of nodal coordinates. These coordinates are translation and rotational displacements defined at joints between structural elements of the beam (beam segments). The dynamic equations for a linear system are conveniently written in matrix notation as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\}$$

where $F(t)$ is the force vector and $[M]$, $[C]$, and $[K]$ are, respectively, the mass, damping, and stiffness matrices of the structure. These matrices are assembled by the appropriate superposition (direct method) of the matrices determined for each beam segment of the structure.

The solution of the dynamic equations (i.e., the response) of a linear system may be found by the modal superposition method. This method requires the determination of the natural frequencies ω_n ($n = 1, 2, 3, \dots, N$) and the corresponding normal modes which are conveniently written as the columns of the modal matrix $[\Phi]$. The linear transformation $\{u\} = [\Phi]\{z\}$ applied to the dynamic equations reduces them to a set of independent equations (uncoupled equations) of the form

$$\ddot{z}_n + 2\omega_n \xi_n \dot{z}_n + \omega_n^2 z_n = P_n(t)$$

where ξ_n is the modal damping ratio and $P_n(t) = \sum_i \phi_{ni} F_i(t)$ is the modal force.

An alternate method for determining the response of linear systems (also valid for nonlinear systems) is the numerical integration of the dynamic equations. Chapter 20 presents the step-by-step linear acceleration method (with a modification introduced by Wilson) which is an efficient method for solving the dynamic equations.

An instructional computer program is also described in this chapter for the dynamic analysis of beams. This program performs the task of assembling and storing in a file the stiffness and mass matrices of the system. These matrices are subsequently used by other programs to calculate natural frequencies or the response of the beam to external excitation.

10.13 Problems

Problem 10.1

A uniform beam of flexural stiffness $EI = 10^9$ (lb. in²) and length 300 in has one end fixed and the other simply supported. Determine the system stiffness matrix considering three beam segments and the free nodal coordinates indicated in Fig.P10.1.

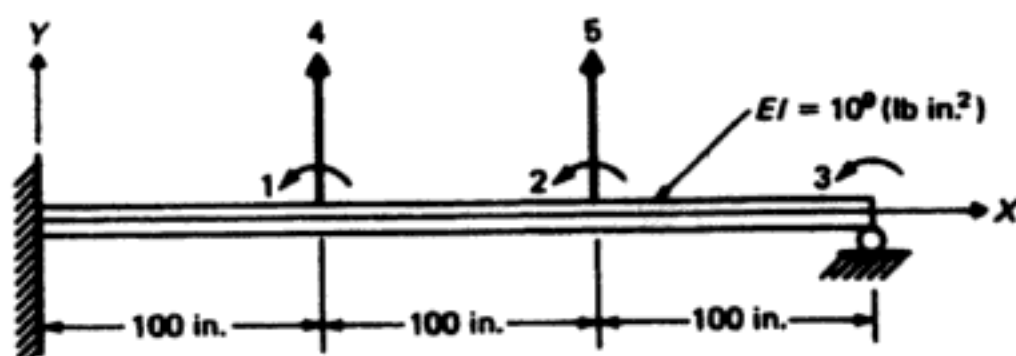


Fig. P10.1.

Problem 10.2

Assuming that the beam shown in Fig. P10.1 carries a uniform weight per unit length $q = 3.86$ lb/in, determine the system mass matrix corresponding to the lumped mass formulation.

Problem 10.3

Determine the system mass matrix for Problem 10.2 using the consistent mass method.

Problem 10.4

For the beam in Problems 10.1 and 10.2, use static condensation to eliminate the massless degrees of freedom and determine the transformation matrix and the reduced stiffness and mass matrices.

Problem 10.5

For the beam in Problems 10.1 and 10.3 use static condensation to eliminate the rotational degrees of freedom. Find the transformation matrix and the reduced stiffness and mass matrices.

Problem 10.6

Determine the natural frequencies and corresponding normal modes using the reduced stiffness and mass matrices obtained in Problem 10.4.

Problem 10.7

Determine the natural frequencies and corresponding normal modes using the reduced stiffness and mass matrices obtained in Problem 10.3.

Problem 10.8

Determine the geometric stiffness matrix for the beam of Problem 10.1 when it is subjected to a constant tensile force of 10,000 lb as shown in Fig. P10.8.

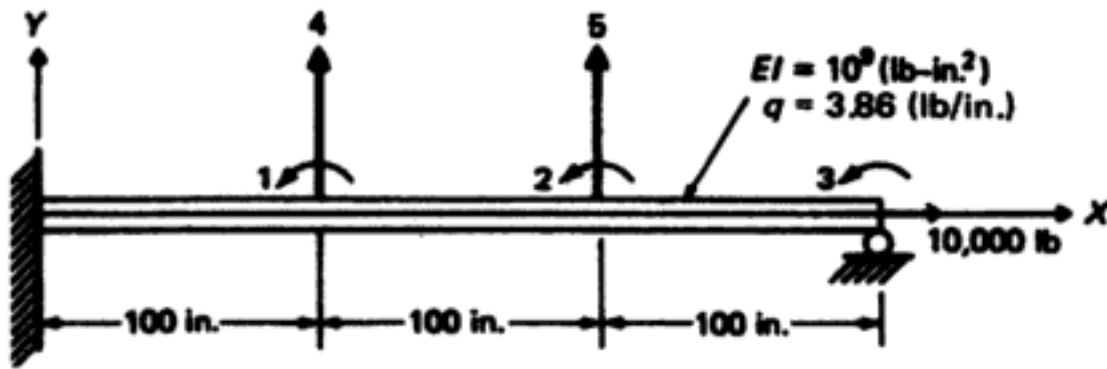


Fig. P10.8.

Problem 10.9

Perform static condensation to reduce the geometric stiffness matrix obtained in Problem 10.8. Eliminate the rotational coordinates.

Problem 10.10

Use results from Problem 10.4 and 10.9 and determine the natural frequencies and corresponding normal modes for the beam shown in Fig. P10.8.

Problem 10.11 Use the results of Problems 10.5 and 10.9 and determine the natural frequencies and corresponding normal modes for the beam shown in Fig. P10.8.

Problem 10.12

Determine the stiffness matrix for a beam segment in which the flexural stiffness has a linear variation as shown in Fig. P10.12.

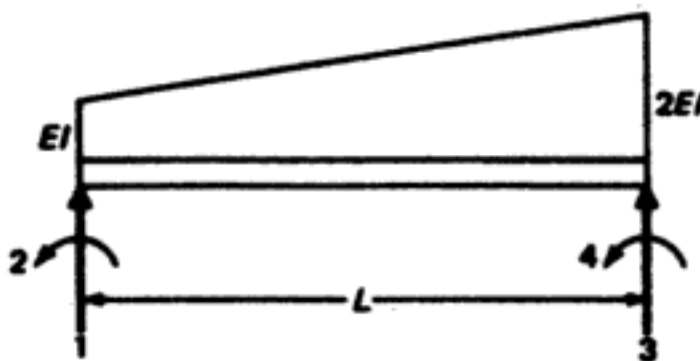


Fig. P10.12.

Problem 10.13

Determine the lumped mass matrix for a beam segment in which the mass has a linear distribution as shown in Fig. P10.13.

Problem 10.14

Determine the consistent mass matrix for the beam segment shown in Fig. P10.13.

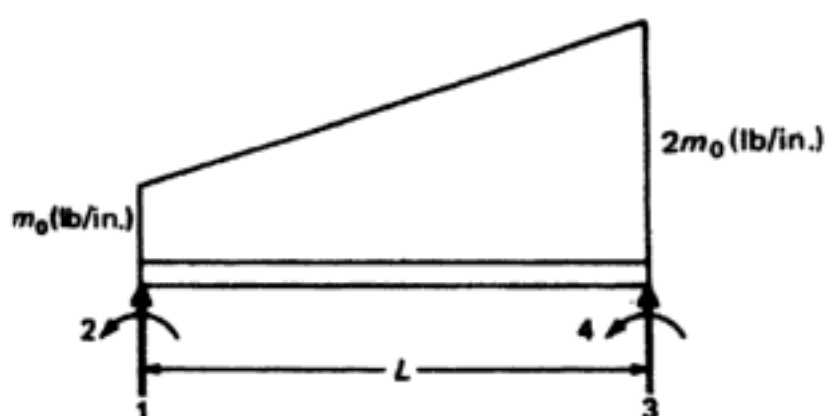


Fig. P10.13.

Problem 10.15

The uniform beam shown in Fig. P10.15 is subjected to a constant force of 5000 lb suddenly applied along the nodal coordinate 4. Use the results obtained in Problem 10.6 to determine the response by the modal superposition method. (Use only the two modes left by the static condensation.)

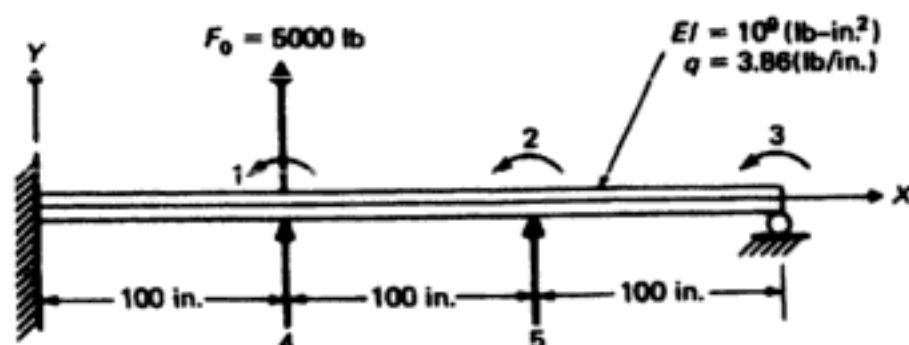


Fig. P10.15

Problem 10.16

Solve Problem 10.15 using the results obtained in Problem 10.7 which are based on the consistent mass formulation.

Problem 10.17

Solve Problem 10.15 using the results obtained in Problem 10.9 which includes the effect of the axial force in the stiffness of the system.

Problem 10.18

Determine the steady-state response for the beam shown in Fig. P10.18 which is acted upon by a harmonic force $F(t) = 5000 \sin 30t$ (lb) as shown in the figure. Eliminate the rotational coordinates by static condensation (Problem 10.5).

Neglect damping in the system.

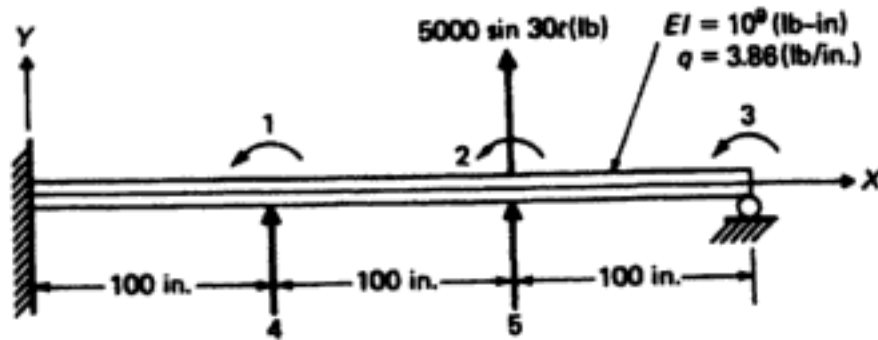


Fig. P10.18.

Problem 10.19

Determine the natural frequencies and corresponding normal modes for the beam shown in Fig. P10.1: (a) condensing the three rotational nodal coordinates; (b) no condensing coordinates. (Use the consistent mass method.)

Problem 10.20

Determine the response for the beam shown in Fig. P10.15. Neglect damping. Do not condense coordinates; (b) condense the three rotational coordinates.

Problem 10.21

Repeat Problem 10.20 assuming 10% damping in all the modes.

Problem 10.22

Determine the steady-state response for the beam shown in Fig. P10.18 when subjected to a harmonic force as shown in the figure. Do not condense coordinates, and neglect damping in the system.

Problem 10.23

Repeat Problem 10.22 assuming that the damping is proportional to stiffness of the system where the constant of proportionality $\alpha_0 = 0.2$.

Problem 10.24

Solve Problem 10.22 after condensing the three rotational coordinates.

Problem 10.25

Repeat Problem 10.24 assuming 15% damping in all the modes.

Problem 10.26

Determine the steady-state response for the beam shown in Fig. P10.15. Do not condense coordinates, and neglect damping in the system.

11 Dynamic Analysis of Plane Frames

The dynamic analysis using the stiffness matrix method for structures modeled as beams was presented in Chapter 10. This method of analysis when applied to beams requires the calculation of element matrices (stiffness, mass, and damping matrices), the assemblage from these matrices of the corresponding system matrices, the formation of the force vector, and the solution of the resultant equations of motion. These equations, as we have seen, may be solved in general by the modal superposition method or by numerical integration of the differential equations of motion. In this chapter and in the following chapters, the dynamic analysis of structures modeled as frames is presented.

We begin in the present chapter with the analysis of structures modeled as plane frames and with the loads acting in the plane of the frame. The dynamic analysis of such structures requires the inclusion of the axial effects in the stiffness and mass matrices. It also requires a coordinate transformation of the nodal coordinates from element or local coordinates to system or global coordinates. Except for the consideration of axial effects and the need to transform these coordinates, the dynamic analysis by the stiffness method when applied to frames is identical to the analysis of beams as discussed in Chapter 10.

11.1 Element Stiffness Matrix for Axial Effects

The inclusion of axial forces in the stiffness matrix of a flexural beam element requires the determination of the stiffness coefficients for axial loads. To derive the stiffness matrix for an axially loaded member, consider in Fig. 11.1 a beam segment

acted on by the axial forces P_1 and P_2 producing axial displacements δ_1 and δ_2 at the nodes of the element. For a prismatic and uniform beam segment of length L and cross-sectional A , it is relatively simple to obtain the stiffness relation for axial effects by the application of Hooke's law. In relation to the beam shown in Fig. 11.1, the displacements δ_1 produced by the force P_1 acting at node 1 while node 2 is maintained fixed ($\delta_2 = 0$) is given by

$$\delta_1 = \frac{P_1 L}{AE} \quad (11.1)$$

From eq. (11.1) and the definition of the stiffness coefficient k_{11} (force at node 1 to produce a unit displacement, $\delta_1 = 1$), we obtain

$$k_{11} = \frac{P_1}{\delta_1} = \frac{AE}{L} \quad (11.2a)$$

The equilibrium of the beam segment acted upon by the force k_{11} requires a force k_{21} at the other end of equal magnitude but in opposite direction, namely

$$k_{21} = -k_{11} = -\frac{AE}{L} \quad (11.2b)$$

Analogously, the other stiffness coefficients due to a unit displacement at node 2 ($\delta_2 = 1$) are:

$$k_{22} = \frac{AE}{L} \quad (11.2c)$$

and

$$k_{12} = -\frac{AE}{L} \quad (11.2d)$$

The stiffness coefficients as given by eqs. (11.2) are the elements of the stiffness matrix relating axial forces and displacements for a prismatic beam segment, that is,

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \quad (11.3)$$

The stiffness matrix corresponding to the nodal coordinates for the beam segment shown in Fig. 11.2 is obtained by combining in a single matrix the stiffness matrix for axial effects, eq. (11.3), and the stiffness matrix for flexural effects, eq. (10.20). The matrix resulting from this combination relates the forces P_i and the displacements δ_i at the coordinates indicated in Fig. 11.2 as

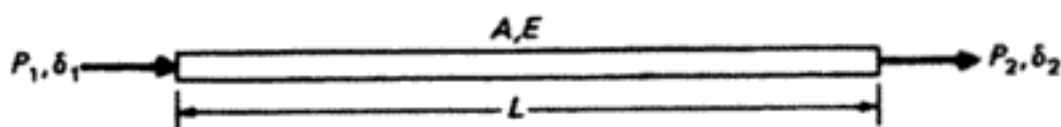


Fig. 11.1 Beam element showing nodal axial loads P_1 , P_2 , and corresponding nodal displacements δ_1 , δ_2 .

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} AL^2/I & & & & & \\ 0 & 12 & & & & \\ 0 & 6L & 4L^2 & & & \\ -AL^2/I & 0 & 0 & AL^2/I & & \\ 0 & -12 & -6L & 0 & 12 & \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \quad (11.4)$$

or, in concise notation,

$$\{P\} = [K]\{\delta\} \quad (11.5)$$

11.2 Element Mass Matrix for Axial Effects

The determination of mass influence coefficients for axial effects of a beam element may be carried out by any of two methods indicated previously for the flexural effects: (1) the lumped mass method and (2) the consistent mass method. In the lumped mass method, the mass allocation to the nodes of the beam element is found from static considerations, which for a uniform beam gives half of the total mass of the beam segment allocated at each node. Then for a prismatic beam segment, the relation between modal axial forces and modal accelerations is given by

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \quad (11.6)$$

where \bar{m} is the mass per unit of length. The combination of the flexural lumped mass coefficient and axial mass coefficients gives, in reference to the modal coordinates in Fig. 11.2, the following diagonal matrix:

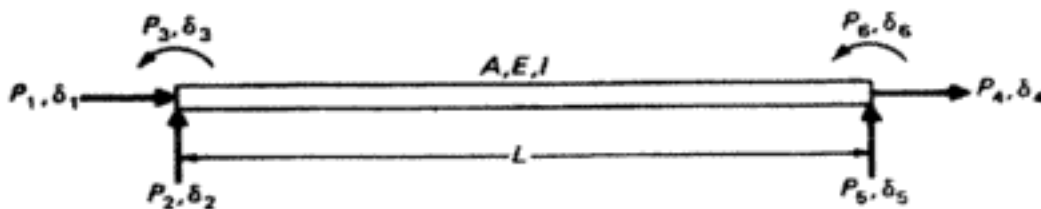


Fig. 11.2 Beam element showing flexural and axial nodal forces and displacements.

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{\delta}_5 \\ \ddot{\delta}_6 \end{Bmatrix} \quad (11.7)$$

To calculate the coefficients for the consistent mass matrix, it is necessary first to determine the displacement functions corresponding to a unit axial displacement at one of the nodal coordinates. Consider in Fig. 11.3 an axial unit displacement $\delta_1 = 1$ of node 1 while the other node 2 is kept fixed so that $\delta_2 = 0$. If $u = u(x)$ is the displacement at section x , the displacement at section $x + dx$ will be $u + du$. It is evident then that the element dx in the new position has changed in length by an amount du , and thus, the strain is du/dx . Since from Hooke's law, the ratio of stress to strain is equal to the modulus of elasticity E , we can write

$$\frac{du}{dx} = \frac{P(x)}{AE} \quad (11.8)$$

Integration with respect to x yields

$$u = \frac{P(x)}{AE}x + C \quad (11.9)$$

in which C is a constant of integration. Introducing the boundary conditions, $u = 1$ at $x = 0$ and $u = 0$ at $x = L$, we obtain the displacement function $u_1(x)$ corresponding to a unit displacement $\delta_1 = 1$ as

$$u_1(x) = 1 - \frac{x}{L} \quad (11.10)$$

Analogously, the displacement function $u_2(x)$ corresponding to a unit displacement $\delta_2 = 1$ is found to be:

$$u_2(x) = \frac{x}{L} \quad (11.11)$$

The application of the principle of virtual work results in a general expression for the calculation of the stiffness coefficients. For example, consider the beam in Fig. 11.3, which is in equilibrium with the forces $P_1 = k_{11}$ and $P_2 = k_{21}$ at its two ends.

Assume that a virtual displacement $\delta_2 = 1$ takes place. Then, according to the principle of virtual work, during this virtual displacement, the work of the external and internal forces are equal. The external force k_{21} performs the work

or

$$W_E = k_{21} \quad (11.12)$$

since $\delta_2 = 1$. The internal force $P(x)$ at any section x is obtained from eq. (11.8) as

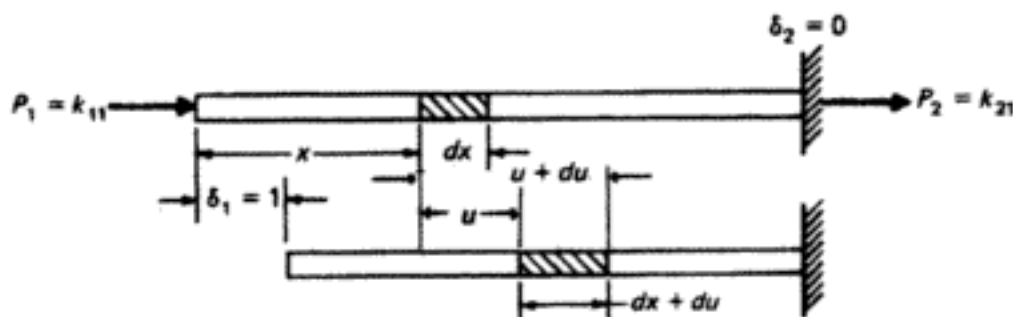


Fig. 11.3 Displacement at node 1 ($\delta_1 = 1$) of a beam element.

$$P(x) = AEu_1'(x) \quad (11.13)$$

in which $u_1'(x) = du_1/dx$.

The incremental displacement δu_2 of element dx during this virtual displacement may be expressed as

$$du_2 = \frac{du_2}{dx} dx \quad (11.14)$$

Hence the internal work for element dx is obtained from eqs. (11.13) and (11.14) as

$$dW_I = AEu_1'(x)u_2'(x)dx$$

and for the beam segment of length L

$$W_I = \int_0^L AEu_1'(x)u_2'(x)dx \quad (11.15)$$

Finally, equating $W_E = W_I$ from eqs. (11.12) and (11.15) gives the stiffness coefficient

$$k_{21} = \int_0^L AEu_1'(x)u_2'(x)dx \quad (11.16)$$

In general, the stiffness coefficient k_{ij} for axial effects may be obtained from

$$k_{ij} = \int_0^L AEu_i'(x)u_j'(x)dx \quad (11.17)$$

Using eq. (11.17), the reader may check the results obtained in eq. (11.3) for a uniform beam. However, eq. (11.17) could as well be used for nonuniform beams in which the cross-sectional area A would in general be a function of x . In practice, the same displacement $u_1(x)$ and $u_2(x)$ obtained for a uniform beam, are also used in eq. (11.17) for a non uniform member. The displacement $u(x,t)$ at any section x of a beam element due to dynamic nodal displacements, $\delta_1(t)$ and $\delta_2(t)$, is obtained by superposition. Hence

$$u(x,t) = u_1(x)\delta_1(t) + u_2(x)\delta_2(t) \quad (11.18)$$

in which $u_1(x)$ and $u_2(x)$ are given by eqs. (11.10) and (11.11).

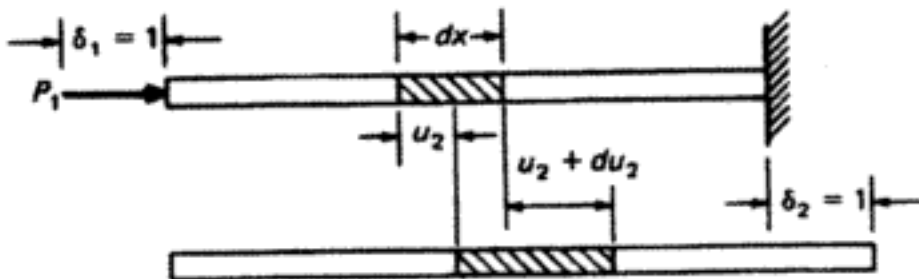


Fig. 11.4 Displacement along of a beam element subjected to axial loading that give a unit displacement ($\delta_2=1$)

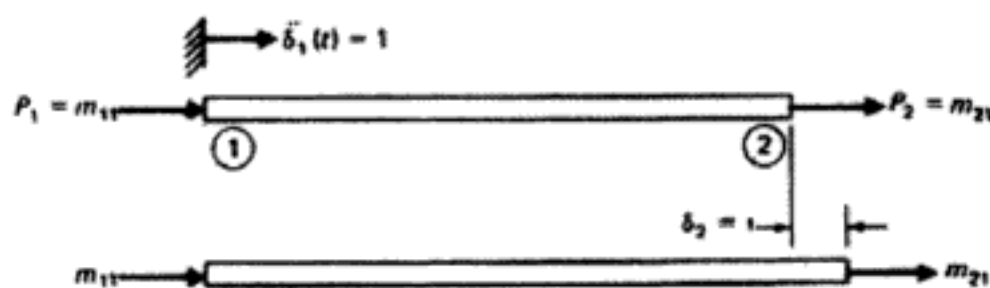


Fig. 11.5 Beam element with unit displacement at node 2 ($\delta_2 = 1$) undergoing a unit axial acceleration at node 1 [$\ddot{\delta}_1(t) = 1$].

Now consider the beam of Fig. 11.5 while undergoing a unit acceleration, $\ddot{\delta}_1(t) = 1$ which by eq. (11.18) results in an acceleration at x given by

$$\ddot{u}_1(x, t) = u_1(x) \ddot{\delta}_1(t)$$

or

$$\ddot{u}_1(x, t) = u_1(x)$$

since $\ddot{\delta}_1(t) = 1$. The inertial force per unit length along the beam resulting from this unit acceleration is

$$f_I = \bar{m}(x) u_1(x) \quad (11.19)$$

where $\bar{m}(x)$ is the mass per unit length along the beam. Now, to determine the mass coefficient m_{21} , we give to the beam shown in Fig. 11.5 a virtual displacement $\delta_2 = 1$. The only external force doing work during this virtual displacement is the reaction m_{21} . This work is then

$$W_E = m_{21} \delta_2$$

or

$$W_E = m_{21} \quad (11.20)$$

since $\delta_2 = 1$. The internal work per unit length along the beam performed by the inertial force f_I during this virtual displacement is

$$\delta W_I = f_I u_2(x)$$

or, from eq. (11.19)

$$\delta W_I = \bar{m}(x) u_1(x) u_2(x)$$

Hence the total internal work is

$$W_I = \int_0^l \bar{m}(x) u_1(x) u_2(x) dx \quad (11.21)$$

Finally, equating eqs. (11.20) and (11.21) yields

$$m_{21} = \int_0^l \bar{m}(x) u_1(x) u_2(x) dx \quad (11.22)$$

or, in general,

$$m_{ij} = \int_0^L \bar{m}(x) u_i(x) u_j(x) dx \quad (11.23)$$

The application of eq. (11.23) to the special case of a uniform beam results in

$$m_{11} = \int_0^L \bar{m} \left(1 - \frac{x}{L}\right)^2 dx = \frac{\bar{m}L}{3} \quad (11.24)$$

Similarly,

$$m_{22} = \frac{\bar{m}L}{3}$$

and

$$m_{12} = m_{21} = \int_0^L \bar{m} \left(1 - \frac{x}{L}\right) \left(\frac{x}{L}\right) dx = \frac{\bar{m}L}{6} \quad (11.25)$$

In matrix form, the axial inertial force relationship for a uniform beam may be written as

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \quad (11.26)$$

Finally, combining the mass matrix eq. (10.34) for flexural effects with eq. (11.26) for the axial effects, we obtain the consistent mass matrix for a uniform element of a plane frame in reference to the modal coordinates shown in Fig. 11.2 as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 140 & & & & & \\ & 0 & 156 & & & \\ & 0 & 22L & 4L^2 & & \\ 70 & 0 & 0 & 140 & & \\ & 0 & 54 & 13L & 0 & 156 \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \quad (11.27)$$

or, in condensed notation,

$$\{P\} = [M_c] \{\ddot{\delta}\}$$

in which $[M_c]$ is the consistent mass matrix for an element of a plane frame.

11.3 Coordinate Transformation

The stiffness matrix for an element of a plane frame in eq. (11.4) as well as the mass matrix in eq. (11.27) are in reference to nodal coordinates defined by coordinate axes fixed on the beam element. These axes are called *local* or *element coordinate axes* while the coordinate axes for the whole structure are known as *global* or *system coordinate axes*. Figure 11.6 shows a beam element with nodal forces P_1, P_2, \dots, P_6 referred to the local coordinate axes x, y, z , and $\bar{P}_1, \bar{P}_2, \dots, \bar{P}_6$ referred to global coordinate set of axes X, Y, Z . The objective is to transform the element matrices (stiffness, mass, etc.) from the reference of local coordinate axes to the global coordinate axes. This transformation is required in order that the matrices for all the

elements refer to the same set of coordinates; hence, the matrices become compatible for assemblage into the system matrices for the structure. We begin by expressing the forces (P_1, P_2, P_3) in terms of the forces $(\bar{P}_1, \bar{P}_2, \bar{P}_3)$. Since these two sets of forces are equivalent, we obtain from Fig. 11.6 the following relationships:

$$\begin{aligned} P_1 &= \bar{P}_1 \cos \theta + \bar{P}_2 \sin \theta \\ P_2 &= -\bar{P}_1 \sin \theta + \bar{P}_2 \cos \theta \\ P_3 &= \bar{P}_3 \end{aligned} \tag{11.28}$$

The equations of eq. (11.28) may be written in matrix notation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{Bmatrix} \tag{11.29}$$

Analogously, we obtain for the forces on the other node the relationships:

$$\begin{aligned} P_4 &= \bar{P}_4 \cos \theta + \bar{P}_5 \sin \theta \\ P_5 &= -\bar{P}_4 \sin \theta + \bar{P}_5 \cos \theta \\ P_6 &= \bar{P}_6 \end{aligned} \tag{11.30}$$

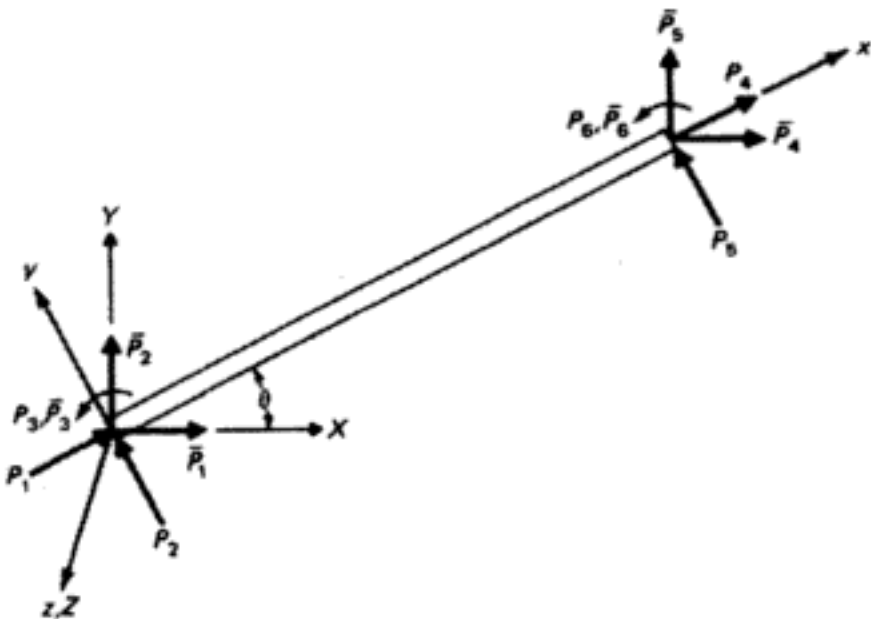


Fig. 11.6 Beam element showing nodal forces P_i in local (x, y, z) and nodal forces \bar{P}_i in global coordinate axes (X, Y, Z) .

Equations (11.28) and (11.30) may conveniently be arranged in matrix form as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \\ \bar{P}_5 \\ \bar{P}_6 \end{Bmatrix} \quad (11.31)$$

or in condensed notation

$$\{P\} = [T]\{\bar{P}\} \quad (11.32)$$

in which $\{P\}$ and $\{\bar{P}\}$ are, respectively, the vectors of the element nodal forces in local and global coordinates and $[T]$ is the transformation matrix given by the square matrix in eq. (11.31).

Repeating the same procedure, we obtain the relation between nodal displacements $(\delta_1, \delta_2, \dots, \delta_6)$ in local coordinates and the components of the nodal displacements in global coordinates $(\bar{\delta}_1, \bar{\delta}_2, \dots, \bar{\delta}_6)$, namely

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{\delta}_1 \\ \bar{\delta}_2 \\ \bar{\delta}_3 \\ \bar{\delta}_4 \\ \bar{\delta}_5 \\ \bar{\delta}_6 \end{Bmatrix} \quad (11.33)$$

or

$$\{\delta\} = [T]\{\bar{\delta}\} \quad (11.34)$$

Now, the substitution of $\{P\}$ from eq. (11.32) and $\{\delta\}$ from eq. (11.34) into the stiffness equation referred to local axes $\{P\} = [K]\{\delta\}$ results in

$$[T]\{\bar{P}\} = [K][T]\{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [T]^{-1}[K][T]\{\bar{\delta}\} \quad (11.35)$$

where $[T]^{-1}$ is the inverse of matrix $[T]$. However, as the reader may verify, the transformation matrix $[T]$ in eq. (11.31) is an orthogonal matrix, that is, $[T]^{-1} = [T]^T$. Hence

$$\{\bar{P}\} = [T]^T[K][T]\{\bar{\delta}\} \quad (11.36)$$

or, in a more convenient notation,

$$\{\bar{P}\} = [\bar{K}]\{\bar{\delta}\} \quad (11.37)$$

in which

$$[\bar{K}] = [T]^T [K] [T] \quad (11.38)$$

is the stiffness matrix for an element of a plane frame in reference to the global system of coordinates.

Repeating the procedure of transformation as applied to the stiffness matrix for the lumped mass, eq. (11.7), or the consistent mass matrix, eq. (11.27), we obtain in a similar manner

$$\{\bar{P}\} = [\bar{M}]\{\bar{\delta}\}$$

in which

$$[\bar{M}] = [T]^T [M] [T] \quad (11.39)$$

is the mass matrix for an element of a plane frame in reference to the global system of coordinates and $[T]$ is the transformation matrix given by the square matrix in eq. (11.33).

Illustrative Example 11.1.

Consider in Fig. 11.7 a plane frame having two prismatic beam elements and three degrees of freedom as indicated in the figure. Using the consistent mass formulation, determine the three natural frequencies and corresponding normal modes for this discrete model of the frame.

Solution:

The stiffness matrix for element 1 or 2 in local coordinates by eq. (11.4) is

$$[K_1] = [K_2] = \begin{bmatrix} 600 & & & & & \\ 0 & 12 & & & & \\ 0 & 600 & 40000 & & & \\ -600 & 0 & 0 & 600 & & \\ 0 & -12 & -600 & 0 & 12 & \\ 0 & 600 & 20000 & 0 & -600 & 40000 \end{bmatrix} \quad \text{Symmetric}$$

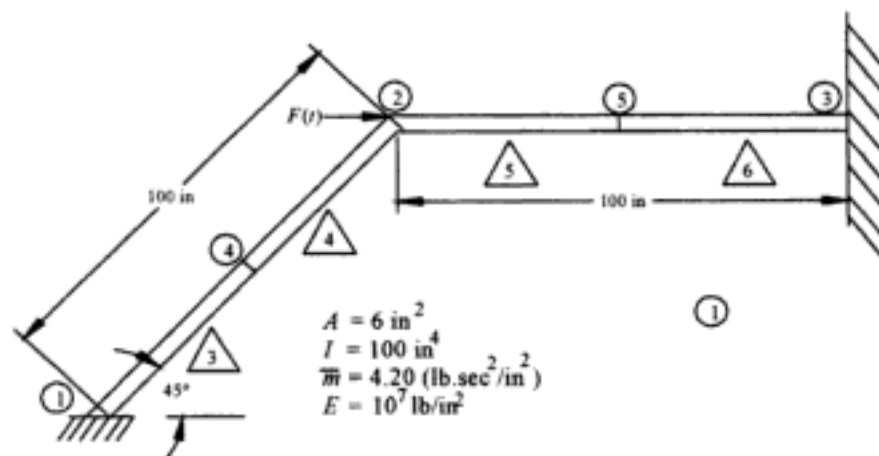


Fig. 11.7 Plane Frame of Illustrative Example 11.1

The transformation matrix for element 1 by eq. (11.31) with $\theta = 45^\circ$ is

$$[T_1] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

and for element 2 with $\theta = 0^\circ$ is the identity matrix:

$$[T_2] = [I]$$

The mass matrix in local coordinates for either of the two elements of this frame from eq. (11.27) is

$$[M] = [M_2] = \begin{bmatrix} 140 & & & & & \\ 0 & 156 & & & & \\ 0 & 2200 & 40000 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 1300 & 0 & 156 & \\ 0 & -1300 & -30000 & 0 & -2200 & 40000 \end{bmatrix} \quad \text{Symmetric}$$

The element stiffness and mass matrices in reference to the global system of coordinates, are, respectively, calculated by eqs. (11.38) and (11.39). For element 1 the stiffness matrix is

$$[K_1] = 10^6 \begin{array}{ccccc} & \begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0.306 \\ 0.294 \\ -0.424 \\ -0.306 \\ -0.294 \\ -0.424 \end{matrix} & \begin{matrix} \\ 0.306 \\ 0.424 \\ -0.294 \\ -0.306 \\ 0.424 \end{matrix} & \begin{matrix} \\ \\ 40000 \\ 0.424 \\ 0.294 \\ 20000 \end{matrix} & \begin{matrix} \\ \\ \\ 0.306 \\ 0.294 \\ 0.424 \end{matrix} & \begin{matrix} \text{Symmetric} \\ \\ \\ 0.306 \\ -0.424 \end{matrix} & \begin{matrix} \\ \\ \\ \\ \\ 40000 \end{matrix} \\ & \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

and the mass matrix

$$[M] = \begin{array}{ccccc} & \begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 148 \\ -8 \\ -1556 \\ 62 \\ 8 \\ 919 \end{matrix} & \begin{matrix} \\ 148 \\ 1556 \\ 8 \\ 62 \\ -919 \end{matrix} & \begin{matrix} \\ \\ 40000 \\ -919 \\ 919 \\ -30000 \end{matrix} & \begin{matrix} \\ \\ \\ 148 \\ -8 \\ 1556 \end{matrix} & \begin{matrix} \text{Symmetric} \\ \\ \\ 148 \\ -1556 \end{matrix} & \begin{matrix} \\ \\ \\ \\ \\ 40000 \end{matrix} \\ & \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

For element 2:

$$[K_2] = 10^6 \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 0600 \\ 0 \\ 0 \\ -0600 \\ 0 \\ 0 \end{matrix} & \begin{matrix} \\ 0.012 \\ 0600 \\ 0 \\ -0012 \\ 0600 \end{matrix} & \begin{matrix} \\ \\ 40000 \\ 0 \\ 0 \\ 20000 \end{matrix} & \begin{matrix} \\ \\ \\ 0600 \\ 0 \\ 0 \end{matrix} & \begin{matrix} \text{Symmetric} \\ \\ \\ 0012 \\ -0600 \end{matrix} & \begin{matrix} \\ \\ \\ \\ \\ 40000 \end{matrix} \\ & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

and

$$[M_2] = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 140 \\ 0 \\ 0 \\ 70 \\ 0 \\ 0 \end{matrix} & \begin{matrix} \\ 156 \\ 2200 \\ 0 \\ 54 \\ -1300 \end{matrix} & \begin{matrix} \\ \\ 40000 \\ 0 \\ 1300 \\ -30000 \end{matrix} & \begin{matrix} \\ \\ \\ 140 \\ 0 \\ 0 \end{matrix} & \begin{matrix} \text{Symmetric} \\ \\ \\ 156 \\ -2200 \end{matrix} & \begin{matrix} \\ \\ \\ \\ \\ 40000 \end{matrix} \\ & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

The system stiffness and mass matrices are assembled by the direct method. As was mentioned before, it is expedient for hand calculation of these matrices to indicate the corresponding system nodal coordinates at the top and right of the element matrices. We thus obtain, considering only the free coordinates, the system stiffness matrix:

$$[\bar{K}] = 10^6 \begin{bmatrix} 0.906 & 0.294 & 0.424 \\ 0.294 & 0.318 & 0.176 \\ 0.424 & 0.176 & 80000 \end{bmatrix}$$

and the system mass matrix as

$$[\bar{M}] = \begin{bmatrix} 288 & -8 & 1556 \\ -8 & 304 & 644 \\ 1556 & 644 & 80000 \end{bmatrix}$$

The natural frequencies are found as the roots of the characteristic equation

$$| [K] - \omega^2 [M] | = 0$$

which, upon substituting the numerical values given for this example, yields

$$10^3 \begin{vmatrix} 906 - 0.288\omega^2 & 294 + 0.008\omega^2 & 424 - 1.556\omega^2 \\ 294 + 0.008\omega^2 & 318 - 0.304\omega^2 & 176 - 0.644\omega^2 \\ 424 - 1.556\omega^2 & 176 - 0.644\omega^2 & 80000 - 80\omega^2 \end{vmatrix} = 0$$

The roots then are found to be:

$$\omega_1^2 = 638.5, \quad \omega_2^2 = 976.6, \quad \omega_3^2 = 4211.6,$$

and the natural frequencies are

$$\omega_1 = 25.26 \text{ rad/sec}, \quad \omega_2 = 31.24 \text{ rad/sec}, \quad \text{and} \quad \omega_3 = 64.90 \text{ rad/sec},$$

or

$$f_1 = 4.02 \text{ cps}, \quad f_2 = 4.97 \text{ cps}, \quad \text{and} \quad f_3 = 10.33 \text{ cps},$$

The normal modes are given as the nontrivial solution of the eigenproblem

$$([K] - \omega^2 [M]) \{a\} = \{0\}$$

Substituting $\omega_1^2 = 638.5$ and setting $a_{11} = 1.0$, we obtain the first mode shape as

$$\{a_1\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -2.38 \\ 0 \end{Bmatrix}$$

which is normalized with the factor

$$\sqrt{\{a_1\}^T [M] \{a_1\}} = 45.81$$

The normalized eigenvector is then

$$\{\phi_1\} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 0.0218 \\ -0.0527 \\ 0 \end{Bmatrix}$$

Analogously, for the other two modes, we obtain:

$$\{\phi_2\} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} = \begin{Bmatrix} 0.00498 \\ 0.00206 \\ 0.00341 \end{Bmatrix} \quad \text{and} \quad \{\phi_3\} = \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} = \begin{Bmatrix} 0.0583 \\ 0.0241 \\ -0.0016 \end{Bmatrix}$$

arranging these modal vectors into columns of the modal matrix, we obtain:

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.00498 & 0.0583 \\ -0.0527 & 0.00206 & 0.0241 \\ 0 & 0.00341 & -0.0016 \end{bmatrix}$$

Illustrative Example 11.2.

Determine the maximum displacement at the nodal coordinates of the frame in Fig. 11.7 when a force of magnitude 100,000 lb is suddenly applied at nodal coordinate 1. Neglect damping.

Solution:

From Illustrative Example 11.1, the natural frequencies are $\omega_1 = 25.26$ rad/sec, $\omega_2 = 31.24$ rad/sec, and $\omega_3 = 64.90$ rad/sec; and the modal matrix is

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.00498 & 0.0583 \\ -0.0527 & 0.00206 & 0.0241 \\ 0 & 0.00341 & -0.0016 \end{bmatrix}$$

The modal equations have the form of

$$\ddot{z}_i + \omega_i^2 z_i = P_i \quad (\text{a})$$

where

$$P_i = \sum_j \phi_{ji} F_j \quad (\text{b})$$

In this example, the nodal applied forces are

$$F_1 = 100,000 \text{ lb}, \quad F_2 = 0, \quad F_3 = 0$$

We thus obtain, after substituting numerical values into eqs. (a) and (b), the modal equations as

$$\begin{aligned}\ddot{z}_1 + 638.5z_1 &= 2180 \\ \ddot{z}_2 + 976.6z_2 &= 498 \\ \ddot{z}_3 + 4211.6z_3 &= 5830\end{aligned}\quad (c)$$

The solutions of these equations by (4.5) are of the form

$$z_i = \frac{P_i}{\omega_i^2} (1 - \cos \omega_i t)$$

Substitution for P_i and ω_i yields

$$\begin{aligned}z_1 &= 3.414(1 - \cos 25.2685t) \\ z_2 &= 0.510(1 - \cos 31.2506t) \\ z_3 &= 1.384(1 - \cos 64.8970t)\end{aligned}\quad (d)$$

The nodal displacements are obtained from

$$\{u\} = [\Phi] \{z\}$$

which results in

$$\begin{aligned}u_1 &= 0.1577 - 0.0744 \cos 25.26t - 0.00254 \cos 31.25t - 0.0807 \cos 64.9t \text{ (in)} \\ u_2 &= -0.1455 + 0.1800 \cos 25.26t - 0.00105 \cos 31.25t - 0.0333 \cos 64.9t \text{ (in)} \\ u_3 &= -0.000475 + 0 \cos 25.26t - 0.00174 \cos 31.25t + 0.0022 \cos 64.9t \text{ (in)}\end{aligned}\quad (e)$$

The maximum possible displacements at the nodal coordinates may then be estimated as the summation of the absolute values of the coefficients in the above expressions. Hence

$$u_{1\max} = 0.3177 \text{ in} \quad u_{2\max} = 0.3589 \text{ in} \quad u_{3\max} = 0.0044 \text{ in}$$

11.4 Program 14—Modeling Structures as Plane Frames

Program 14 serves to determine the stiffness and the mass matrices for a plane frame and to store the coefficients of these matrices in a file. Since the stiffness and mass matrices are symmetric matrices, only the upper triangular portion of these matrices needs to be stored. The program also stores in another file, named by the user, the general information on the frame. The information stored in these files is needed by programs which the user may call to perform dynamic analysis of the frame, such as determination of natural frequencies or calculation of the response of the structure subject to external excitation.

Illustrative Example 11.3.

Use Program 14 to determine the stiffness and mass matrices for the plane frame shown in Fig. 11.7.

Solution:

Input Data and Output Results

PROGRAM 14: MODELING PLANE FRAMES DATA FILE: D14

GENERAL DATA:

NUMBER OF JOINTS	NJ = 3
NUMBER OF BEAM ELEMENTS	NE = 2
NUMBER OF FIXED COORDINATES	NC = 6
MODULUS OF ELASTICITY	E = 1E+07

JOINT DATA:

JOINT #	X-COORDINATE	Y-COORDINATE	CONCENTRATED JOINT MASS
1	0.00	0.00	0.000
2	70.11	70.11	0.000
3	170.71	70.71	0.000

BEAM ELEMENT DATA:

BEAM-ELEMENT#	FIRST JT.#	SECOND JT.#	MASS / LENGTH	SECT.INERTIA	SECT. AREA
1	1	2	4.200	100.00	6.00
2	2	3	4.200	100.00	6.00

FIXED COORDINATES DATA:

JOINT #, DIRECTION # (HORIZONTAL = 1 VERTICAL = 2, ROTATION = 3)

1	1
1	2
1	3
3	1
3	2
3	3

OUTPUT RESULTS:

SYSTEM STIFFNESS MATRIX

9.0600E+05	2.9400E+05	4.2427E+05
2.9400E+05	3.1800E+05	1.7573E+05
4.2427E+05	1.7573E+05	8.0000E+07

SYSTEM MASS MATRIX

2.8800E+02	-7.9999E+00	1.5556E+03
-7.9999E+00	3.0400E+02	6.4440E+02
1.5556E+03	6.4440E+02	7.9999E+04

Illustrative Example 11.4.

For the structure shown in Fig. 11.7 which was modeled in Illustrative Example 11.3, determine: (a) natural frequencies and modal shapes; (b) the response to a force of magnitude 100,000 lb suddenly applied at nodal coordinate 2. The solution requires the use of files prepared during the execution of Program 14 to model the structure as a plane frame followed by the execution of Programs 8 and 9 to determine natural frequencies and to obtain the response using the modal superposition method.

Solution:

(a) Execution of Program 8:

PROGRAM 8: NATURAL FREQUENCIES DATA FILE: SK
 OUTPUT RESULTS

EIGENVALUES:

6.385E+02 9.766E+02 4.212E+03

NATURAL FREQUENCIES (C.P.S.):

4.021 4.973 10.328
 b

EIGENVECTORS BY ROWS

-0.02183 0.05270 0.00000
 0.00498 0.00206 0.00341
 0.05831 0.02415 -0.00163

(b) Execution of Program 9:

PROGRAM 9: MODAL DATA FILE: D15.4B

NATURAL FREQUENCIES (C.P.S.):

4.021 4.973 10.328

MODAL SHAPES BY ROW (EIGENVECTORS):

-0.021.83 0.05270 0.00000
 0.00498 0.00206 0.00341
 0.05831 0.02415 -0.00163

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM

ND = 3

NUMBER OF EXTERNAL FORCES

NF = 1

TIME STEP OF INTEGRATION

H = .01

GRAVITATIONAL INDEX

G = 0

PRINT TIME HISTORY (NPRT = 1; ONLY MAX. VALUES NPRT = 0) NPRT = 0

FORCE #. COORD.# WHERE FORCE IS APPLIED. NUM. OF POINT DEFINING THE FORCE

1 1 2

EXCITATION FUNCTION FOR FORCE # 1

TIME	EXCITATION
0.00	100000.00
0.20	100000.00

MAXIMUM RESPONSE

COORD.	MAX.DISPL.	MAX.VELOC.	MAX.ACC.
1	0.303	7.025	390.10
2	0.335	6.576	230.79
3	0.003	0.188	10.99

As expected results using these instructional programs agreed with those obtained by hand calculations in Illustrative Examples 11.2 and 11.3.

11.5 Dynamic Analysis of Plane Frames Using SAP2000

Illustrative Example 11.5

Use SAP2000 to: (1) Model the plane frame shown in Fig. 11.8 using a total of four beam elements, (2) determine the first six natural frequencies; and (3) calculate the response due to the force $F(t)$ shown in Fig. 11.9.

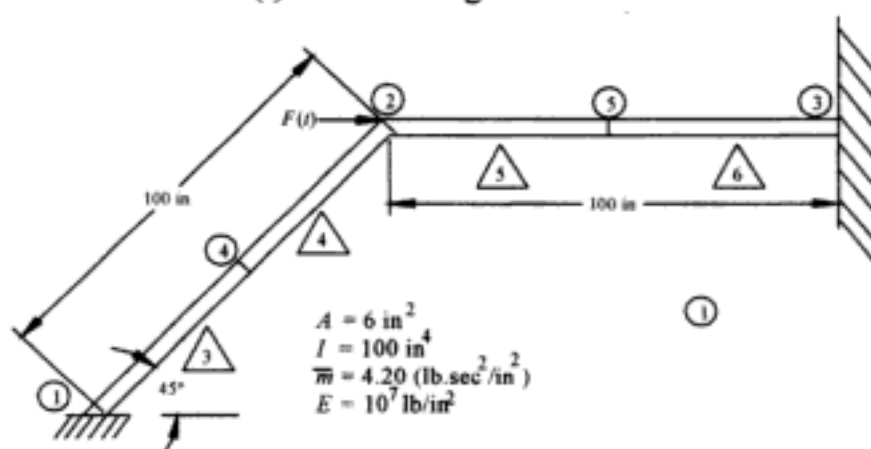


Fig. 11.8 Plane Frame of Illustrative Example 11.5.

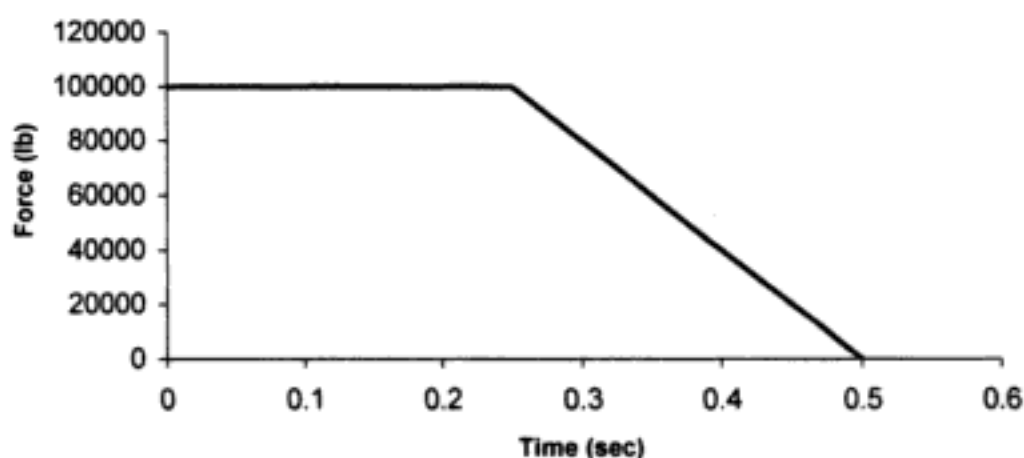


Fig. 11.9 Forcing function applied at joint 2 of the plane frame of Example 11.5.

Solution:

The following commands are implemented in SAP2000:

Begin: Open SAP2000 (already installed)

Hint: Maximize both windows for a full view of the screen.

Select: In the lower right-hand corner of the screen use the drop-down menu to select "lb-in".

- Select:** From the Main Menu bar:
 FILE>NEW MODEL
 Number of Grid Spaces:
 X – direction = 2
 Y – direction = 0
 Z - direction = 1
 Grid Spaces
 X – direction = 70.7
 Y – direction = 1
 Z – direction = 70.7. Then OK.
- Edit:** Click on XZ icon in the Menu Bar
 Maximize the XZ window and use the PAN icon in the toolbar to drag the figure to the center of the screen.
- Edit:** DRAW>EDIT GRID
 Click on Direction X
 In “X Location” highlight the entry 70.7 and change it to 100
 Click on “Move Grid Line”, accept all other default entries.
 Then OK.
- Draw:** From the Main Menu enter:
 DRAW>DRAW FRAME ELEMENT
 Click on (-70.7,0,0) and drag cursor to (0,0,70.7), located at intersections of the grid lines. (One frame section completed)
 Click on (0,0,70.7) and drag cursor to (100,0,70.7) (second frame element completed)
 Select both frame elements by clicking on them.
 Enter: EDIT>DIVIDE FRAMES
 Enter: “Divide into “ = 2
 Last/First ratio = 1. Then OK.
- Label:** From the Main Menu enter:
 VIEW>SET ELEMENTS
 Click on Joint Labels and Frame Labels. Then OK.
 Use the PAN icon on the toolbar to center the figure on the screen.
- Material:** DEFINE>MATERIALS
 Under “Materials” select OTHER
 Click to Modify/Show Material. Then enter:
 Material Name = OTHER
 Mass per Unit Volume = 0.7
 Weight per Unit Volume = 0
 Modulus of Elasticity = 10E6
 Accept remaining default values. Then OK, OK.
- Frame Sections:** DEFINE>FRAME SECTIONS
 Click on Add I/Wide Flange

Click on "Add General" from the drop down menu
Enter the following in the "Property Data" Window
Cross Sectional Area = 6
Moment of Inertia about 3 axis = 100
Shear Area in directions 2 and 3 = 0
Accept remaining default values.
Material select OTHER. Then OK, OK, OK.

Assign: Select all the elements of the frame by clicking on them.
ASSIGN>FRAME>SECTIONS
Select FSEC2. Then OK.

Boundary: Select joints 1 and 3 by clicking on them and enter:
ASSIGN>JOINT>RESTRAINTS
Select restraints in all directions. Then OK.

Load: DEFINE>STATIC LOAD CASES
Enter: Load = LOAD1
Type = LIVE
Self Weight Multiplier = 0
Click on Add New Load. Then OK.

Functions: DEFINE>TIME HISTORY FUNCTIONS
Click on Add New Function and accept default name FUNC1
Enter: Time = 0

Value = 100000 Then press ADD
Enter: Time = 0.25
Value = 100000 Then press ADD
Enter: Time = 0.5
Value = 0 then press ADD, and OK, OK.

Load Case: DEFINE>TIME HISTORY CASES
Enter: Add New History
Analysis Type = Linear
Number of Output Time Steps = 51
Time Steps = 0.01.
Click on "Envelopes"
Load Assignments:
Load = LOAD1
Function = FUNC1. Accept all other default values.
Then press ADD. Enter OK, OK.

Assign Loads: Click on Joint 2 and enter:
ASSIGN>JOINT STATIC LOADS>FORCES
For LOAD1: Enter Load, Force Global X = 1. Then OK.

Analyze: ANALYZE>SET OPTIONS
Check only UX, RY, UZ in Available DOF's (Degrees of Freedom)

[Alternatively, click on picture of Plane Frame (Quick DOF's)]
 Check Dynamic Analysis and then click on Set Dynamic Parameters
 Enter: Number of Modes = 6.
 Then OK, OK.

ANALYZE>RUN

Enter filename: EXD 11.5. Then SAVE.

At the conclusion of the calculations, after checking for errors, press OK.
 Select X-Z view.

Print Input Tables: FILE>PRINT INPUT TABLES

CLICK "Print to File" and accept all other default values.
 Then OK.

Use Notepad or other editor to retrieve and edit the input table file.

Note: Using Notepad or similar word editor the file EXD 11.5.txt may be opened, edited and printed.

(The edited Input Tables of Illustrative Example 11.5 are given in Table 11.1)

Table 11.1 Edited Input Data for Illustrative Example 11.5.

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	LIVE	0.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	TIME STEP INCREMENT
HIST1	LINEAR	51	0.01000

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	- 70.70000	0.00000	0.00000	1 1 1 1 1
2	0.00000	0.00000	70.70000	0 0 0 0 0
3	100.00000	0.00000	70.70000	1 1 1 1 1
4	- 35.35000	0.00000	35.35000	0 0 0 0 0
5	50.00000	0.00000	70.70000	0 0 0 0 0

FRAME ELEMENT DATA

FRAME	JNT1	JNT2	SECT	ANGLE	RELEASES	SEG	R1	R2	FACTOR	LENGTH
3	1	4	FSEC2	0.000	000000	2	0.000	0.000	1.000	49.992
4	4	2	FSEC2	0.000	000000	2	0.000	0.000	1.000	49.992
5	2	5	FSEC2	0.000	000000	4	0.000	0.000	1.000	50.000
6	5	3	FSEC2	0.000	000000	4	0.000	0.000	1.000	50.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
2	1.000	0.000	0.000	0.000	0.000	0.000

Plot Displacement Function:

DISPLAY>SHOW TIME HISTORY TRACES
Click On Define Functions
Click to "Add Joint Disps/Forces"
Joint ID = 2
Vector Type = Displacement
Component = UX.
Mode Number = Include All. Then OK.
Click On Define Functions
Click to "Add Joint Disps/Forces"
Joint ID = 2
Vector Type = Displacement
Component = UZ.
Mode Number = Include All. Then OK, OK.
Under List of Functions click once on "Joint2",
Click on ADD to move "Joint2" to Plot Functions column
Repeat these two steps for "Joint 2-1".
(Both functions will then appear on the Time History Traces Plot)
Axis Labels
 Vertical = Displacement Components X, Z at Joint 2 (in)
Click "Display" to view Displacement functions
FILE>PRINT GRAPHICS to obtain a printed version of these plots.

(Fig. 11.10 reproduces the plot shown on the screen for the displacement components X and Z at Joint 2 of the plane frame of Illustrative Example 11.5)

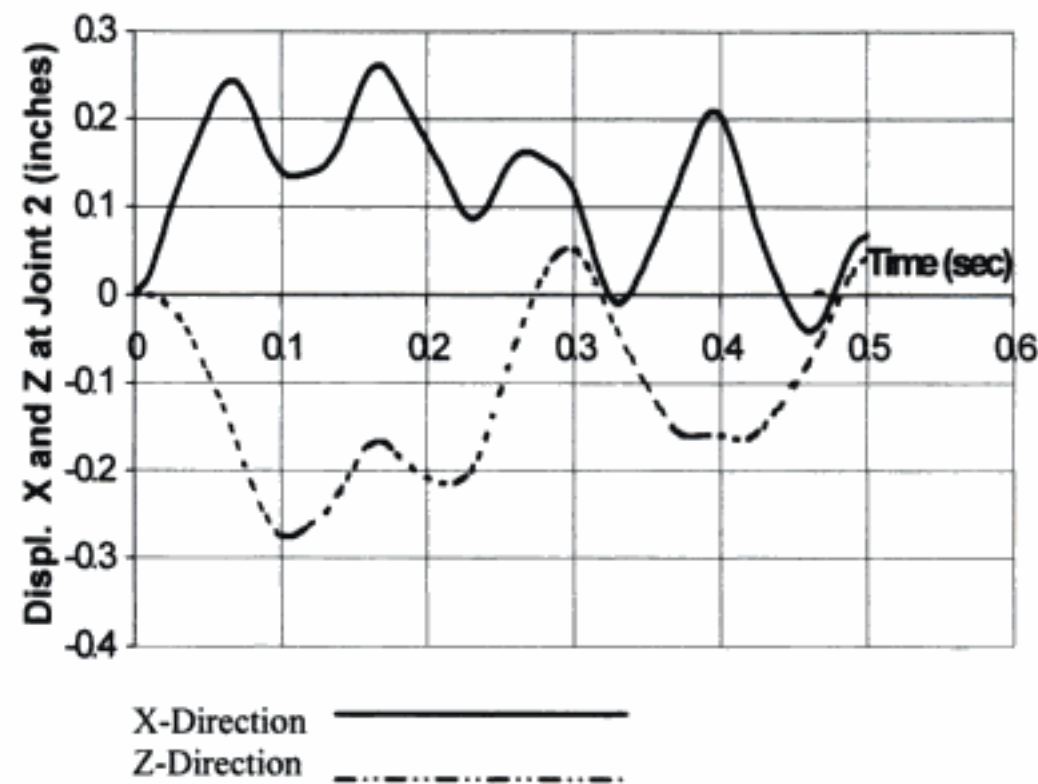


Fig. 11.10 Displacements UX and UZ at node 2 for the plane frame of Illustrative Example 11.5.

Note: Displacement values may be read on the screen by moving the cursor along the curves shown on the screen. Table 11.2 provides the maximum displacement components in the X and Z directions at joint 2 for the plane frame of Illustrative Example 11.5.

Table 11.2 Maximum Displacements at node 2 in X and Z directions for the plane frame of Illustrative Example 11.5.

<i>Direction</i>	<i>Maximum Displacement (in)</i>
X	0.260
Z	0.274

(Table 11.3 contains the Displacement Component Functions for Joint 2, in the X and Z directions for the plane frame of Illustrative Example 11.5)

Alternatively, a table of the displacement component functions in the X and Z directions of joint 2 may be obtained by entering:

FILE>PRINT TABLES TO FILE

Then using Notepad or similar word processor the displacement functions may be accessed and edited to obtain a table of these functions. An abbreviated table of the displacement component functions in the X and Z directions for joint 2 are reproduced in 11.3. Table The values in this table may be plotted using Excel or similar software.

Table 11.3 Displacement Component Functions at Joint 2 in the X and Z directions for the plane frame of Illustrative Example 11.1.

Time (sec)	Displ. Components at Joint 2	
	X-direction (in)	Z-direction (in)
0.00	0	0
0.01	0.02218	-0.00005
0.02	0.07267	-0.00675
0.03	0.12346	-0.02569
0.04	0.16516	-0.05606
0.05	0.20431	-0.09073
-----	-----	-----
0.49	0.05393	0.02457
0.50	0.06737	0.04241

Natural Periods and Frequencies:

Using Notepad or other word editor, open the following file:

C:/SAP2000e/EXD 11.5.OUT

The Natural Period and Frequency table may be found in this file.

(Table 11.3 contains the edited Natural Period and Frequency values of Illustrative Example 11.5)

Table 11.4 Natural Period and Frequencies for the plane frame of Illustrative Example 11.5.

NATURAL PERIODS AND FREQUENCIES

MODE	PERIOD (SEC)	FREQUENCY (CYC/SEC)	FREQUENCY (RAD/SEC)	EIGENVALUE (RAD/SEC)**2
1	0.288	3.471	21.806	475.502
2	0.275	3.632	22.819	520.722
3	0.154	6.486	40.754	1660.873
4	0.113	8.849	55.603	3091.686
5	0.056	17.702	111.222	12370.304
6	0.0467	21.422	134.596	18116.000

11.6 Summary

The dynamic analysis of plane frames by the stiffness method requires the inclusion of the axial effects in the system matrices (stiffness, mass, etc.). It also requires a transformation of coordinates in order to refer all the element matrices to the same coordinate system, so that the appropriate superposition can be applied to assemble the system matrices.

The required matrices for consideration of axial effects as well as the matrix required for the transformation of coordinates are developed in this chapter. A computer program for modeling structures as plane frames is also presented. This program is organized following the pattern of the BEAM program of the preceding chapter.

11.7 Problems

The following problems are intended for hand calculation, though it is recommended that whenever possible solutions should also be obtained using Program 14 to model the structure as a plane frame, Program 8 to determine natural frequencies and modal shapes, and Program 9 to calculate the response using modal superposition method.

Problem 11.1

For the plane frame shown in Fig. P11.1 determine the system stiffness and mass matrices. Base the analysis on the four free nodal coordinates indicated in the figure. Use consistent mass method.

Problem 11.2

Use the results obtained in Problem 11.1 in performing the static condensation to eliminate the rotational degree of freedom at the support to determine the transformation matrix and the reduced stiffness and mass matrices.

Problem 11.3

Determine the natural frequencies and corresponding normal modes for the reduced system in Problem 11.2.

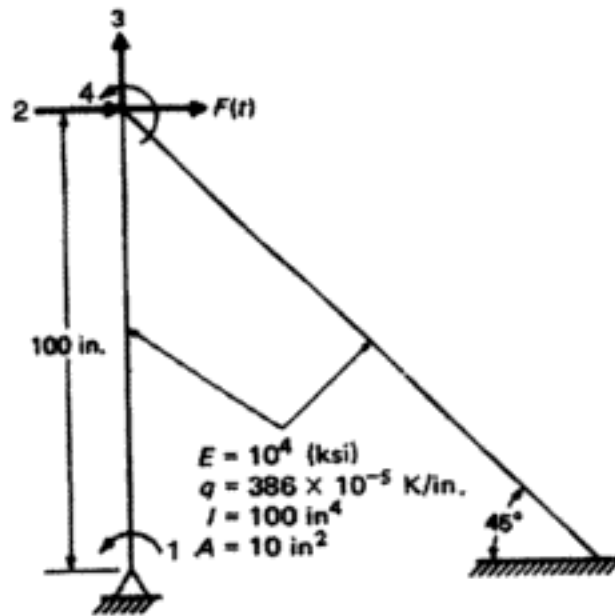


Fig. P11.1.

Problem 11.4

Determine the response of the frame shown in Fig. P11.1 when it is acted upon by a force $F(t) = 1.0$ Kip suddenly applied at nodal coordinate 2 for 0.05 sec. Use results of Problem 11.3 to obtain the modal equations. Neglect damping in the system.

Problem 11.5

Determine the maximum response of the frame shown in Fig. P11.1 when subjected to the triangular impulsive load (Fig. P11.5) along the nodal coordinate 2. Use results of Problem 11.3 to obtain the modal equations and use the appropriate response spectrum to find maximum modal response (Fig. 4.5). Neglect damping in the system.

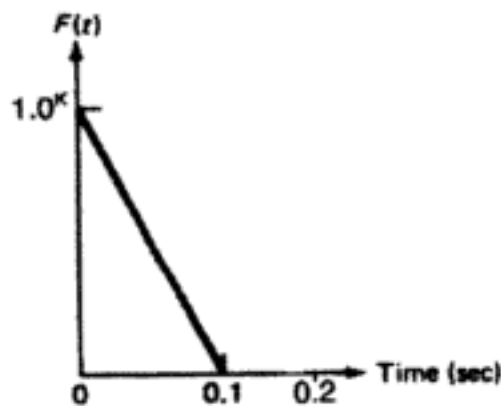


Fig. P11.5.

Problem 11.6

Determine the steady-state response of the frame shown in Fig. P11.1 when subjected to harmonic force $F(t) = 10 \sin 30t$ (Kip) along nodal coordinate 2. Neglect damping in the system.

Problem 11.7

Repeat Problem 11.6 assuming that the damping is proportional to the stiffness matrix of the system, $[C] = a_0 [K]$, where $a_0 = 0.2$.

Problem 11.8

The frame shown in Fig. P11.8 is acted upon by the dynamic forces shown in the figure. Determine the equivalent nodal forces corresponding to each member of the frame.

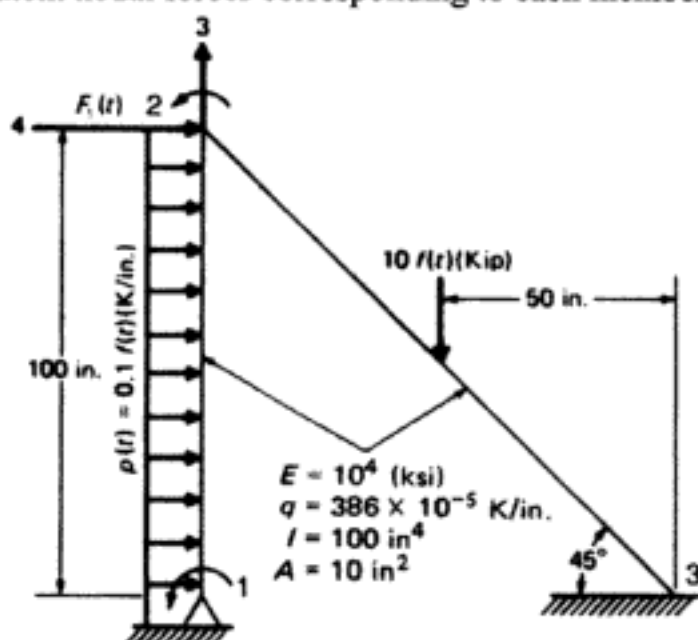


Fig. P11.8.

Problem 11.9

Assemble the system equivalent nodal forces $\{F_e\}$ from equivalent member nodal forces which were calculated in Problem 11.8.

Problem 11.10

Determine the natural frequencies and corresponding normal modes for the frame shown in Fig. P11.8.

Problem 11.11

Determine the response for the frame shown in Fig. P11.11 (a) when subjected to the force $F_3(t)$ [Fig. 11.11(b)] acting along nodal coordinate 1. Assume 5% damping in all the modes.

12 Dynamic Analysis of Grid Frames

In Chapter 11 consideration was given to the dynamic analysis of the plane frame when subjected to forces acting on the plane of the structure. When the planar structural system is subjected to loads applied normally to its plane, the structure is referred to as a grid frame. This structure can also be treated as a special case of the three-dimensional frame to be presented in Chapter 13. The reason for considering the planar frame, whether loaded in its plane or normal to its plane, as a special case, is the immediate reduction of unknown nodal coordinates for an element of these special structures, hence a considerable reduction in the number of unknown displacements for the structural system.

When analyzing the planar frame under action of loads in the plane, the possible components of joint displacements that had to be considered were translations in the X and Y directions and rotation about the Z axis. However, if a plane frame is loaded normal to the plane of the structure, the components of joint displacements required to describe the displacements of a joint are a translation in the Z direction and rotations about the X and Y axes. Thus treating the planar grid structure as a special case, it will be necessary to consider only three components of nodal displacements at each end of a typical element of a grid frame.

12.1 Local and Global Coordinate Systems

For an element of a grid frame, the local orthogonal axes will be established such that the x defines the longitudinal centroidal axis of the member and the x - y plane will coincide with the plane of the structural system. In this case, the z axis will define the minor principal axis of the cross section while the y axis will define the major axis of

the cross section. It will be assumed that the shear center of the cross section coincides with the centroid of the cross section. The grid member may have either a variable or constant cross section along its length.

The possible nodal displacements with respect to the local or to the global systems of coordinates are identified in Fig. 12.1. It can be seen that the linear displacements along the z direction for local axes and along the Z direction for the global system are identical since the two axes coincide. However, in general, rotational components at the nodal coordinates differ for these two coordinate systems. Hence, a transformation of coordinates will be required to transform the element matrices from the local to the global coordinates.

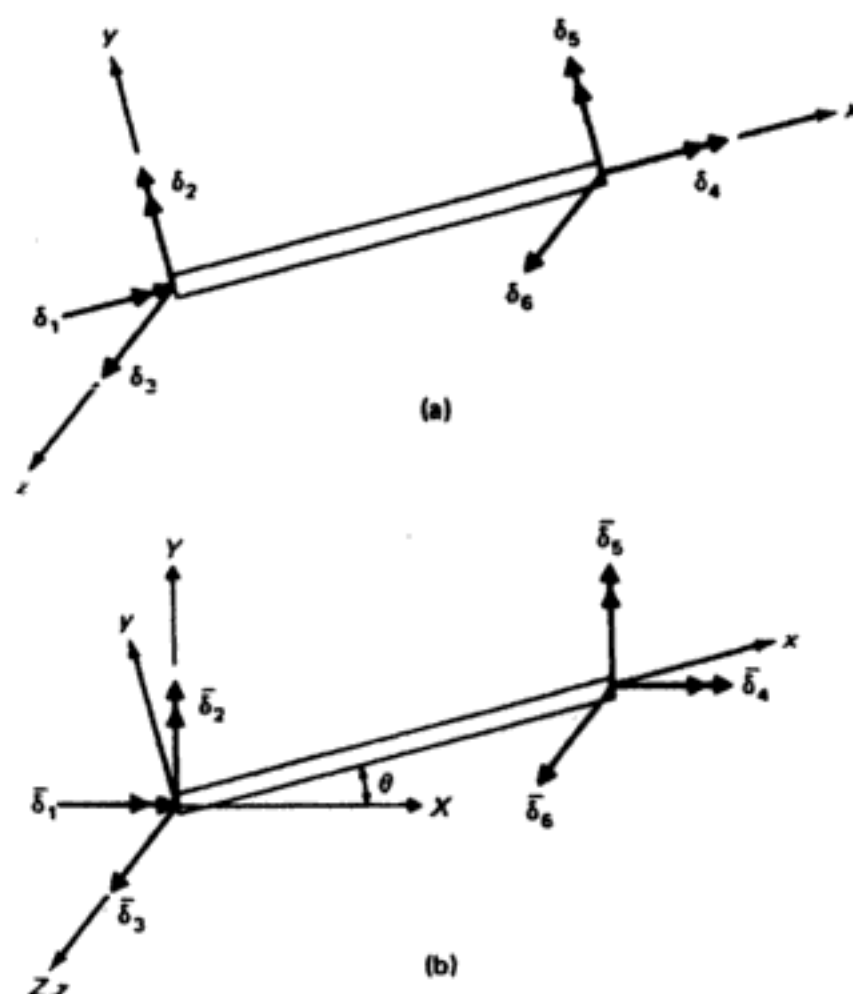


Fig. 12.1 Components of nodal displacements for a grid member. (a) Local coordinate system. (b) Global coordinate system.

12.2 Torsional Effects

The dynamic analysis by the stiffness method for grid frames, that is, for plane frames subjected to normal loads, requires the determination of the torsional stiffness and mass coefficients for a typical element of the grid frame. The derivation of these coefficients is essentially identical to the derivation of the stiffness and mass coefficients for axial effects on a beam element. Similarity between these two derivations occurs because the differential equations for both problems have the same

mathematical form. For the axial problem, the differential equation for the displacement function is given by eq. (11.8) as

$$\frac{du}{dx} = \frac{P}{AE} \quad (11.8 \text{ (repeated)})$$

Likewise, the differential equation for torsional displacement is

$$\frac{d\theta}{dx} = \frac{T}{JG} \quad (12.1)$$

in which θ is the angular displacement, T is the torsional moment, G is the modulus of elasticity in shear, and J is the torsional constant of the cross section (polar moment of inertia for circular sections).

As a consequence of the analogy between eqs. (11.8) and (12.1), we can write the following results already obtained for axial effects. The displacement functions for the torsional effects are the same as the corresponding functions giving the displacements for axial effects; hence by analogy to eqs. (11.10) and (11.11) and in reference to the nodal coordinates of Fig. 12.2, we obtain

$$\theta_1(x) = (1 - \frac{x}{L}) \quad (12.2)$$

and

$$\theta_2(x) = \frac{x}{L} \quad (12.3)$$

in which the angular displacement function $\theta_1(x)$ corresponds to a unit angular displacement $\delta_1 = 1$ at nodal coordinate 1 and $\theta_2(x)$ corresponds to the displacement

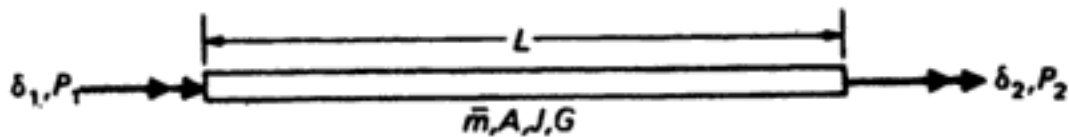


Fig. 12.2 Nodal torsional coordinates for a beam element.

function resulting from a unit angular displacement $\delta_2 = 1$ at nodal coordinate 2. Analogous to eq. (11.17), the stiffness coefficients for torsional effects may be calculated from

$$k_{ij} = \int_0^L JG \theta'_i(x) \theta'_j(x) dx \quad (12.4)$$

in which $\theta'_i(x)$ and $\theta'_j(x)$ are the derivatives with respect to x of the displacement functions $\theta_1(x)$ and $\theta_2(x)$. Also analogous to eq. (11.23), the consistent mass matrix coefficients for torsional effects are given by

$$m_{ij} = \int_0^L I_{\bar{m}} \theta_i(x) \theta_j'(x) dx \quad (12.5)$$

in which $I_{\bar{m}}$ is the polar mass moment of inertia, per unit length along the beam element. This moment of inertia may conveniently be expressed as the product of the mass \bar{m} per unit length times the radius of gyration squared, k^2 . The radius of gyration may, in turn, be calculated as the ratio I_0/A . Therefore, the mass polar moment of inertia per unit length $I_{\bar{m}}$ is given by

$$I_{\bar{m}} = \bar{m} \frac{I_0}{A} \quad (12.6)$$

in which I_0 is the polar moment of inertia of the cross-sectional area and A the cross-sectional area.

The application of eqs. (12.4) and (12.5) for a uniform beam yields the stiffness and mass matrices for torsional effects as

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{JG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \quad (12.7)$$

and

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{I_{\bar{m}} L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \quad (12.8)$$

in which $I_{\bar{m}}$ is given by eq. (12.6), and T_1, T_2 are torsional moments at the ends of the beam element labeled in Fig. 12.2 as P_1 and P_2 .

12.3 Stiffness Matrix for a Grid Element

The torsional stiffness matrix, eq. (12.7), is combined with the flexural stiffness matrix, eq. (10.20), to obtain the stiffness matrix for a typical element of a grid frame. In reference to the local coordinate system indicated in Fig. 12.1 (a), the stiffness equation for a uniform element is then

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} JGL^2/EI & & & & & \\ & 0 & 4L^2 & & & \\ & 0 & -6L & 12 & & \\ -JGL^2/EI & & 0 & 0 & JGL^2/EI & \\ & 0 & 2L^2 & -6L & 0 & 4L^2 \\ & 0 & 6L & -12 & 0 & 6L \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \quad (12.9)$$

or in condensed form

$$\{P\} = [K]\{\delta\} \quad (12.10)$$

12.4 Consistent Mass Matrix for a Grid Element

The combination of the consistent mass matrix for flexural effects (10.34) with the consistent mass matrix for torsional effects (12.8) results in the consistent mass matrix for a typical member of a grid, namely

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 14I_0/A & & & & & \\ & 4L^2 & & & & \\ & 0 & 2L & 156 & & \\ 70I_0/A & 0 & 0 & 14I_0/A & & \\ & 0 & -3L^2 & -1L & 0 & 4L^2 \\ & 0 & 1L & 54 & 0 & -2L & 156 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \quad (12.11)$$

or in concise notation

$$\{P\} = [M_c] \{\delta\} \quad (12.12)$$

in which $[M_c]$ is the mass matrix for a typical uniform member of a grid structure.

12.5 Lumped Mass Matrix for a Grid Element

The lumped mass allocation to the nodal coordinates of a typical grid member is obtained from static considerations. For a uniform member having a uniform distributed mass along its length, the nodal mass is simply one-half of the total rotational mass $I_{\bar{m}}L$. The matrix equation for the lumped mass matrix corresponding to the torsional effects is then

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{I_{\bar{m}}L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \quad (12.13)$$

in which $I_{\bar{m}}$ is given by eq. (12.6). The combination of the lumped torsional mass and the lumped translatory mass results in the diagonal matrix which is the lumped mass matrix for the grid element. This matrix, relating forces and accelerations at nodal coordinates, is given by the following equation:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\overline{m}L}{2} \begin{bmatrix} I_0/A & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & I_0/A & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} \delta_1'' \\ \delta_2'' \\ \delta_3'' \\ \delta_4'' \\ \delta_5'' \\ \delta_6'' \end{Bmatrix} \tag{12.14}$$

or briefly

$$\{P\} = [M_L] \{\delta\} \tag{12.15}$$

in which $[M_L]$ is, in this case, the diagonal lumped mass matrix for a grid element.

12.6 Transformation of Coordinates

The stiffness matrix, eq. (12.9), as well as the consistent and the lumped mass matrices in eqs. (12.11) and (12.14), respectively, are in reference to the local system of coordinates. Therefore, it is necessary to transform the reference of these matrices to the global system of coordinates before their assemblage in the corresponding matrices for the structure. As has been indicated, the z axis for the local coordinate system coincides with the Z axis for the global system. Therefore, the only step left to perform is a rotation of the coordinates in the x-y plane. The corresponding matrix for this transformation may be obtained by establishing the relationship between components of the moment at the nodes expressed in these two systems of coordinates. In reference to Fig. 12.3, these relations when written for node ① are

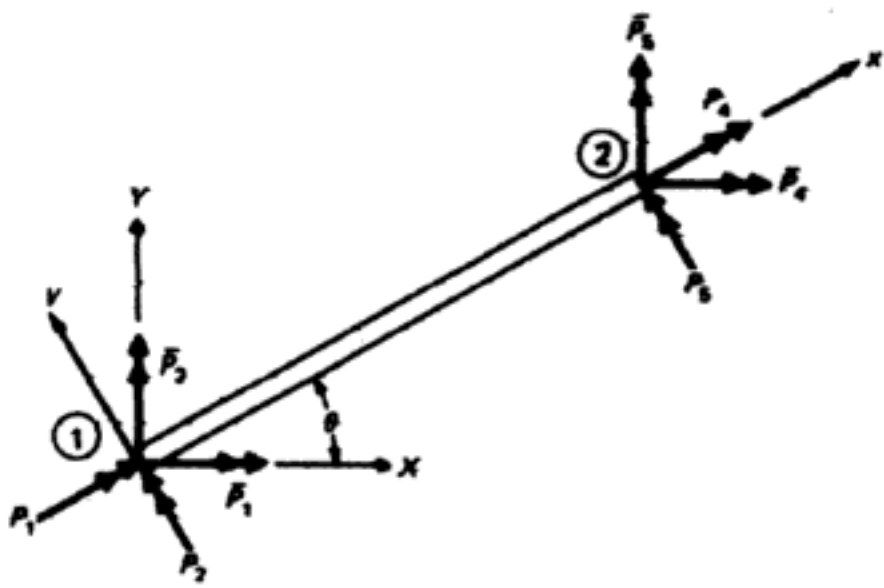


Fig. 12.3 Components of the nodal moments in local and global coordinates.

$$\begin{aligned}
 P_1 &= \bar{P}_1 \cos \theta + \bar{P}_2 \sin \theta \\
 P_2 &= -\bar{P}_1 \sin \theta + \bar{P}_2 \cos \theta \\
 P_3 &= \bar{P}_3
 \end{aligned}
 \tag{12.16a}$$

and for node ②

$$\begin{aligned}
 P_4 &= \bar{P}_4 \cos \theta + \bar{P}_5 \sin \theta \\
 P_5 &= -\bar{P}_4 \sin \theta + \bar{P}_5 \cos \theta \\
 P_6 &= \bar{P}_6
 \end{aligned}
 \tag{12.16b}$$

The identical form of these equations with those derived for the transformation of coordinates for nodal forces of an element of a plane frame, eqs. (11.28) and (11.30), should be noted. Equations (12.16) may be written in matrix notation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & \sin \theta & 0 \\ 0 & 0 & 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \\ \bar{P}_5 \\ \bar{P}_6 \end{Bmatrix}
 \tag{12.17}$$

or in short notation

$$\{P\} = [T]\{\bar{P}\}
 \tag{12.18}$$

in which $\{P\}$ and $\{\bar{P}\}$ are, respectively, the vectors of the nodal forces of a typical grid member in local and global coordinates and $[T]$ the transformation matrix. The same transformation matrix $[T]$ serves also to transform the nodal components of the displacements from a global to a local system of coordinates. In condensed notation, this relation is given by

$$\{\delta\} = [T]\{\bar{\delta}\}
 \tag{12.19}$$

where $\{\delta\}$ and $\{\bar{\delta}\}$ are, respectively, the components of nodal displacements in local and global coordinates. The substitution of eqs. (12.18) and (12.19) in the stiffness relation eq. (12.10) yields the element stiffness matrix in reference to the global coordinate system, that is,

$$[T]\{\bar{P}\} = [K][T]\{\bar{\delta}\}$$

or, since $[T]$ is an orthogonal matrix $[(T)^{-1} = (T)^T]$, it follows that

$$\{\bar{P}\} = [T]^T [K] [T] \{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [\bar{K}]\{\bar{\delta}\}
 \tag{12.20}$$

in which

$$\{\bar{K}\} = [T]^T [K] [T] \quad (12.21)$$

is the stiffness matrix of an element of a grid frame in reference to the global system of coordinates.

Analogously, for the mass matrix, we find

$$\{\bar{P}\} = [\bar{M}] \{\delta\} \quad (12.22)$$

in which

$$\{\bar{M}\} = [T]^T [M] [T] \quad (12.23)$$

is the transformed mass matrix.

Illustrative Example 12.1.

Figure 12.4 shows a grid frame in a horizontal plane consisting of two prismatic beam elements with a total of three degrees of freedom as indicated. Determine the natural frequencies and corresponding mode shapes. Use the consistent mass formulation.

Solution:

The stiffness matrix for elements 1 or 2 of the grid frame in reference to the local system of coordinates, by eq. (12.9), is

$$[K_1] = 10^6 \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 40 & 0 & 0 & -40 & 0 & 0 \\ 0 & 200 & -5 & 0 & 100 & 5 \\ 0 & -5 & 0.167 & 0 & -5 & -0.167 \\ -40 & 0 & 0 & 40 & 0 & 0 \\ 0 & 100 & -5 & 0 & 200 & 5 \\ 0 & 5 & -0.167 & 0 & 5 & 0.167 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The transformation matrix for element Δ with $\theta = 0^\circ$ is simply the unit matrix $[T_1] = [I]$. Hence

$$[\bar{K}_1] = [T_1]^T [K_1] [T_1] = [K_1]$$

and for element (2) with $\theta = 90^\circ$

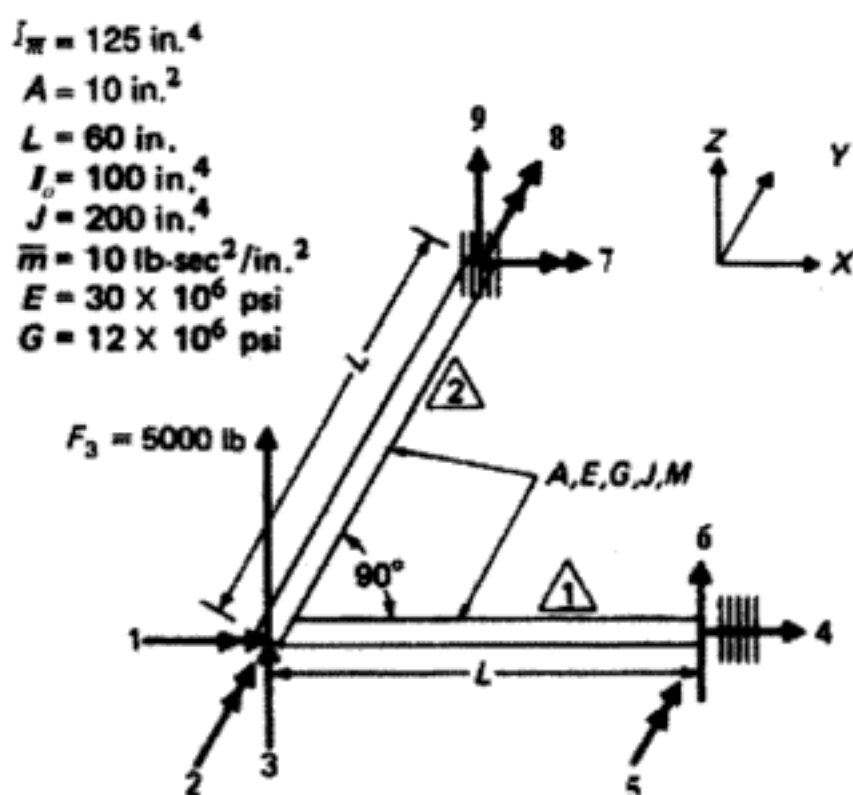


Fig. 12.4. Grid frame of illustrative Example 12.1

$$[T_2] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that

$$[\bar{K}_2] = [T_2]^T [K_2] [T_2]$$

$$[\bar{K}_2] = 10^6 \begin{bmatrix} 200 & 0 & 5 & 100 & 0 & 5 \\ 0 & 40 & 0 & 0 & -40 & 0 \\ 5 & 0 & 0.167 & -5 & 0 & 0.167 \\ 100 & 0 & 5 & 200 & 0 & -5 \\ 0 & -40 & 0 & 0 & 40 & 0 \\ -5 & 0 & -0.167 & -5 & 0 & 0.167 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The system matrix $[K_s]$ assembled from $[\bar{K}_1]$ and $[\bar{K}_2]$ is

$$[K_s] = 10^6 \begin{bmatrix} 240 & 0 & 5 \\ 0 & 240 & -5 \\ 5 & -5 & 0.333 \end{bmatrix}$$

Analogously, for the mass, we have from eq.(12.11)

$$[M_1] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2500 & 0 & 0 & 1250 & 0 & 0 \\ 0 & 20,570 & 1886 & 0 & -15,430 & 1114 \\ 0 & 1886 & 223 & 0 & -1114 & 77 \\ 1250 & 0 & 0 & 2500 & 0 & 0 \\ 0 & -15,430 & -1114 & 0 & 20,570 & -1886 \\ 0 & 1114 & 77 & 0 & -1886 & 223 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

We then calculate using eq.(12.23)

$$[\bar{M}_1] = [M_1]$$

since

$$[T_1] = [I]$$

and analogously

$$[\bar{M}_2] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 20,570 & 0 & -1886 & 15,430 & 0 & 1114 \\ 0 & 2500 & 0 & 0 & 1250 & 0 \\ -1886 & 0 & 223 & 1114 & 0 & 77 \\ 15,430 & 0 & 1114 & 20,570 & 0 & 1886 \\ 0 & 1250 & 0 & 0 & 2500 & 0 \\ 1114 & 0 & 77 & 1886 & 0 & 223 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

From $[\bar{M}_1]$ and $[\bar{M}_2]$ we assemble the system mass matrix and obtain

$$[M_s] = \begin{bmatrix} 23,070 & 0 & 1886 \\ 0 & 23,070 & -1886 \\ 1886 & -1886 & 446 \end{bmatrix}$$

The natural frequencies and mode shapes are obtained from the solution of the

eigenproblem

$$([K_s] - \omega^2 [M_s])\{a\} = \{0\}$$

which gives the eigenvalues (squares of the natural frequencies)

$$\omega_1^2 = 396,35, \quad \omega_2^2 = 10,402, \quad \text{and} \quad \omega_3^2 = 23,866$$

Then

$$\omega_1 = 19.91 \text{ rad/sec}, \quad \omega_2 = 101.99 \text{ rad/sec}, \quad \text{and} \quad \omega_3 = 154.49 \text{ rad/sec}$$

and the eigenvectors ordered in the column of the modal matrix:

$$[a] = \begin{bmatrix} -1.000 & 1.000 & -1.000 \\ 1.000 & 1.000 & 1.000 \\ 54.285 & 0 & 7.765 \end{bmatrix}$$

The eigenvectors are conveniently normalized by dividing the columns of the modal matrix, respectively, by the factors

$$\begin{aligned} \sqrt{\{a_1\}^T [M_s] \{a_1\}} &= 974.75 \\ \sqrt{\{a_2\}^T [M_s] \{a_2\}} &= 214.81 \\ \sqrt{\{a_3\}^T [M_s] \{a_3\}} &= 120.20 \end{aligned}$$

The normalized eigenvectors are arranged in columns of the modal matrix, so that

$$[\Phi] = 10^{-2} \begin{bmatrix} -0.1026 & 0.4655 & -0.8320 \\ 0.1026 & 0.4655 & 0.8320 \\ 5.5691 & 0 & 06.4603 \end{bmatrix}$$

Illustrative Example 12.2.

Determine the response of the grid frame shown in Fig. 12.4 when subjected to a suddenly applied force $F_3 = 5000 \text{ lb}$ as shown in the figure.

Solution:

The natural frequencies and modal shapes for this structure were calculated in Example 12.1. The modal equation is given in general as

$$\ddot{z}_n + \omega_n^2 z_n = P_n \quad (n = 1, 2, 3)$$

where

$$P_n = \sum_{i=1}^3 \phi_{in} F_i$$

and F_i the external forces at the nodal coordinates which for this example are $F_1 = F_2 = 0$ and $F_3 = 5000$ lb. Hence, we obtain

$$\ddot{z}_1 + 396.35z_1 = 278.46$$

$$\ddot{z}_2 + 10,402z_2 = 0$$

$$\ddot{z}_3 + 23,866z_3 = 323.01$$

The solution of these equations for zero initial conditions is

$$z_1 = \frac{278.46}{396.35} (1 - \cos 19.91t)$$

$$z_2 = 0$$

$$z_3 = \frac{323.01}{23,866} (1 - \cos 154.49t)$$

The displacements at the nodal coordinates are calculated from

$$\{u\} = [\Phi]\{z\}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 10^{-2} \begin{bmatrix} -0.1026 & 0.4655 & -0.8320 \\ 0.1026 & 0.4655 & 0.8320 \\ 5.5691 & 0 & 06.4603 \end{bmatrix} \begin{bmatrix} 0.7026 & (1 - \cos 19.91t) \\ 0 \\ 0.0135 & (1 - \cos 154.49t) \end{bmatrix}$$

and finally

$$u_1 = 10^{-3} (-0.8332 + 0.7209 \cos 19.91t + 0.1123 \cos 154.49t) \text{ in}$$

$$u_2 = 10^{-3} (-0.8332 - 0.7209 \cos 19.91t - 0.1123 \cos 154.49t) \text{ in}$$

$$u_3 = 10^{-3} (40 - 39.13 \cos 19.91t - 0.87 \cos 154.49t) \text{ in}$$

12.7 Program 15—Modeling Structures as Grid Frames

Program 15 calculates the stiffness and mass matrices for a grid frame and stores the coefficients of these matrices in a file. Since the stiffness and the mass matrices are symmetric, only the upper triangular portion of these matrices needs to be stored. The program also stores in another file, named by the user, the general information of the grid frame. The information stored in these files is needed by programs that the user may call to perform a dynamic analysis, such as determination of natural frequencies or calculations of the response of the structure subjected to external excitation.

Illustrative Example 12.3.

For the grid frame shown in Fig. 12.4 and analyzed in the previous examples, (a) use Program 15 to model this structure, (b) use Program 8 to calculate the natural frequencies and mode shapes, and (c) use Program 9 to determine the response to a constant force of 5000 lb suddenly applied for 0.1 sec as indicated in the figure.

Solution:

Problem Data:

Modulus of elasticity:	$E = 30 \times 10^6 \text{ psi}$
Modulus of rigidity:	$G = 12 \times 10^6 \text{ psi}$
Distributed mass:	$\overline{m} = 10 \text{ lb}\cdot\text{sec}^2/\text{in}/\text{in}$
Cross-sectional polar moment of inertia:	$I_0 = 125 \text{ in}^4$
Cross-sectional torsional constant:	$J = 200 \text{ in}^4$
Cross-sectional moment of inertia:	$I = 100 \text{ in}^4$
Cross-sectional area:	$A = 10 \text{ in}^2$

(a) Input Data and Output Results: Modeling the Structure

PROGRAM 15: MODELING GRID FRAMES

DATA FILE: D15

INPUT DATA:

NUMBER OF JOINTS	NJ = 3
NUMBER OF BEAM COORDINATES	NE = 2
NUMBER OF FIXED COORDINATES	NC = 6
MODULUS OF ELASTICITY	E = 3E+07
MODULUS OF RIGIDITY	GR = 1.2E+07

JOINT DATA:

JOINT #	X-COORDINATE	Y-COORDINATE	CONCENTRATED JOINT MASS
1	0.00	0.00	0.00
2	60.00	0.00	0.00
3	0.00	60.00	0.00

BEAM ELEMENT DATA:

BEAM #	FIRST JOINT #	SECOND JOINT #	MASS/LENGTH	SECTION INERTIA	POLAR INERT / AREA	TORSIONAL CONSTANT
1	1	2	10.000	100.00	12.50	200.00
2	1	3	10.000	100.00	12.50	200.00

FIXED COORDINATES DATA:

JOINT #,	DIRECTION #	(X-ROT = 1 Y-ROT = 2 Z-DISPL = 3)
2	1	
2	2	
2	3	
3	1	
3	2	
3	3	

OUTPUT RESULTS:

SYSTEM STIFFNESS MATRIX

2.4000E+08	0.0000E+00	5.0000E+06
0.0000E+00	2.4000E+08	-5.0000E+06
5.0000E+06	-5.0000E+06	3.3333E+05

SYSTEM MASS MATRIX

2.3071E+04	0.0000E+00	1.8857E+03
0.0000E+00	2.3071E+04	-1.8857E+03
1.8857E+03	-1.8857E+03	4.4571E+02

(b) Output Results: Natural Frequencies and Modes

PROGRAM 8: NATURAL FREQUENCIES

DATA FILE: SK

EIGENVALUES:

3.963E+02	1.040E+04	2.387E+04
-----------	-----------	-----------

NATURAL FREQUENCIES (C.P.S.):

3.168	16.232	24.587
-------	--------	--------

EIGENVECTORS BY ROWS

-0.00103	0.00103	0.05569
0.00466	0.00466	-0.00000
-0.00832	0.00832	0.06460

(c) Input Data and Output Results: Response by Modal Superposition

PROGRAM 9: PROGRAM 9: MODAL

DATA FILE: D16.3C

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM	D = 3
NUMBER OF EXTERNAL FORCES	NF = 1
TIME STEP OF INTEGRATION	H = .01
GRAVITATIONAL INDEX (G = 0 for forces applied).	G = 0
PRINT TIME HISTORY NPRT = 1; ONLY MAX. VALUES NPRT = 0	NPRT = 0

FORCE #,	COORD. #	WHERE FORCE IS APPLIED,	NOM.	OF POINTS DEFINING THE
1		3		2

EXCITATION FUNCTION FOR FORCE #1:

TIME	EXCITATION
0.00	5000.00
0.10	5000.00

MODAL DAMPING RATIOS:

0	0	0
---	---	---

MAXIMUM RESPONSE :

COORD.	MAX. DISPL.	MAX. VELOC.	MAX. ACC.
1	0.00123	0.0309	2.97
2	0.00123	0.0309	2.97
3	0.05680	0.8914	36.37

The results given by the computer compare very closely to those obtained in Illustrative Examples 12.1 and 12.2. For example, the displacement given by the computer (not printed) at nodal coordinate 3, a time $t = 0.1$ sec, is

$$u_3(t = 0.1) = 0.0568 \text{ in}$$

while the hand calculation from Illustrative Example 12.2 results in

$$u_3(t = 0.1) = 10^{-3} (40 - 39.13 \cos 1.991 - 0.87 \cos 15.45) = 0.0568 \text{ in}$$

12.8 Dynamic Analysis of Grid Frames using SAP2000

Illustrative Example 12.4

Use SAP2000 to: (1) Model the plane grid frame shown in Fig. 12.5 using a total of four beam elements, (2) Determine the first three natural frequencies and corresponding mode shapes, and (3) Calculate the response due to the force $F(t) = 5000 \text{ lb}$ applied in the Z direction at joint 1 suddenly for 0.1 sec and then decreasing linearly to zero at time 0.5 sec.

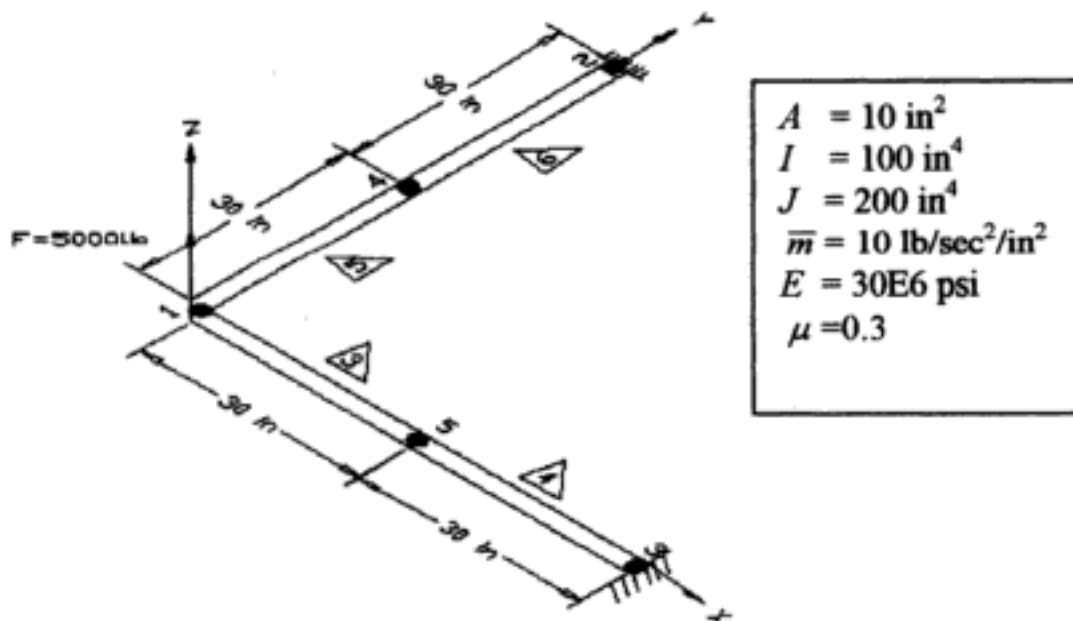


Fig. 12.5 Grid frame of Illustrative Example 12.4.

Solution:

The following commands are implemented in SAP2000:

Begin: Open SAP2000

Hint: Maximize both windows for a full view of the screen.

Select: Use the drop-down menu located in the lower right-hand corner of the screen to select "lb-in".

Select: From the Main Menu bar:
FILE>NEW MODEL
Number of Grid Spaces:
 X – direction = 2
 Y – direction = 2
 Z – direction = 0
Grid Spaces
 X – direction = 60
 Y – direction = 60
 Z – direction = 1. Then OK.

Edit: Click on 3D View icon in the Menu Bar
Maximize the 3D View window and use the PAN icon in the toolbar to drag the figure to the center of the screen.

Draw: From the Main Menu enter:
DRAW>DRAW FRAME ELEMENT
Click on (0,0,0) and drag cursor to (60,0,0), located at intersections of the grid lines. (One frame section completed)
Click on (0,0,0) and drag cursor to (0,60,0) (second frame element completed)

Select both frame elements by clicking on them.
Enter: EDIT>DIVIDE FRAMES
Enter: "Divide into " = 2
 Last/First ratio = 1. Then OK.

Label: From the Main Menu enter:
VIEW>SET ELEMENTS
Click on Joint Labels and Frame Labels. Then OK.
Use the PAN icon on the toolbar to center the figure on the screen.

Material: DEFINE>MATERIALS

Under "Materials" select STEEL

Click to Modify/Show Material

Enter:

Material Name = STEEL

Mass per Unit Volume = 1

Weight per Unit Volume = 0

Modulus of Elasticity = 30E6

Poisson's Ratio = 0.3

Accept remaining default values. Then OK, OK.

Frame Sections: DEFINE>FRAME SECTIONS

Click on Add I/Wide Flange

Click on "Add General" from the drop down menu

Enter the following in the "Property Data" Window

Cross Sectional Area = 10

Torsional Constant = 200

Moment of Inertia about 3 axis = 100

Moment of Inertia about 2 axis = 100

Shear Area in directions 2 and 3 = 0

Accept remaining default values.

Material select STEEL. Then OK, OK, OK.

Assign: Select all the elements of the frame by clicking on them.

ASSIGN>FRAME>SECTIONS

Select FSEC2. Then OK.

Boundary: Select joints 2 and 3 by clicking on them and enter:

ASSIGN>JOINT>RESTRAINTS

Select restraints in all directions. Then OK.

Load: DEFINE>STATIC LOAD CASES

Enter: Load = LOAD1

Type = LIVE

Self Weight Multiplier = 0

Click on Add New Load. Then OK.

Functions: DEFINE>TIME HISTORY FUNCTIONS

Click on Add New Function and accept default name FUNC1

Enter: Time = 0

Value = 5000 Then press ADD

Enter: Time = 0.1

Value = 5000 Then press ADD

Enter: Time = 0.5

Value = 0 then press ADD, and OK, OK.

Load Case: DEFINE>TIME HISTORY CASES

Enter: Add New History
Analysis Type = Linear
Number of Output Time Steps = 51
Time Steps = 0.01.
Click on "Envelopes"
Load Assignments:
Load = LOAD1
Function = FUNC1. Accept all other default values.
Then press ADD. Enter OK, OK.

Assign Loads: Click on Joint 1 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES
For LOAD1: Enter Load, Force Global Z = 1. Then OK.

Analyze: ANALYZE>SET OPTIONS

Check only UZ, RX, RY in Available DOF's (Degrees of Freedom)
Check Dynamic Analysis and then click on Set Dynamic Parameters
Enter: Number of Modes = 3.
Then OK, OK.

ANALYZE>RUN

Enter filename: EXD 12.4. Then SAVE.
At the conclusion of the calculations, after checking for errors, press OK.
Select 3D view.

Print Input Tables: FILE>PRINT INPUT TABLES

CLICK "Print to File" and accept all other default values.
Then OK.
Use Notepad or other word editor to retrieve and edit the input table file
EXD 12.4.txt.

(The edited Input Tables of Illustrative Example 12.4 are reproduced in Table 12.1)

Table 12.1 Edited Input Tables for Illustrative Example 12.4.

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	LIVE	0.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	TIME STEP INCREMENT
HIST1	LINEAR	50	0.01000

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	0.00000	0.00000	0.00000	0 0 0 0 0
2	60.00000	0.00000	0.00000	1 1 1 1 1
3	0.00000	60.00000	0.00000	1 1 1 1 1
4	30.00000	0.00000	0.00000	0 0 0 0 0
5	0.00000	30.00000	0.00000	0 0 0 0 0

FRAME ELEMENT DATA

FRAME	JNT1	JNT2	SCT	ANGLE	RLSS	SGMNT	R1	R2	FACTOR	LENGTH
3	1	4	FSEC2	0.000	000000	4	0.000	0.000	1.000	30.000
4	4	2	FSEC2	0.000	000000	4	0.000	0.000	1.000	30.000
5	1	5	FSEC2	0.000	000000	4	0.000	0.000	1.000	30.000
6	5	3	FSEC2	0.000	000000	4	0.000	0.000	1.000	30.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
1	0.000	0.000	1.000	0.000	0.000	0.000

Plot Displacement Function:

DISPLAY>SHOW TIME HISTORY TRACES

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 1

Vector Type = Displacement

Component = UZ.

Mode Number = Include All. Then OK.

Under List of Functions click once on "Joint1",

Click on ADD to move "Joint1" to Plot Functions column

Axis Labels

Horizontal = Time (sec)

Vertical = Displacement Component Z at Joint 1 (in)

Click "Display" to view the Displacement function

Enter: FILE>PRINT GRAPHICS to obtain a printed version of this plot.

(Fig. 12.6 reproduces the plot shown on the screen for the displacement component Z at Joint 1 of the plane grid structure of Illustrative Example 12.4)

Note: Displacement values may be read on the screen by moving the cursor along the curves shown on the screen. Table 12.2 provides the maximum displacement components in the Z direction at joint 1 for the plane grid structure of Illustrative Example 12.4.

Table 12.2 Maximum Displacements at node 1 in Z direction for the plane grid structure of Illustrative Example 12.4.

Direction	Maximum Displacement (in)
Z	.0784

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Natural Periods and Frequencies:

Using Notepad or other word editor, open the following file:

C:/SAP2000c/EXD 12.4.OUT

The Natural Period and Frequency table may be found in this file.

(Table 12.4 contains the edited Natural Period and Frequency values of Illustrative Example 12.4)

Table 12.4 Natural Period and Frequency values for the plane grid structure of Illustrative Example 12.4.

NATURAL PERIODS AND FREQUENCIES

MODE	PERIOD (SEC)	FREQUENCY (CYC/SEC)	FREQUENCY (RAD/SEC)	EIGENVALUE (RAD/SEC)**2
1	0.346017	2.890	18.159	329.735
2	0.085	11.757	73.868	5456.546
3	0.076	13.144	82.589	6820.980

Plot Bending Moment:

Enter: DISPLAY>SHOW TIME HISTORY TRACES

Click on "Define Functions"

Click to "Add Frame Element Forces"

Element ID = 3

Station = 1

Component = Moment 3-3

Accept all other default settings. Then OK, OK.

Click once on "Frame3" and then ADD to move "Frame3" to the Plot Function column

Highlight "Joint1" and click remove "Joint1" back to List of Functions column

Axis Labels

Horizontal = Time (sec)

Vertical = Bending Moment at joint 1 of Element 3 (lb-in)

Click Display to view the plot of the Bending moment and its maximum and minimum values.

Max = 2.256E04

Min = -6.191E04

(Fig. 12.7 contains a plot of the bending moment time function for frame element 3 of the plane grid structure of Illustrative Example 12.4)

Note: Bending moment values may be read on the screen by moving the cursor along the curve shown on the screen.

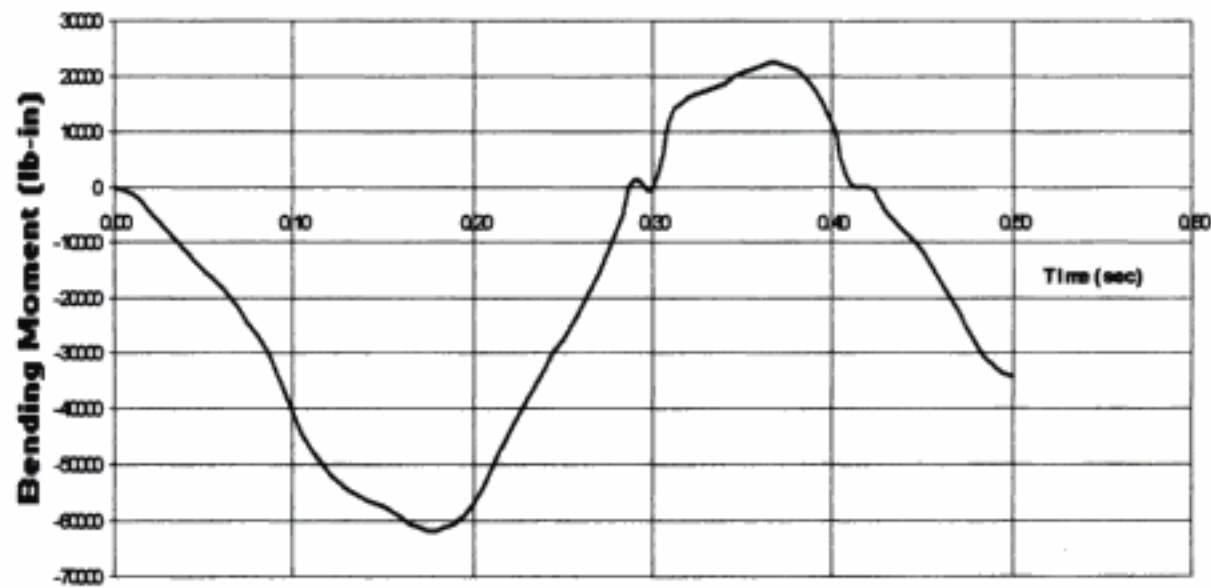


Fig 12.7 Bending moment time function for element 3, at section 1, of the plane grid of Illustrative Example 12.4.

Bending Moment Function:

A table for the bending moment function (or other internal force) for a member may be obtained by entering:

FILE > PRINT TABLES TO FILE

At the prompt enter filename “bending moment joint 1 element 3 EXD 12.4.txt”)

Then using Notepad or a similar word processor, the file “bending moment EXD 12.4.txt” may be edited to obtain a table of the Bending Moment and Time History Function. (An abbreviated table of the Bending Moment History Function for frame element 3 of the plane grid structure of Illustrative Example 12.4 is shown in Table 12.5)

Table 12.5 Bending moment time function for joint 1, element 3, of the plane grid structure of Illustrative Example 12.4.

Bending Moment Frame 3	
Time (sec)	Value (lb-in)
0.00	0.0
0.01	-1195.0
0.02	-4318.0
0.03	-8263.0
0.04	-11971.0
0.05	-15056.0
0.49	-32194.0
0.50	-34139.0

12.9 Summary

This chapter has presented the dynamic analysis of structures modeled as grid supporting loads applied normally to its plane. The dynamic analysis of grids requires the inclusion of torsional effects in the element stiffness and mass matrices. It also requires a transformation of coordinates of the element matrices previous to the assembling of the system matrix. The required matrices for torsional effects are developed and a computer program for the dynamic analysis of grids is presented. This program is also organized along the same pattern of the programs in the two preceding chapters for the dynamic analysis of beams and plane frames.

12.10 Problems

The following problems are intended for hand calculation, though it is recommended that whenever possible solutions should also be obtained using Program 15 to model the structure as a grid frame and Programs 8 and 9, respectively, to calculate natural frequencies and to solve for the response.

Problem 12.1

For the grid shown in Fig. P12.1 determine the system stiffness and mass matrices. Base the analysis on the three nodal coordinates indicated in the figure. Use consistent mass method.

Problem 12.2

Use static condensation to eliminate the rotational degrees of freedom and determine the transformation matrix and the reduced stiffness and mass matrices in Problem 12.1.

Problem 12.3

Determine the natural frequency for the reduced system in Problem 12.2.

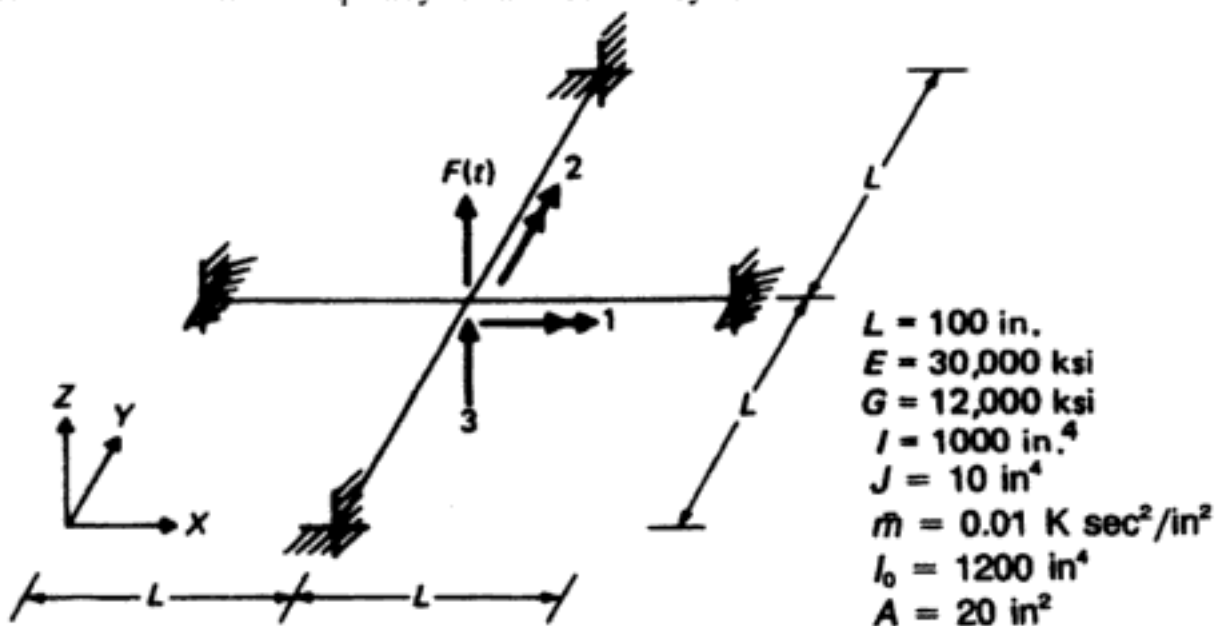


Fig. P12.1

Problem 12.4

Determine the natural frequencies and corresponding normal modes for the grid analyzed in Problem 12.1.

Problem 12.5

Determine the response of the grid shown in Fig. P12.1 when acted upon by a force $F(t) = 10$ Kip suddenly applied for one second at the nodal coordinate 3 as shown in the figure. Use results of Problem 12.2 to obtain the equation of motion for the condensed system. Assume 10% modal damping.

Problem 12.6

Use results from Problem 12.4 to solve Problem 12.5 on the basis of the three nodal coordinates as indicated in Fig. P12.1.

Problem 12.7

Determine the steady-state response of the grid shown in Fig. P12.1 when subjected to harmonic force $F(t) = 10 \sin 50t$ (Kip) along nodal coordinate 3. Neglect damping in the system.

Problem 12.8

Repeat Problem 12.7 assuming that the damping is proportional to the stiffness of the system, $[C] = a_0 [K]$, where $a_0 = 0.3$.

Problem 12.9

Determine the equivalent nodal forces for a member of a grid loaded with a dynamic force, $P(t) = P_0 f(t)$, uniformly distributed along its length.

Problem 12.10

Determine the equivalent nodal forces for a member of a grid supporting a concentrated dynamic force $F(t)$ as shown in Fig. P12.10.

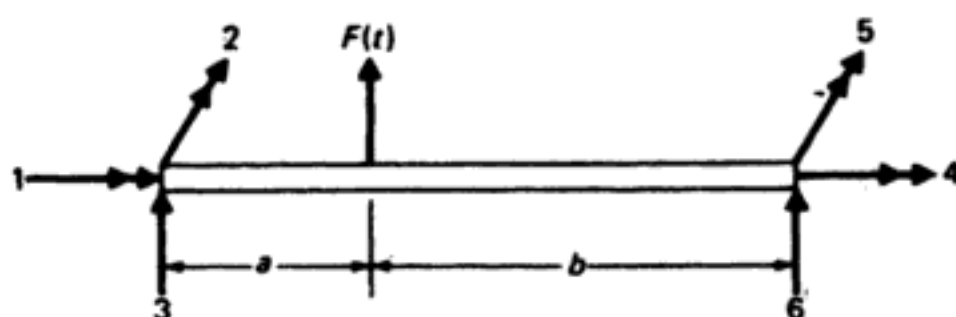


Fig. P12.10.

The following problems are intended for computer solution using Program 15, to model the structure and Program 8 to determine natural frequencies and nodal shapes and Program 9 to calculate the response.

Problem 12.11

Determine the natural frequencies and corresponding normal modes for the grid shown in Fig. P12.1.

Problem 12.12

Determine the response of the grid shown in Fig. P12.1 when acted upon by the force depicted in Fig. P12.12 acting along nodal coordinate 3. Neglect damping in the system.

Problem 12.13

Repeat Problem 12.12 for 15% damping in all the modes. Use modal super-position method.

Problem 12.14

Repeat Problem 12.12. Use step-by-step linear acceleration method, Program 19. Neglect damping.

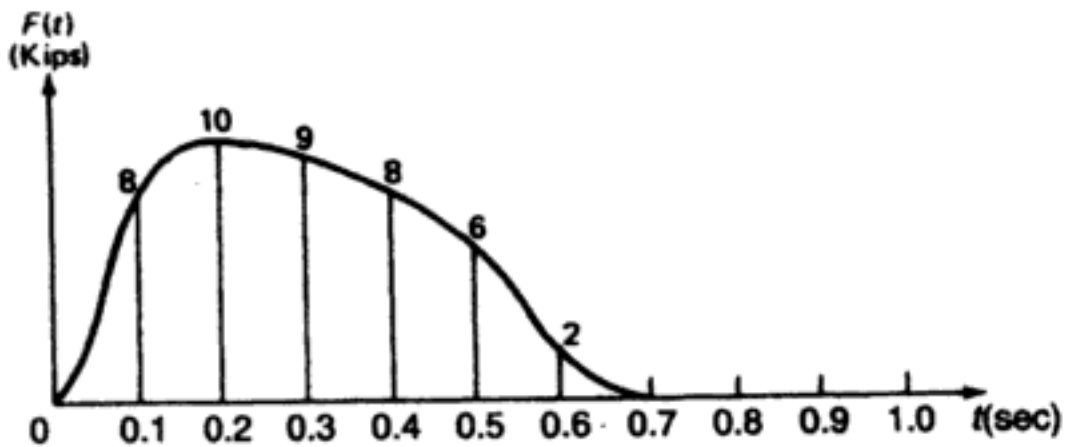


Fig. P12.12.

13 Dynamic Analysis of Three-dimensional Frames

The stiffness method for dynamic analysis of frames presented in Chapter 11 for plane frames and in Chapter 12 for grid frames can readily be expanded for the analysis of three-dimensional space frames. Although for the plane frame or for the grid there were only three nodal coordinates at each joint, the three-dimensional frame has a total of six possible nodal displacements at each unconstrained joint: three translation components along the x , y , z axes and three rotational components about these axes. Consequently, a beam element of a three-dimensional frame or a space frame has for its two joints a total of 12 nodal coordinates; hence the resulting element matrices will be of dimension 12×12 .

The dynamic analysis of three-dimensional frames resulting in a comparatively longer computer program in general, requiring more input data as well as substantially more computational time. However, except for size, the analysis of three-dimensional frames by the stiffness method of dynamic analysis is basically identical to the analysis of plane frames or grid frames.

13.1 Element Stiffness Matrix

Figure 13.1 shows a beam segment of a space frame with its 12 nodal coordinates numbered consecutively. The convention adopted is to label first the three translatory displacements of the first joint followed by the three rotational displacements of the same joint, then to continue with the three translatory displacements of the second joint and finally the three rotational displacements of

this second joint. The double arrows used in Fig. 13.1 serve to indicate rotational nodal coordinates; thus, these are distinguished from translational nodal coordinates for which single arrows are used.

The stiffness matrix for a three-dimensional uniform beam segment is readily written by the superposition of the axial stiffness matrix from eq. (11.3), the torsional stiffness matrix from eq. (12.6), and the flexural stiffness matrix from eq. (10.20). The flexural stiffness matrix is used twice in forming the stiffness matrix of a three-dimensional beam segment to account for the flexural effects in the two principal planes of the cross section. Proceeding to combine in an appropriate manner these matrices, we obtain in eq. (13.1) the stiffness equation for a uniform beam segment of a three-dimensional frame, namely

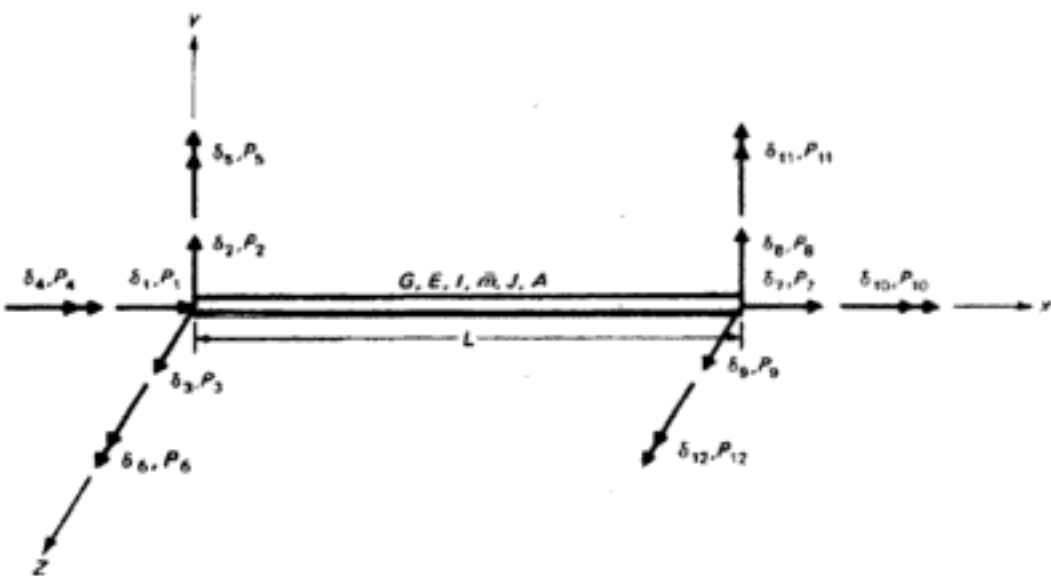


Fig. 13.1 Beam segment of a space frame showing forces and displacements at the nodal coordinates.

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & & & & & & & & & & & \\ 0 & \frac{12EI_y}{L^3} & & & & & & & & & & \\ 0 & 0 & \frac{12EI_z}{L^3} & & & & & & & & & \\ 0 & 0 & 0 & \frac{GI}{L} & & & & & & & & \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_z}{L} & & & & & & & \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_y}{L} & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & & & & & \\ \frac{-EA}{L} & \frac{-12EI_y}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_x}{L^3} & & & & \\ 0 & 0 & \frac{-12EI_z}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_z}{L^3} & & & \\ 0 & 0 & 0 & \frac{-GI}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI}{L} & & \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_z}{L} & 0 & 0 & 0 & \frac{6EI_z}{L^2} & 0 & \frac{4EI_y}{L} & \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_y}{L} & 0 & \frac{-6EI_x}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{bmatrix} \quad (13.1)$$

Symmetric

or in condensed notation

$$\{P\} = [K] \{\delta\} \quad (13.2)$$

in which I_y and I_z are, respectively, the cross-sectional moments of inertia with respect to the principal axes labeled as y and z in Fig. 13.1, and L , A , and J are respectively the length, cross-sectional area, and torsional constant of the beam element.

13.2 Element Mass Matrix

The lumped mass matrix for the uniform beam segment of a three-dimensional frame is simply a diagonal matrix in which the coefficients corresponding to translatory and torsional displacements are equal to one-half of the total inertia of the beam segment while the coefficients corresponding to flexural rotations are assumed to be zero. The diagonal lumped mass matrix for the uniform beam of distributed mass \bar{m} and polar mass moment $I_{\bar{m}} = \bar{m}I_o / A$ of inertia per unit of length may be written conveniently as

$$[M_L] = \frac{\bar{m}L}{2} \begin{bmatrix} 1 & 1 & 1 & I_o/A & 0 & 0 & 1 & 1 & 1 & I_o/A & 0 & 0 \end{bmatrix} \quad (13.3)$$

in which I_o is the polar moment of inertia of the cross-sectional area A .

The consistent mass matrix for a uniform beam segment of a three-dimensional frame is readily obtained combining the consistent mass matrices, eq. (11.26) for axial effects, eq. (12.8) for torsional effects, and eq. (10.34) for flexural effects. The appropriate combination of these matrices results in the consistent mass matrix for the uniform beam segment of a three-dimensional frame, namely,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 140 & & & & & & & & & & & \\ 0 & 156 & & & & & & & & & & \\ 0 & 0 & 156 & & & & & & & & & \\ 0 & 0 & 0 & \frac{140I_o}{A} & & & & & & & & \\ 0 & 0 & -22L & 0 & 4L^2 & & & & & & & \\ 0 & 22L & 0 & 0 & 0 & 4L^2 & & & & & & \\ 70 & 0 & 0 & 0 & 0 & 0 & 140 & & & & & \\ 0 & 54 & 0 & 0 & 0 & 13L & 0 & 156 & & & & \\ 0 & 0 & 54 & 0 & -13L & 0 & 0 & 0 & 156 & & & \\ 0 & 0 & 0 & \frac{70I_o}{A} & 0 & 0 & 0 & 0 & 0 & \frac{140I_o}{A} & & \\ 0 & 0 & 13L & 0 & -3L^2 & 0 & 0 & 0 & 22L & 0 & 4L^2 & \\ 0 & -13L & 0 & 0 & 0 & -3L^2 & 0 & -22L & 0 & 0 & 0 & 4L^2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{bmatrix} \quad (13.4)$$

or in condensed notation

$$\{P\} = [M] \{\ddot{\delta}\} \quad (13.5)$$

13.3 Element Damping Matrix

The damping matrix for a uniform beam segment of a three-dimensional frame may be obtained in a manner entirely similar to those of the stiffness, eq. (13.1), and mass, eq. (13.4), matrices. Nevertheless, as was discussed in Section 10.5, in practice, damping is generally expressed in terms of damping ratios for each mode of vibration. Therefore, if the response is sought using the modal superposition method, these damping ratios are directly introduced in the modal equations. When the damping matrix is required explicitly, it may be determined from given values of damping ratios by the methods presented in Chapter 22.

13.4 Transformation of Coordinates

The stiffness and the mass matrices, respectively, given by eqs. (13.1) and (13.4), are referred to local coordinates axes fixed on the beam segment. Inasmuch as the elements of these matrices corresponding to the same nodal coordinates of the structure should be added to obtain the system stiffness and mass matrices, it is necessary first to transform these matrices to the same reference system, the global system of coordinates. Figure 13.2 shows these two reference systems, the x, y, z axes representing the local system of coordinates and the X, Y, Z axes representing the global system of coordinates. Also shown in this figure is a general vector A with its components X, Y, Z along the global coordinates. This vector A may represent any force or displacement at the nodal coordinates of one of the joints of the structure. To obtain the components of vector A along one of the local axes x, y, z , it is necessary to add the projections along that axis of the components X, Y, Z . For example, the component x of vector A along the x coordinate is given by

$$x = X \cos xX + Y \cos xY + Z \cos xZ \quad (13.6a)$$

in which $\cos xY$ is the cosine of the angle between axes x and Y and corresponding definitions for other cosines. Similarly, the y and z components of A are

$$y = X \cos yX + Y \cos yY + Z \cos yZ \quad (13.6b)$$

$$z = X \cos zX + Y \cos zY + Z \cos zZ \quad (13.6c)$$

Equations (13.6) are conveniently written in matrix notation as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{Bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \quad (13.7)$$

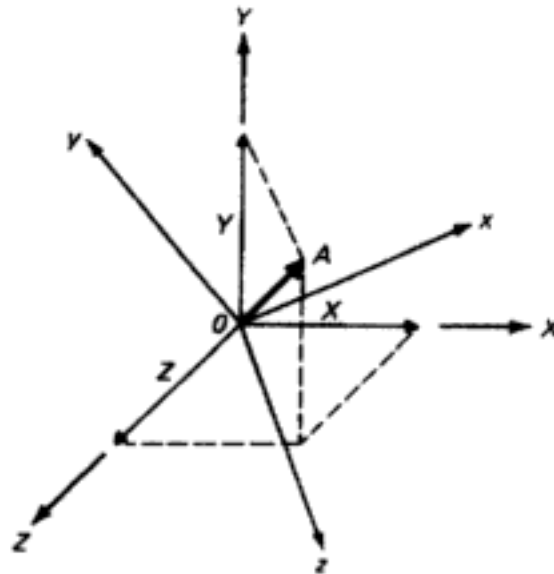


Fig. 13.2 Components of a general vector A in local and global coordinates.

or in short notation

$$\{A\} = [T_1] \{\bar{A}\} \quad (13.8)$$

in which $\{A\}$ and $\{\bar{A}\}$ are, respectively, the components in the local and global systems of the general vector A and $[T_1]$ the transformation matrix given by

$$[T_1] = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix} \quad (13.9)$$

The cosines required in the transformation matrix $[T_1]$, are usually calculated in computer codes from the global coordinates of three points. The two points defining the two ends of the beam element along the local x axis and any third point located in x - y local plane in which y is one of the principal axes of the cross-sectional area of the member. The input data containing the global coordinates of these three points are sufficient for the evaluation of all the cosine terms in eq. (13.9). To demonstrate this fact let us designate by x_i, y_i, z_i and x_j, y_j, z_j the coordinates of point ① and ② at the two ends of a beam element and by x_p, y_p, z_p the coordinates of a point P placed on the local x - y plane. Then the direction cosines of local axis x along the beam element are given by

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$$y_x \hat{i} + y_y \hat{j} + y_z \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos xX & \cos xY & \cos xZ \\ \cos zX & \cos zY & \cos zZ \end{vmatrix} \quad (13.17)$$

Therefore,

$$\cos yX = \frac{y_x}{|Y|}, \quad \cos yY = \frac{y_y}{|Y|}, \quad \cos yZ = \frac{y_z}{|Y|}$$

where

$$\begin{aligned} y_x &= \cos xY \cos zZ - \cos xZ \cos zY \\ y_y &= \cos zX \cos zX - \cos xX \cos zZ \\ y_z &= \cos xX \cos zY - \cos xY \cos zX \end{aligned} \quad (13.18)$$

and

$$|Y| = \sqrt{y_x^2 + y_y^2 + y_z^2}$$

We have, therefore, shown that knowledge of the coordinates of points at the two ends of an element of a point P on the local plane x - y suffices to calculate the direction cosines of the transformation matrix $[T_1]$ in eq. (13.9). The choice of point P is generally governed by the geometry of the structure and the orientation of the principal directions of the cross section of the member. Quite often point P is selected as a known point in the structure, which is placed on the local axis y , although, as it has been shown, the point P could be any point in the plane formed by the local x - y axes.

Alternatively, the direction cosines in the transformation matrix, eq. (13.9) may be calculated from the nodal coordinates (x_i, y_i, z_i) and (x_j, y_j, z_j) at the two ends of the beam element and the knowledge of an angle known as the angle of rolling. This angle provides the rational information of the principal axes of the cross-sectional area with respect to an orientation defined as the standard orientation of these axes. The analytical development to implement the calculation of the direction cosines using the angle of rolling is presented at the end of this chapter as Problems 13.1 and 13.2.

We have, therefore, shown that the knowledge of the coordinates at the two ends of an element, together with the knowledge of either a third point located in the x - y local plane in which y is one of the principal axes of the cross-sectional area of the member, or alternatively, the knowledge of the angle of rolling, suffices to calculate the direction cosines of the transformation matrix $[T_1]$ in eq. (13.9).

For the beam segment of a three-dimensional frame, the transformation of the nodal displacement vectors involve the transformation of linear and angular displacement vectors at each joint of the segment. Therefore, a beam element of a space frame requires, for the two joints, the transformation of a total of four displacement vectors. This transformation of the 12 nodal displacements $\{\bar{\delta}\}$ in

global coordinates to the displacement $\{\delta\}$ in local coordinates may be written in abbreviated form as

$$\{\delta\} = [T] \{\bar{\delta}\} \quad (13.19)$$

in which

$$[T] = \begin{bmatrix} [T_1] & & & \\ & [T_1] & & \\ & & [T_1] & \\ & & & [T_1] \end{bmatrix}$$

Analogously, the transformation from nodal forces $\{\bar{P}\}$ in global coordinates to nodal forces $\{P\}$ in local coordinates is given by

$$\{P\} = [T] \{\bar{P}\} \quad (13.20)$$

Finally, to obtain the stiffness matrix $[\bar{K}]$ and the mass matrix $[\bar{M}]$ in reference to the global system of coordinates, we simply substitute, into eq (13.2), $\{\delta\}$ from eq. (13.19) and $\{P\}$ from eq. (13.21) to obtain

$$[T] \{\bar{P}\} = [K] [T] \{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [T]^T [K] [T] \{\bar{\delta}\} \quad (13.21)$$

since $[T]$ is an orthogonal matrix. From eq. (13.21), we may write

$$\{\bar{P}\} = [\bar{K}] \{\bar{\delta}\} \quad (13.22)$$

in which $[\bar{K}]$ is defined as

$$[\bar{K}] = [T]^T [K] [T] \quad (13.25)$$

Analogously, the mass matrix in eq. (13.5) is transformed from local to global coordinates by

$$[\bar{M}] = [T]^T [M] [T] \quad (13.24)$$

and the damping matrix $[C]$ by

$$[\bar{C}] = [T]^T [C] [T] \quad (13.25)$$

13.5 Differential Equation of Motion

The direct method which was explained in detail in Chapter 10 may also be used to assemble the stiffness, mass, and damping matrices from the corresponding matrices for a three-dimensional beam segment, eqs. (13.23), (13.24) and (13.25), which are referred to the global system of coordinates. The differential equations of motion

referred to the global system of coordinates. The differential equations of motion which are obtained by establishing the dynamic equilibrium among the inertial, damping, and elastic forces with the external forces may be expressed in matrix notation as

$$[M] \{\ddot{u}\} + [C] \{\dot{u}\} + [K] \{u\} = \{F(t)\} \quad (13.26)$$

in which $[M]$, $[C]$, and $[K]$ are, respectively, the system mass, damping, and stiffness matrices, $\{\ddot{u}\}$, $\{\dot{u}\}$, and $\{u\}$ are the system acceleration, velocity, and displacement vectors, and $\{F(t)\}$ is the force vector which includes the forces applied directly to the joints of the structure and the equivalent nodal forces for the forces not applied at the joints.

13.6 Dynamic Response

The integration of the differential equations of motion, eq. (13.26), may be accomplished by any of the methods presented in previous chapters to obtain the response of structures modeled as beams, plane frames, or grids. The selection of the particular method of solution depends, as discussed previously, on the linearity of the differential equation, that is, whether the stiffness matrix $[K]$ or any other coefficient matrix is constant, and also depends on the complexity of the excitation as a function of time. When the differential equations of motion, eq. (13.26), are linear, the modal superposition method is applicable. This method, as we have seen in the preceding chapters, requires the solution of an eigenproblem to uncouple the differential equations resulting in the modal equations of motion.

If the structure is assumed to follow an elastoplastic behavior or any other form of nonlinearity, it is necessary to resort to some kind of numerical integration in order to solve the differential equations of motion, eq. (13.26). In Chapter 16, the linear acceleration method with a modification known as the Wilson- θ method is presented for analysis of linear structures with an elastic behavior.

13.7 Program 16—Modeling Structures as Space Frames

Program 16 calculates the stiffness and mass matrices for a three-dimensional frame and stores the coefficients of these matrices in a file. Since the stiffness and mass matrices are symmetric matrices, it is only necessary to store the coefficients in the upper triangle of these matrices. The program also stores in another file the general data of the space frame. The information stored in these files is needed by the programs to undertake the dynamic analysis of the structure, such as the determination of natural frequencies or the response of the structure subjected to external excitation.

Illustrative Example 13.1

For the three-dimensional frame shown in Fig. 13.3, determine (a) the stiffness and mass matrices, and (b) the natural frequencies and corresponding modal shapes. Model the structure with four beam elements and use consistent mass matrix formulation.

Solution:
As the first step in the analysis, the frame is divided in four elements, as indicated in Fig. 13.3. This division of the structure results in five nodes with a total of 30 nodal coordinates, of which 24 are fixed. The numerical values needed for analysis of this structure are given in Table 13.1.

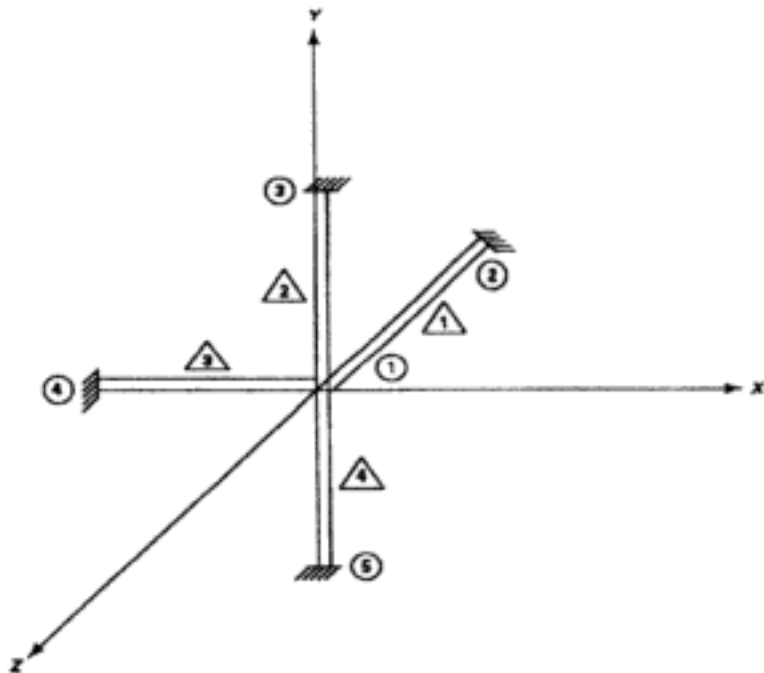


Fig. 13.3 Space Frame of Example 13.1.

Table 13.1 Input Data for Illustrative Example 13.1

Quantity	Members 1,3	Members 2,4
Modulus of elasticity (psi)	30×10^6	30×10^6
Modulus of rigidity (psi)	12×10^6	12×10^6
Distributed mass (lb · sec ² /in ²)	0.2	0.1
Cross-sectional y moment of inertia (in ⁴)	200	64
Cross-sectional z moment of inertia (in ⁴)	200	64
Torsional constant (in ⁴)	40.0	12.8
Cross-sectional area (in ²)	50	28

Input Data and Output Results

(a) Modeling the structure

PROGRAM 16: MODELING SPACEFRAMES

DATA FILE: D16

NUMBER OF JOINTS

NJ = 5

NUMBER OF BEAM ELEMENTS

NE = 4

NUMBER OF FIXED COORDINATES

NC = 24

MODULUS OF ELASTICITY

E = 3E+07

MODULUS OF RIGIDITY

R = 1.2E+07

JOINT DATA:

JOINT #	X-COORD.	Y-COORD.	Z-COORD.	JOINT MASS
1	0.00	0.00	0.00	0
2	0.00	0.00	-200.00	0
3	0.00	200.00	0.00	0
4	-200.00	0.00	0.00	0
5	0.00	-200.00	0.00	0

BEAM ELEMENT DATA:

MEMBER'S ROLLING ANGLES (DEGREES)

1	0.000
2	0.000
3	0.000
4	0.000

ELEM #	FIRST JOINT#	SECOND JOINT#	MASS/ LENGTH	SECT-Y INERTIA	SECT-Z INERTIA	TORSION CONST	SECTION AREA
1	1	2	0.200	200.00	200.00	40.00	50.00
2	1	3	0.100	64.00	64.00	12.80	28.00
3	4	1	0.200	200.00	200.00	40.00	50.00
4	5	1	0.100	64.00	64.00	12.80	28.00

FIXED COORDINATES DATA:

DIRECTION # (X=1, Y=2, Z=3, X-ROT=4, Y-ROT=5, Z-ROT=6)

NODE DIR.	NODE DIR.	NODE DIR.	NODE DIR.	NODE DIR.	NODE DIR.	NODE DIR.	NODE DIR.
2	1	2	2	2	3	2	4
3	2	3	3	3	4	3	5
4	3	4	4	4	5	4	6
5	4	5	5	5	6	5	1

OUTPUT RESULTS:

SYSTEM STIFFNESS MATRIX

0.7515E+07	0.0000E+00	0.0000E+00	0.0000E+00	-.9000E+06	0.0000E+00
0.0000E+00	0.8418E+07	0.0000E+00	0.9000E+06	0.7813E-02	-.9000E+06
0.0000E+00	0.0000E+00	0.7515E+07	0.0000E+00	0.9000E+06	0.7813E-02
0.0000E+00	0.9000E+06	0.0000E+00	0.1992E+09	0.1000E+01	0.1000E+01
-.9000E+06	0.7813E-03	0.9000E+06	0.1000E+01	0.2415E+09	-.1000E+01
0.0000E+00	-.9000E+06	0.1813E-02	0.1000E+01	-.1000E+01	0.1992E+09

****SYSTEM MASS MATRIX****

0.4305E+02	0.0000E+00	-.1192E-06	0.0000E+00	-0.4190E+03	0.0000E+00
0.0000E+00	0.4305E+02	0.1191E-06	0.4190E+03	0.0000E+00	-0.4190E+03
-.1192E-06	0.1192E-06	0.4305E+02	0.0000E+00	0.4190E+03	0.0000E+00
0.0000E+00	0.4190E+03	0.0000E+00	0.3058E+05	0.0000E+00	0.0000E+00
-0.4190E+03	0.0000E+00	0.4190E+03	0.0000E+00	0.3054E+05	0.0000E+00
0.0000E+00	-0.4190E+03	0.0000E+00	0.0000E+00	0.0000E+00	0.3058E+05

(b) Natural frequencies and modal shapes**EIGENVALUES:**

6.420E+03 6.502E+03 7.833E+03 1.746E+05 2.396E+05 2.675E+05

NATURAL FREQUENCIES (C.P.S.):

.1275E+02 .1283E+02 .1409E+02 .6650E+02 .1190E+02 .8232E+02

EIGENVALUES BY ROWS:

0.00000	0.00143	0.00000	0.00562	0.00000	-0.00090
0.00000	-0.00105	0.00000	0.00093	0.00000	0.00563
-0.00188	0.00000	0.00188	0.00000	0.00567	0.00000
0.10777	0.00000	0.10777	0.00000	-0.00000	0.00000
0.12588	0.00000	-0.12588	0.00000	0.00354	0.00000
0.00000	0.17198	0.00000	0.00248	0.00000	0.00248

13.8 Dynamic Response of Three Dimensional Frames Using SAP2000**Illustrative Example 13.2**

For the three-dimensional frame shown in Fig. 13.4(a), determine: (a) The first 6 natural frequencies and corresponding modal shapes and (b) The dynamic response when the frame is acted upon by the impulsive force depicted in Fig. 13.4(b) applied at joint 8 of the frame along the Y- direction as shown in the figure.

Problem Data (for all members):**Modulus of elasticity:**

$$E = 30 \times 10^6 \text{ psi}$$

Modulus of rigidity:

$$G = 12 \times 10^6 \text{ psi}$$

Distributed mass:

$$m = 0.2 \text{ lb} \cdot \text{sec}^2/\text{in}$$

Concentrated masses:

$$m = 10 \text{ lb} \cdot \text{sec}^2/\text{in}$$

Cross-sectional y moment of inertia:

$$I_y = 300 \text{ in}^4$$

Cross-sectional z moment of inertia:

$$I_z = 400 \text{ in}^4$$

Torsional constant:

$$J = 500 \text{ in}^4$$

Cross-sectional area:

$$A = 20 \text{ in}^2$$

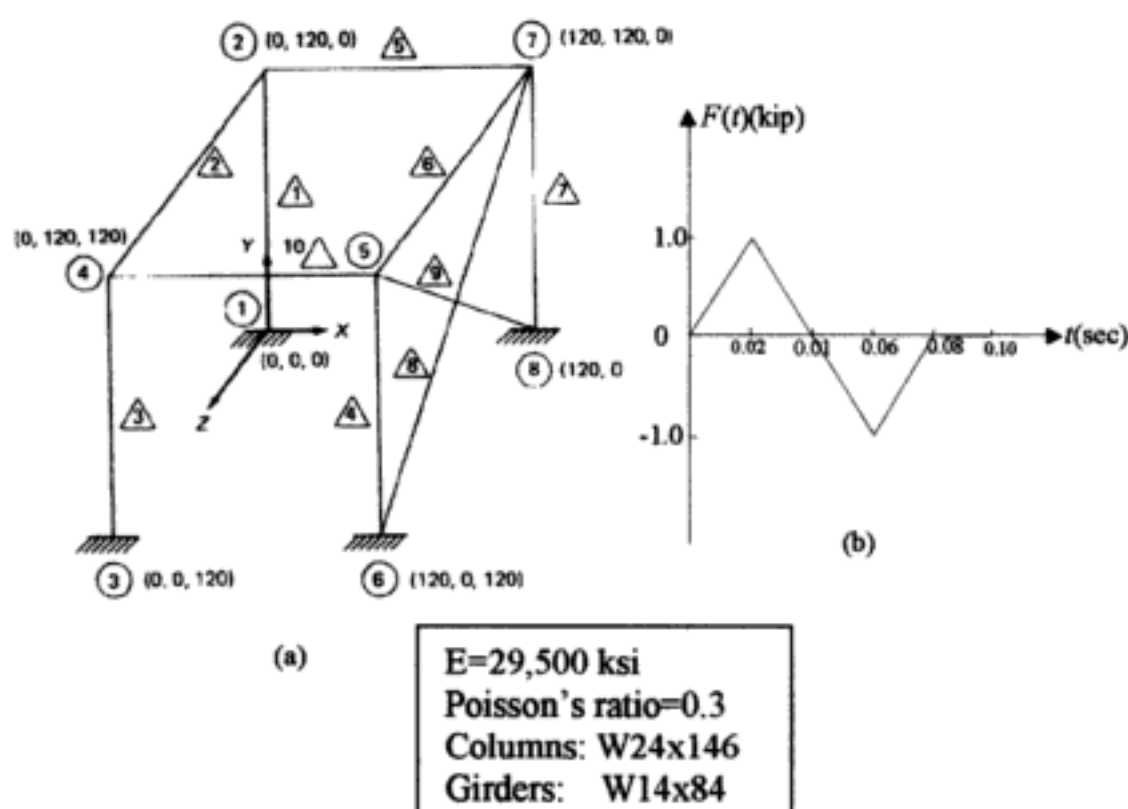


Fig 13.4 Space frame for Illustrative Example 13.2

Solution:

The following commands are implemented in SAP2000:

Begin: Open SAP2000

Hint: Maximize both windows for a full view of the screen.

Select: In the lower right-hand corner of the screen use the drop-down menu to select "kip-in"

Model: From the main menu bar:
 FILE>NEW MODEL FROM TEMPLATE
 Click on the space frame shown in the screen
 Enter:

Number of grid spaces
 Number of stories=1
 Number of Bays along X=1
 Number of Bays along Y=1
 Grid Distance:
 Story Height=180
 Bay width along X=240
 Bay width along Y=114. Then OK

Edit: Maximize the 3-D screen and use the PAN-icon in the tool bar to drag the figure to the center of the screen.
Then enter from the main menu:
VIEW>3-D VIEW
Rotate plan view to approximately 30 degrees to have a frontal view of the X axis.

Label: From the Main Menu enter:
VIEW>SET ELEMENTS
Click on Joint Labels on Frame labels, then OK

Material: From the Main Menu enter:
DEFINE>MATERIALS
Select STEEL
Click on modify/show material
Set: Mass per unit of volume= 0
Weight per unit of volume= 0
Modules of elasticity= 29,500
Poisson ratio= 0.3, then OK,OK

Restraints: Select joint 1,3,5,and 7 with a click of the mouse.
Enter from the Main Menu:
ASSIGN>JOINT>RESTRAINTS
Select: "Restraint all." Then OK.

Sections: From the Main Menu enter
DEFINE>FRAME SECTIONS
Click on Import/Wide flange
Select section properties .and click. OPEN
Scroll on the sections to select:
(Hold the control key)
W24x146 and W14x82; then OK,OK,OK

Assign: Select on all the columns by clicking on them.
From the Main Menu enter:
ASSIGN>FRAME>SECTIONS
Select section W24*146 then OK
Select all the girders by clicking on them
From the Main Menu enter:
ASSIGN>FRAME>SECTION
Select section W14*82 then OK.

Mass: Select Joints 2,4,6,and 8 and enter:
ASSIGN>JOINT>MASSES
Enter Masses:
Direction 1= 0.001

Direction 2= 0.001
Direction 3= 0.001; then OK

Load: From the Main Menu enter:
DEFINE>STATIC LOAD CASES
Change type of load to LIVE
And self weight multiplies to 0
Then click on change load and OK

Load Function: DEFINE>TIME HISTORY FUNCTIONS
Click on "Add new function"
Click on "add" to register the first point (0,0)
Enter= Time= 0.0, Value= 1.0, then press ADD
Enter= Time=0.04, Value= 0.0, then press ADD
Enter= Time= 0.06, Value= -1.0, then press ADD
Enter=Time= 0.08, Value= 0.0, then press ADD
Enter= Time= 2.0, Value= 0.0, then press ADD and OK,OK

Load Case: DEFINE>TIME HISTORY CASES
Click on Add new history
Enter: Number of output time steps= 50
Output time step= 0.004
Check: Envelopes
Load Assignments
Load= LOAD1
FUNCTION=FUNCI
Accept all of the default values
Then press ADD and OK, OK

Assign Load: Click on joints and enter=
ASSIGN>JOINT STATIC LOADS>FORCES
Enter: Force Global Y= 1.0; then OK

Analyze: ANALYZE>SET OPTIONS
Accept check on all available DOFs
Click on Dynamic Analyze and set Dynamic Parameters
Enter: Numbers of Modes=6 then OK,OK
ANALYZE>RUN
Enter Filename: EXD13.2, then SAVE
At the conclusion of the calculations,
After checking for errors, press OK.

Print Input Tables
FILES>PRINT INPUT TABLES
Check: "Print to File". Then OK.
Use NOTEPAD or another editor to retrieve, edit and print
the file C:\SAP2000\EXD13.2

(The edited Input Tables for Illustrative Example 13.2 is given in Table 13.2)

Table 13.2 Edited Input Tables for Illustrative Example 13.2

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
-------------	-----------	----------------

LOAD1	LIVE	0.0000
-------	------	--------

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	INCREMENT
--------------	--------------	----------------------	-----------

HIST1	LINEAR	50	0.05000
-------	--------	----	---------

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-576.00000	0.00000	0.00000	1 1 1 0 0 0
2	0.00000	0.00000	0.00000	0 0 0 0 0 0
3	576.00000	0.00000	0.00000	0 1 1 0 0 0
4	-288.00000	0.00000	384.00000	0 0 0 0 0 0
5	288.00000	0.00000	384.00000	0 0 0 0 0 0
6	-288.00000	0.00000	0.00000	0 0 0 0 0 0
7	288.00000	0.00000	0.00000	0 0 0 0 0 0
8	0.00000	0.00000	384.00000	0 0 0 0 0 0

FRAME ELEMENT DATA

FRAME	JNT-1	JNT-2	SECTION	ANGLE	RELEASES	SEGMENTS	LENGTH
4	1	4	FSEC2	0.000	000000	2	480.000
5	2	5	FSEC2	0.000	000000	2	480.000
6	2	4	FSEC2	0.000	000000	2	480.000
7	3	5	FSEC2	0.000	000000	2	480.000
8	6	4	FSEC2	0.000	000000	2	384.000
9	2	8	FSEC2	0.000	000000	2	384.000
10	7	5	FSEC2	0.000	000000	2	384.000
11	7	8	FSEC2	0.000	000000	2	480.000
12	1	6	FSEC2	0.000	000000	4	288.000
13	6	2	FSEC2	0.000	000000	4	288.000
14	2	7	FSEC2	0.000	000000	4	288.000
15	7	3	FSEC2	0.000	000000	4	288.000
16	4	8	FSEC2	0.000	000000	4	288.000
17	8	5	FSEC2	0.000	000000	4	288.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	-Y	-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
2	0.000	0.000	-1.000	0.000	0.000	0.000

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Table 13.3 Continued**5 LOAD1**

0.00	0.00	3.320E-02	-2.986E-02	-3.006E-03	-3.58	3.98
60.00	0.00	3.320E-02	-2.986E-02	-3.006E-03	-1.79	1.99
120.00	0.00	3.320E-02	-2.986E-02	-3.006E-03	0.00	0.00
180.00	0.00	3.320E-02	-2.986E-02	-3.006E-03	1.79	-1.99
240.00	0.00	3.320E-02	-2.986E-02	-3.006E-03	3.58	-3.98

6 LOAD1

0.00	0.00	-3.320E-02	-2.994E-02	-3.025E-03	-3.59	-3.98
60.00	0.00	-3.320E-02	-2.994E-02	-3.025E-03	-1.80	-1.9
120.00	0.00	-3.320E-02	-2.994E-02	-3.025E-03	0.00	0.00
180.00	0.00	-3.320E-02	-2.994E-02	-3.025E-03	1.80	1.99
240.00	0.00	-3.320E-02	-2.994E-02	-3.025E-03	3.59	3.98

7 LOAD1

0.00	3.919E-05	4.584E-02	6.245E-02	-1.604E-03	3.56	2.61
28.50	3.919E-05	4.584E-02	6.245E-02	-1.604E-03	1.78	1.31
57.00	3.919E-05	4.584E-02	6.245E-02	-1.604E-03	-4.629E-03	-1.338E-05
85.50	3.919E-05	4.584E-02	6.245E-02	-1.604E-03	-1.78	-1.31
114.00	3.919E-05	4.584E-02	6.245E-02	-1.604E-03	-3.56	-2.61

8 LOAD1

0.00	4.992E-01	7.214E-01	6.245E-02	-1.604E-03	3.56	41.05
28.50	4.992E-01	7.214E-01	6.245E-02	-1.604E-03	1.78	20.49
57.00	4.992E-01	7.214E-01	6.245E-02	-1.604E-03	-4.629E-03	-6.803E-02
85.50	4.992E-01	7.214E-01	6.245E-02	-1.604E-03	-1.78	-20.63
114.00	4.992E-01	7.214E-01	6.245E-02	-1.604E-03	-3.56	-41.19

Plot Displacement Function:

DISPLAY>SHOW TIME HISTORY TRACES

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 8

Vector Type = Displacement

Component = UX.

Mode Number = Include All. Then OK.

Under List of Functions click once on Joint8,

Click on ADD to move Joint8 to Plot Functions column

Axis Labels

Horizontal = Time (sec)

Vertical = X-Displacements Joint 8 (in)

Click "Display" to view Displacement functions

Enter: FILE>PRINT GRAPHICS to obtain a printed version of this plot.

(Figure 13.5 represents the plot of the Displacement Function Values for Joints 2, 3 and 4 of Illustrative Example 13.2.)

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13.10 Problems

Problem 13.1

For the three-dimensional frame of Illustrative Example 13.2 reproduced for convenience in Fig. P13.1, determine the dynamic response when the frame is acted upon by the harmonic force $F(t) = 2000 \sin 5.3t$ (lb) applied at joint 2 of the frame along the Y-direction as shown in the figure.

Problem Data (for all members):

$$E = 30 \times 10^6 \text{ psi}$$

$$G = 12 \times 10^6 \text{ psi}$$

$$\bar{m} = 0.2 \text{ lb sec}^2/\text{in/in}$$

$$m = 10 \text{ lb sec}^2/\text{in}$$

$$I_y = 300 \text{ in}^4$$

$$I_z = 400 \text{ in}^4$$

$$J = 500 \text{ in}^4$$

$$A = 20 \text{ in}^2$$

Columns: W24x146

Girders: W14x84

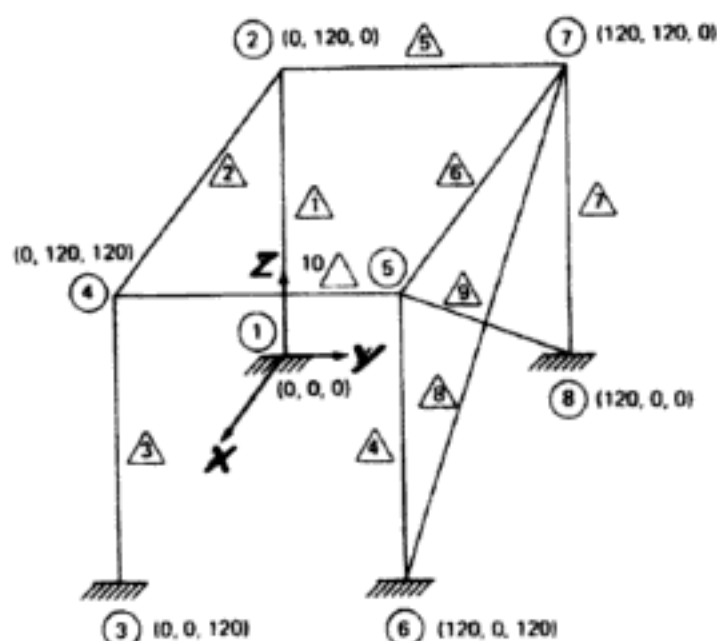


Fig. P13.1

Problem 13.2

Determine the dynamic response of the three dimensional frame of Problem 13.2 subjected to the harmonic excitation of acceleration $a(t) = 0.3 \sin 5.3t$ applied at support 3 in the Y-direction.

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14 Dynamic Analysis of Trusses

The static analysis of trusses whose members are pin-connected reduces to the problem of determining the bar forces due to a set of loads applied at the joints. When the same trusses are subjected to the action of dynamics forces, the simple situation of only axial stresses in the members is no longer present. The inertial forces developed along the members of the truss will, in general, produce flexural bending in addition to axial forces. The bending moments at the ends of the truss members will still remain zero in the absence of external joint moments. The dynamic stiffness method for the analysis of trusses is developed as in the case of framed structures by establishing the basic relations between external forces, elastic forces, damping forces, inertial forces and the resulting displacements, velocities, and accelerations at the nodal coordinates, that is, by determining the stiffness, damping, and mass matrices for a member of the truss. The assemblage of system stiffness, damping, and mass matrices of the truss as well as the solution for the displacements at the nodal coordinates follows along the standard method presented in the preceding chapters for framed structures.

14.1 Stiffness and Mass Matrices for the Plane Truss

A member of a plane truss has two nodal coordinates at each joint, that is, a total of four nodal coordinates (Fig. 14.1). For small deflections, it may be assumed that the force-displacement relationship for the nodal coordinates along the axis of the member (coordinates 1 and 3 in Fig. 14.1) are independent of the transverse displacements along nodal coordinates 2 and 4. This assumption is equivalent to

stating that a displacement along nodal coordinates 1 or 3 does not produce forces along nodal coordinates 2 or 4 and vice versa.

The stiffness and mass coefficients corresponding to the axial nodal coordinates were derived in Chapter 11 and are given, in general, by eq. (11.17) for the stiffness coefficients and by eq. (11.23) for consistent mass coefficients. Applying these equations to a uniform beam element, we obtain, using the notation of Fig. 14.1, the following coefficients:

$$k_{11} = k_{33} = \frac{AE}{L}, \quad k_{13} = k_{31} = -\frac{AE}{L} \quad (14.1)$$

$$m_{11} = m_{33} = \frac{\bar{m}L}{3}, \quad m_{13} = m_{31} = \frac{\bar{m}L}{6} \quad (14.2)$$

in which \bar{m} is the mass per unit length, A is the cross-sectional area, and L is the length of the element.

The stiffness coefficients, for pin-ended elements, corresponding to the nodal coordinates 2 and 4 are all equal to zero, since a force is not required to produce displacements at these coordinates. Therefore, arranging the coefficients given by eq. (14.1), we obtain the stiffness equation for a uniform member of a truss as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \quad (14.3)$$

or in condensed notation

$$\{P\} = [K]\{\delta\} \quad (14.4)$$

in which $[K]$ is the element stiffness matrix.

The consistent mass matrix is obtained, as previously demonstrated, using expressions for static displacement functions in the application of the principle of

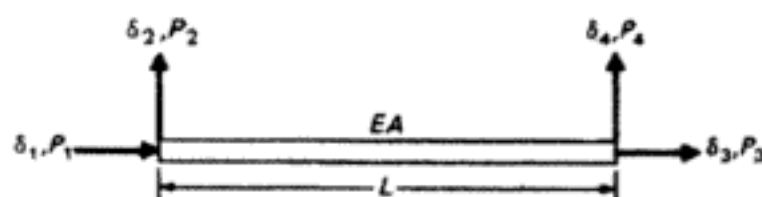


Fig. 14.1 Member of a plane truss showing nodal displacements and forces.

virtual work. The displacement functions corresponding to a unit deflection at nodal coordinates 2 and 4 indicated in Fig. 14.2 are given by

$$u_2 = 1 - \frac{x}{L} \quad (14.5)$$

and

$$u_4 = \frac{x}{L} \quad (14.6)$$

The consistent mass coefficients are given by the general expression, eq. (11.23), which is repeated here for convenience, namely,

$$m_{ij} = \int \bar{m}(x) u_i(x) u_j(x) dx \quad (14.7)$$

For a uniform member of mass \bar{m} per unit length, the substitution of eqs. (14.5) and (14.6) into eq. (14.7) yields

$$\begin{aligned} m_{22} = m_{44} &= \frac{\bar{m}L}{3} \\ m_{24} = m_{42} &= \frac{\bar{m}L}{6} \end{aligned} \quad (14.8)$$

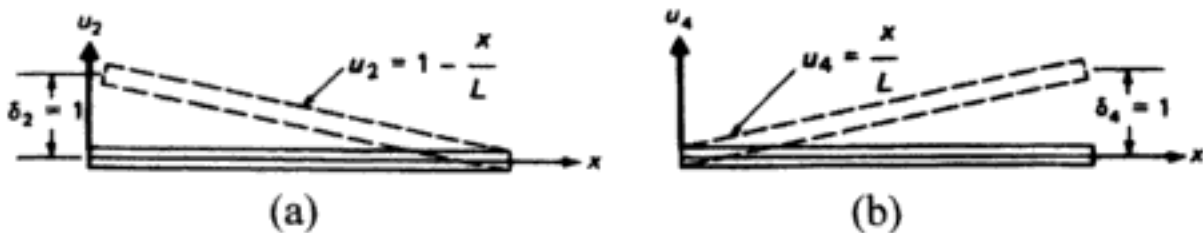


Fig. 14.2 Displacement functions. (a) For a unit displacement $\delta_2 = 1$. (b) For a unit displacement $\delta_4 = 1$.

Finally, the combination of the mass coefficients from eqs. (14.2) and (14.8) forms the consistent mass matrix relating forces to accelerations at the nodal coordinates for a uniform member of a plane truss, namely,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{\bar{m}E}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \end{Bmatrix} \quad (14.9)$$

or in concise notation

$$\{P\} = [M] \{\ddot{\delta}\} \quad (14.10)$$

14.2 Transformation of Coordinates

The stiffness matrix, eq. (14.3), and the mass matrix, eq. (14.9), were derived in reference to nodal coordinates associated with the local or element system of coordinates. As discussed before in the chapters on framed structures, it is necessary to transform these matrices to a common system of reference, the global coordinate system. The transformation of displacements and forces at the nodal coordinates is accomplished, as was demonstrated in Chapter 11, performing a rotation of coordinates. Deleting the angular coordinates in eq. (11.31) and relabeling the remaining coordinates result in the following transformation for the nodal forces:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \end{Bmatrix} \quad (14.11)$$

or in condensed notation ,

$$\{P\} = [T] \{\bar{P}\} \quad (14.12)$$

in which $\{P\}$ and $\{\bar{P}\}$ are the nodal forces in reference to local and global coordinates, respectively, and $[T]$ the transformation matrix defined in eq. (14.11). The same transformation matrix $[T]$ also serves to transform the nodal displacement vector $\{\bar{\delta}\}$ in the global coordinate system to the nodal displacement vector $\{\delta\}$ in local coordinates:

$$\{\delta\} = [T] \{\bar{\delta}\} \quad (14.13)$$

The substitution of eqs. (14.12) and (14.13) into the stiffness equation (14.4) gives

$$[T][\bar{P}] = [K][T] \{\bar{\delta}\}$$

Since $[T]$ is an orthogonal matrix ($[T]^{-1} = [T]^T$), it follows that

$$\{\bar{P}\} = [T]^T [K] [T] \{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [\bar{K}] \{\bar{\delta}\} \quad (14.14)$$

in which

$$[\bar{K}] = [T]^T [K] [T] \quad (14.15)$$

is the element stiffness matrix in the global coordinate system. Analogously, substituting eqs. (14.12) and (14.13) into eq. (14.10) results in

$$\{\bar{P}\} = [T]^T [M] [T] \{\bar{\delta}\} \quad (14.16)$$

or

$$\{\bar{P}\} = [\bar{M}] \{\bar{\delta}\} \quad (14.17)$$

$$[\bar{M}] = [T]^T [M] [T] \quad (14.18)$$

in which $[\bar{M}]$ is the element mass matrix referred to the global system of coordinates.

However, there is no need to use eq. (14.18) to calculate matrix $[\bar{M}]$. This matrix is equal to the mass matrix $[M]$ in reference to local axes of coordinates. To verify this fact, we substitute into eq. (14.18) matrices $[M]$ and $[T]$, respectively, from eqs. (14.9) and (14.11) to obtain

$$\begin{aligned} [\bar{M}] &= \frac{\bar{m}L}{6} \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & c & s & c \end{bmatrix} \\ [\bar{M}] &= \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = [M] \end{aligned} \quad (14.19)$$

in which we use the notation $c = \cos \theta$, $s = \sin \theta$ and the fact that $\cos^2 \theta + \sin^2 \theta = 1$.

A similar relationship is also obtained for the element damping matrix, namely,

$$[\bar{C}] = [T]^T [C] [T] \quad (14.20)$$

in which $[\bar{C}]$ and $[C]$ are the damping matrices referred, respectively, to the global and the local systems of coordinates.

Illustrative Example 14.1.

The plane truss shown in Fig. 14.3 which has only three members is used to illustrate the application of the stiffness method for trusses. For this truss determine the system stiffness and the system consistent mass matrices.

Solution:

The stiffness matrix, eq. (14.3), the mass matrix, eq. (14.9), and the transformation matrix, eq. (14.11), are applied to the three members of this truss. For member 1, $\theta = 90^\circ$,

$$[K_1] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\bar{M}_1] = [M_1] = \frac{\bar{m}E}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

and

$$[T_1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

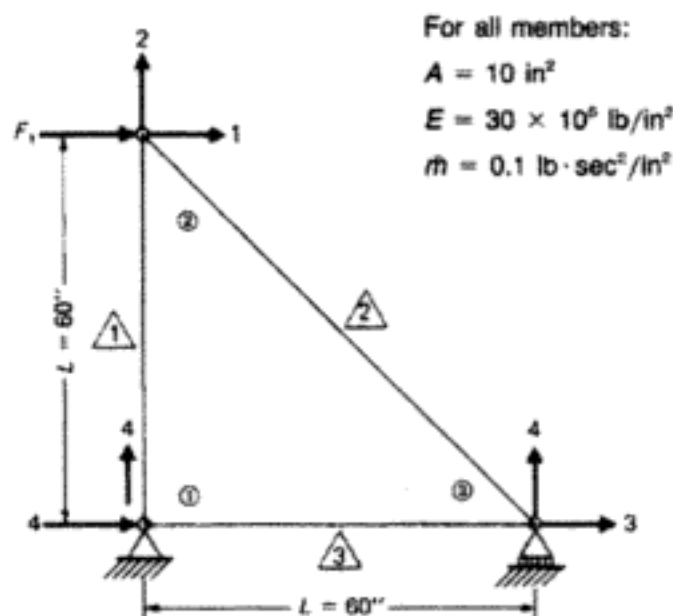


Fig. 14.3 Plane truss for Illustrative Example 14.1.

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substituting the proper numerical values for this example; $L = 60$ in, $A = 10$ in², $\bar{m} = 0.1$ lb · sec² /in², $E = 30 \times 10^6$ lb /in², and following the rules of the direct method of assembling the system stiffness and mass matrix from the above element matrices, we obtain

$$[K_s] = 10^6 \begin{bmatrix} 1.768 & -1.768 & -1.768 \\ -1.768 & 6.768 & 1.768 \\ -1.768 & 1.768 & 6.768 \end{bmatrix}$$

$$[M_s] = \begin{bmatrix} 4.828 & 0 & 1.414 \\ 0 & 4.828 & 0 \\ 1.414 & 0 & 4.828 \end{bmatrix}$$

where $[K_s]$ and $[M_s]$ are, respectively, the system stiffness and mass matrices for the truss shown in Fig. 14.3.

Illustrative Example 14.2.

Determine the natural frequencies and normal modes for the truss of Example 14.1.

Solution:

The differential equations of motion for this system are

$$[M_s]\{\ddot{u}\} + [K_s]\{u\} = 0 \quad (a)$$

Substituting $\{u\} = \{a\} \sin \alpha t$, we obtain

$$([K_s] - \omega^2[M_s])\{a\} = \{0\} \quad (b)$$

For the nontrivial solution, we require

$$| [K_s] - \omega^2[M_s] | = 0 \quad (c)$$

Substituting from Example 14.1 $[K_s]$ and $[M_s]$ and expanding the above determinant give a cubic equation in $\lambda = \omega^2 \bar{m} L^2 / 6AE$, which has the following roots

$$\begin{array}{lll} \lambda_1 = 0.00344 & \text{or} & \omega_1 = 415 \text{ rad/sec} \\ \lambda_2 = 0.0214 & \text{or} & \omega_2 = 1034 \text{ rad/sec} \\ \lambda_3 = 0.0466 & \text{or} & \omega_3 = 1526 \text{ rad/sec} \end{array}$$

Substituting in turn ω_1 , ω_2 and ω_3 into eq. (b), setting $a_1 = 1$, and solving for a_2 and a_3 give the modal vectors

$$\{a_1\} = \begin{Bmatrix} 1.000 \\ 0.216 \\ 0.274 \end{Bmatrix}, \{a_2\} = \begin{Bmatrix} 1.000 \\ 5.488 \\ -4.000 \end{Bmatrix}, \{a_3\} = \begin{Bmatrix} 1.000 \\ -1.000 \\ -1.524 \end{Bmatrix}$$

which may be normalized using the factors

$$\sqrt{\{a_1\}^T [M_s] \{a_1\}} = 2.489, \sqrt{\{a_2\}^T [M_s] \{a_2\}} = 14.695 \\ \sqrt{\{a_3\}^T [M_s] \{a_3\}} = 4.066$$

This normalization results in

$$\{\phi_1\} = \begin{Bmatrix} 0.402 \\ 0.087 \\ 0.110 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} 0.068 \\ 0.373 \\ -0.272 \end{Bmatrix}, \{\phi_3\} = \begin{Bmatrix} 0.246 \\ -0.246 \\ -0.375 \end{Bmatrix}$$

These normalized eigenvectors form the modal matrix:

$$[\Phi] = \begin{bmatrix} 0.402 & 0.068 & 0.246 \\ 0.087 & 0.373 & -0.246 \\ 0.110 & -0.272 & -0.375 \end{bmatrix}$$

Illustrative Example 14.3.

Determine the response of the truss in Examples 14.1 and 14.2 when a constant force $F_1 = 5000$ lb is suddenly applied along coordinate 1 as shown in Fig. 14.3.

Solution:

The modal equations are given in general [eq. (8.6)] by

$$\ddot{z}_n + \omega_n^2 z_n = P_n$$

in which the modal force

$$P_n = \sum_i \phi_{in} F_i$$

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440 Program 17 – Modeling Structures as Plane Trusses

SYSTEM MASS MATRIX

4.8284E+00	0.0000E+00	1.4142E+00
0.0000E+00	4.8284E+00	0.0000E+00
1.4142E+00	0.0000E+00	4.8284E+00

PROGRAM 8: NATURAL FREQUENCIES

INPUT DATA:

OUTPUT RESULTS:

EIGENVALUES:

1.725E+05	1.068E+06	2.329E+06
-----------	-----------	-----------

NATURAL FREQUENCIES (C.P.S.):

66.12	164.52	242.87
-------	--------	--------

EIGENVECTORS BY ROWS:

0.40178	0.08681	0.11035
0.06805	0.37346	-0.27173
-0.24594	0.24516	0.37487

PROGRAM 9: MODAL

DATA FILE: D9.3

INPUT DATA:

NUMBER OF DEGREES OF REEDOM	ND = 3
NUMBER OF EXTERNAL FORCES	NF = 1
TIME STEP OF INTEGRATION	H = .05
GRAVITATIONAL INDEX	G = 0
PRINT TIME HISTORY NPRT = 1;	ONLY MAX. VALUES NPRT = 0
	NPRT = 0
FORCE #, COORD. # WHERE FORCE IS APPLIED, NUM. OF POINTS DEFINE THE FORCE	
1 1 2	

EXCITATION FUNCTION FOR FORCE # 1:

TIME	EXCITATION
0.00	5000.00
2.00	5000.00

MODAL DAMPING RATIOS:

0	0	0
---	---	---

OUTPUT RESULTS:

MAXIMUM RESPONSE:

COORD.	MAX. DISPL.	MAX. VELOC.	MAX. ACC.
1	0.00959	2.002	1132.70
2	0.00211	0.644	517.41
3	0.00236	0.842	651.70

14.4 Stiffness and Mass Matrices for Space Trusses

The stiffness matrix and the mass matrix for a space truss can be obtained as an extension of the corresponding matrices for the plane truss. Figure 14.4 shows the nodal coordinates in the local system (unbarred) and in the global system (barred) for a member of a space truss. The local x axis is directed along the longitudinal axes of the member while the y and z axes are set to agree with the principal directions of the cross section of the member. The following matrices may then be written for a uniform member of a space truss as an extension of the stiffness, eq. (14.3), and the mass, eq. (14.9), matrices for a member of a plane truss.

Stiffness matrix:

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14.21)$$

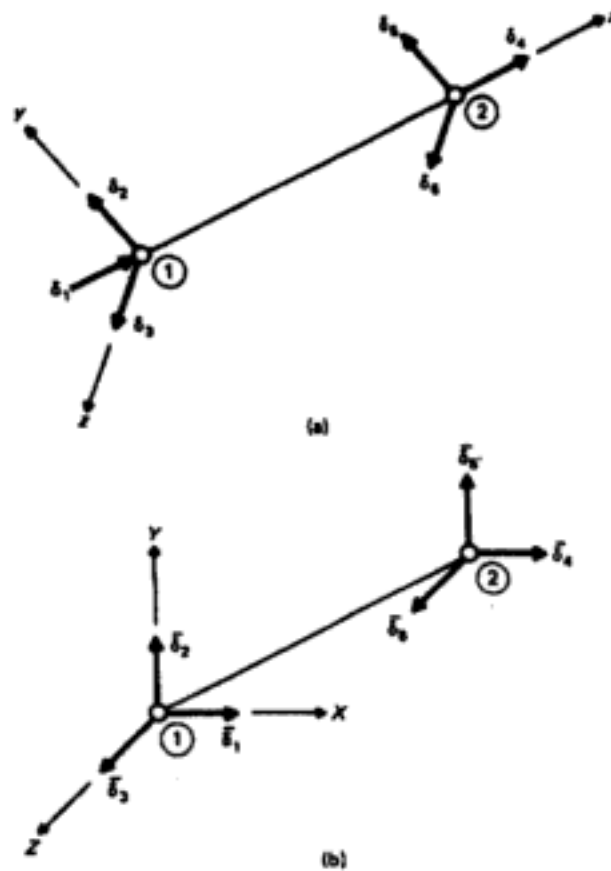


Fig. 14.4 Member of a space truss showing nodal coordinates.

(a) In the local system (unbarred). (b) In the global system (barred).

Consistent mass matrix:

$$[M_c] = \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \quad (14.22)$$

Lumped mass matrix:

$$[M_L] = \frac{\bar{m}L}{2} [1 \ 1 \ 1 \ 1 \ 1 \ 1] \quad (14.23)$$

The transformation matrix $[T_1]$ corresponding to three nodal coordinates at a joint is given by eq. (13.9). It is repeated here for convenience.

$$[T_1] = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix} \quad (14.24)$$

The direction cosines in the first row of eq. (14.24), $c_1 = \cos xX$, $c_2 = \cos xY$, and $c_3 = \cos xZ$, may be calculated from the coordinates of the two points $P_1 (X_1, Y_1, Z_1)$ and $P_2 (X_2, Y_2, Z_2)$ at the ends of the truss element, that is

$$c_1 = \frac{X_2 - X_1}{L}, \quad c_2 = \frac{Y_2 - Y_1}{L} \quad \text{and} \quad c_3 = \frac{Z_2 - Z_1}{L} \quad (14.25)$$

with

$$L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \quad (14.26)$$

The transformation matrix for the nodal coordinates at the two ends of a truss member is then given by

$$[T] = \begin{bmatrix} [T_1] & [0] \\ [0] & [T_1] \end{bmatrix} \quad (14.27)$$

in which $[T_1]$ is given by eq. (14.24). The following transformations are then required to obtain the member stiffness matrix $[\bar{K}]$ and the member mass matrix $[\bar{M}]$ in reference to the global system of coordinates:

$$[\bar{K}] = [T]^T [K] [T] \quad (14.28)$$

and

$$[\bar{M}] = [T]^T [M] [T] \quad (14.29)$$

In the case of an element of a space truss, it is only necessary to calculate the direction cosines of the centroidal axis of the element which are given by eq. (14.25). The other direction cosines in eq. (14.24) do not appear in the final expression for the element stiffness matrix $[\bar{K}]$ as may be verified by substituting eqs. (14.24) and (14.27) into eq. (14.28) and proceeding to multiply the matrices indicated in this last equation. The final result of this operation may be written as follows.

$$[K] = \frac{AE}{L} \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 & -c_1^2 & -c_1 c_2 & -c_1 c_3 \\ c_2 c_1 & c_2^2 & c_2 c_3 & -c_2 c_1 & -c_2^2 & -c_2 c_3 \\ c_3 c_1 & c_3 c_2 & c_3^2 & -c_3 c_1 & -c_3 c_2 & -c_3^2 \\ -c_1^2 & -c_1 c_2 & -c_1 c_3 & c_1^2 & c_1 c_2 & c_1 c_3 \\ -c_2 c_1 & -c_2^2 & -c_2 c_3 & c_2 c_1 & c_2^2 & c_2 c_3 \\ -c_3 c_1 & -c_3 c_2 & -c_3^2 & c_3 c_1 & c_3 c_2 & c_3^2 \end{bmatrix} \quad (14.30)$$

Consequently, the determination of the stiffness matrix for an element of a space truss, in reference to the global system of coordinates by eq. (14.30), requires the evaluation by eq. (14.25) of only the direction cosines of the local axis x along the element.

Also, analogously, to eq. (14.19) for an element of a plane truss, the mass matrix $[\bar{M}]$ for an element of a space truss in reference to global coordinates is equal to the mass matrix of the element $[M]$ in local coordinates. Thus, there is no need to perform the operations indicated in eq. (14.29). The substitution into this equation of $[M]$ from eq. (14.22) and $[T]$ from eqs. (14.24) and (14.27) results, after performing the multiplications established by eq. (14.29), in the same matrix $[M]$, that is,

$$[\bar{M}] = [M] \quad (14.31)$$

14.5 Equation of Motion for Space Trusses

The dynamic equilibrium conditions at the nodes of the space truss result in the differential equations of motion which in matrix notation may be written as follows:

$$[\bar{M}]\{\ddot{u}\} + [\bar{C}]\{\dot{u}\} + [\bar{K}]\{u\} = \{F(t)\} \quad (14.32)$$

in which $\{u\}$, $\{\dot{u}\}$, and $\{\ddot{u}\}$ are, respectively, the displacement, velocity, and acceleration vectors at the nodal coordinates, $\{F(t)\}$ is the vector of external nodal forces, and $[\bar{M}]$, $[\bar{C}]$, and $[\bar{K}]$ are the system mass, damping, and stiffness matrices.

In the stiffness method of analysis, the system matrices in eq. (14.32) are obtained by appropriate superposition of the corresponding member matrices using the direct method as we have shown previously for the framed structures. As was discussed in the preceding chapters, the practical way of evaluating damping is to prescribe damping ratios relative to the critical damping for each mode. Consequently, when eq. (14.32) is solved using the modal superposition method, the specified modal damping ratios are introduced directly into the modal equations. In this case, there is no need for explicitly obtaining the system damping matrix $[C]$. However, this matrix is required when the solution of eq. (14.32) is sought by other methods of solution, such as the step-by-step integration method. In this case, the system damping matrix $[C]$ can be obtained from the specified modal damping ratios by any of the methods discussed in Chapter 22.

14.6 Program 18—Modeling Structures as Space Trusses

Program 18 calculates the stiffness and the mass matrices for the space truss and stores the coefficients in a file. Since the stiffness and mass matrices are symmetric, only the upper triangular portion of these matrices needs to be stored. The program also stores in another file, which is named by the user, the general information on the space truss. The information stored in these files is needed by programs, which the user may call, to perform dynamic analysis of the truss, such as determination of natural frequencies or calculation of the response of the structure subjected to external excitation.

Illustrative Example 14.5.

Determine the stiffness and mass matrices for the space truss shown in Fig. 14.5. The mass per unit length of any member is $\bar{M} = 0.1 \text{ (lb} \cdot \text{sec}^2/\text{in}^2\text{)}$. Also determine the first three natural frequencies of the truss.

Solution:

Input Data and Output Results

PROGRAM 18: MODELING SPACE TRUSSES

DATA FILE:D 18

NUMBER OF JOINTS
NUMBER OF BEAM ELEMENTS
NUMBER OF FIXED COORDINATES
MODULUS OF ELASTICITY

NJ = 5
NE = 7
NC = 12
E = 3E+7

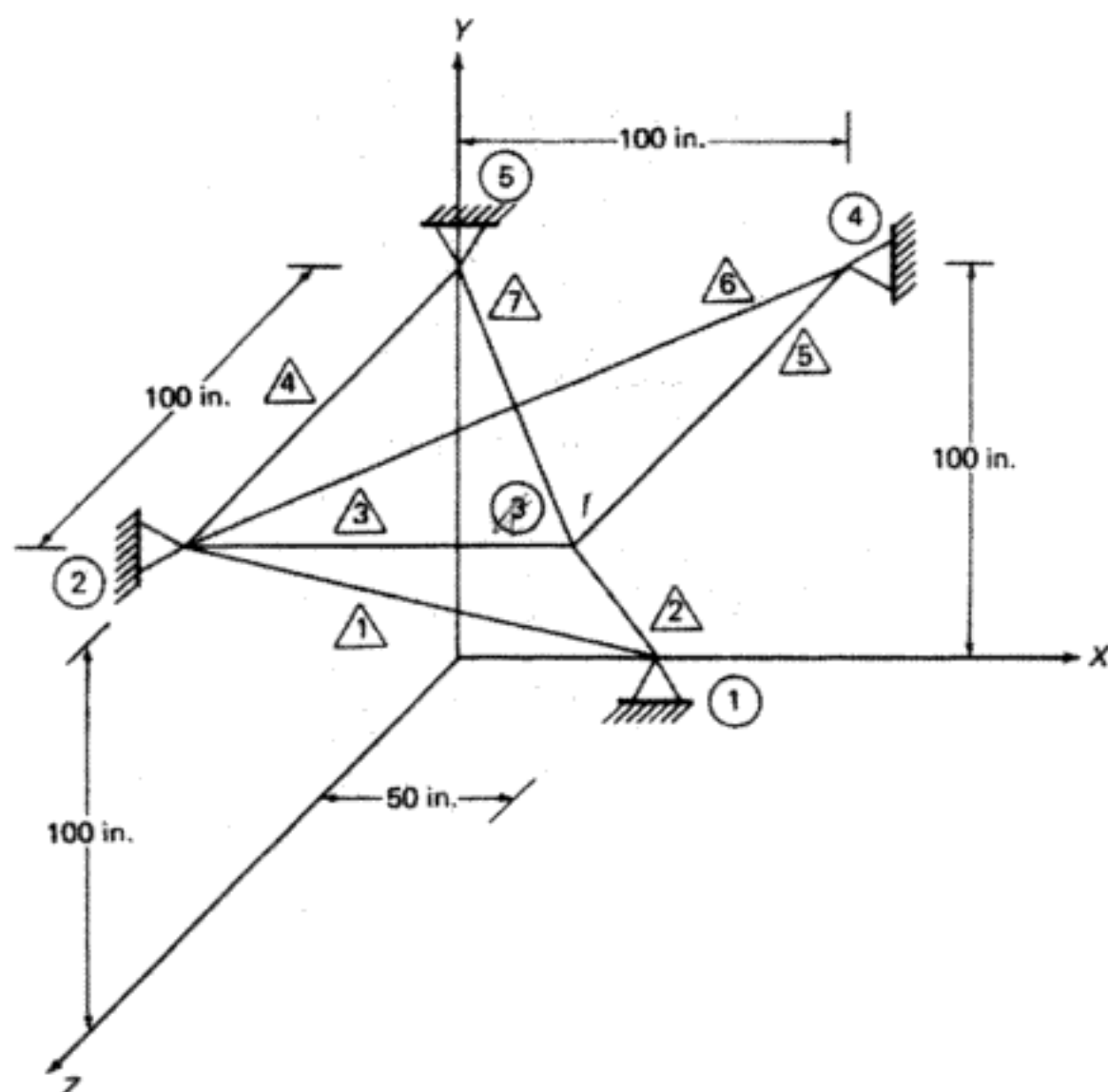


Fig. 14.5 Space truss for Example 14.5

JOINT DATA:

JOINT #.	X-COORD.	Y-COORD.	Z-COORD.	JOINT MASS.
1	50.00	0.00	0.00	0
2	0.00	100.00	100.00	0
3	100.00	100.00	100.00	0
4	100.00	100.00	0.00	0
5	0.00	100.00	0.00	0

INPUT BEAM ELEMENT DATA:

ELEMENT #.	FIRST JOINT	SECOND JOINT	MASS/LENGTH	SECT. AREA
1	2	1	0.100	10.00
2	3	1	0.100	10.00
3	2	3	0.100	10.00
4	2	5	0.100	10.00
5	3	4	0.100	10.00
6	2	4	0.100	10.00
7	5	3	0.100	10.00

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FIXED COORDINATES DATA:
DIRECTION # (X = 1, Y = 2, Z = 3)

NODE	DIR.	NODE	DIR.	NODE	DIR.	NODE	DIR.	NODE	DIR.	NODE	DIR.	NODE	DIR.	NODE	DIR.
1	1	1	2	1	3	2	1	2	2	2	3	4	1	4	2
4	3	5	1	5	2	5	3								

OUTPUT RESULTS:

****SYSTEM STIFFNESS MATRIX****

4.2829E+06	4.4444E+05	1.5051E+06
4.4444E+05	8.8889E+05	8.8889E+05
1.5051E+06	8.8889E+05	4.9495E+06

****SYSTEM MASS MATRIX****

1.6381E+01	0.0000E+00	0.0000E+00
0.0000E+00	1.6381E+01	0.0000E+00
0.0000E+00	0.0000E+00	1.6381E+01

PROGRAM 8: Natural Frequencies

EIGENVALUES:

4.258E+04	1.888E+05	3.865E+05
-----------	-----------	-----------

NATURAL FREQUENCIES (C.P.S.):

.3284E+02	.6915E+02	.9895E+02
-----------	-----------	-----------

EIGENVECTORS BY ROWS:

-0.01030	0.24234	-0.04701
0.19602	-0.02057	-0.14900
0.15006	0.04351	0.19141

14.7 Dynamic Analysis of Trusses Using SAP2000

Illustrative Example 14.6

Use SAP2000 to perform the structural dynamic analysis of the plane truss shown in Fig. 14.6. For all the member of the truss the cross sectional area is $A=2 \text{ in}^2$ and modulus of elasticity $E= 30,000 \text{ ksi}$. Assign concentrated weights at all the joints of the truss, $W = 3.86 \text{ kip}$. The truss is acted by a time force function applied at the center joint on the lower cord in the vertical direction.

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again at this location and proceed to drag the pointer horizontally to the next grid line, click twice to create the second element. Repeat this process to create next two horizontal element of the truss. Press the Esc key to stop drawing elements.

Using the draw command and proceed to create all the other members of the truss shown in Fig. 14.6.

Label: From the Main Menu enter:
VIEW>SET ELEMENTS
Click on "Joint Labels" on "Frame labels", then OK

Restraints: Select joint 1, and 5 of the truss with a click of the mouse on these joints.

Enter from the Main Menu:
ASSIGN>JOINT>RESTRAINTS
Select Restraint translations 1, 2 and 3. Then OK.

Select joints 2 and 4 at the two rollers and enter:
ASSING>JOINT.RESTRAINTS
Select Restraint only the translation 3. Then OK.

Select joints 3,6,7, and 8 and enter:
ASSIGN>JOINT>RESTRAINTS
Select no restraints by a click on the dot icon. Then OK.

Sections: From the Main Menu enter
DEFINE>FRAME SECTIONS
Click on Import/Wide flange
Select section properties and click "open"
Scroll on the sections to select:
"General Section" and enter:
Area = 2.0 and zero values for other properties of the cross-section.
Then OK, OK.

Assign: Select all the members of the truss (click on every member or faster just click at a point outside the truss and drag the pointer to envelope in dotted lines the whole truss. Then from the Main Menu enter:
ASSIGN>FRAME>SECTIONS
Select section SSEC2. Then OK

Mass: Select all the joints in the truss by clicking on them.
ASSIGN>JOINT>MASSES
Enter Masses:
Direction 1= 0.01
Direction 2= 0.01. Then OK.

Load: From the Main Menu enter:
 DEFINE>STATIC LOAD CASES
 Change type of load to LIVE
 And self weight multiplier to 0
 Then click on change load and OK

Load Function: DEFINE>TIME HISTORY FUNCTIONS
 Click on "Add new function"
 Click on "add" to register the first point (0,0)
 Enter: Time= 0.2, Value= 50.0, then press ADD
 Enter: Time=0.4, Value= 50.0, then press ADD
 Enter: Time=0.6, Value= 0.0, then press ADD
 Enter: Time= 0.8, Value= 0.0, then press ADD. And OK, OK.

Load Case: DEFINE>TIME HISTORY CASES
 Click on Add New History
 Enter: Number of output time steps= 80
 Output time step= 0.01
 Check: Envelopes
 Load Assignments:
 Load= LOAD1
 FUNCTION=FUNC1
 Accept all of the other default values
 Then press ADD and OK, OK

Assign Load: Select joint 3 by a click and enter:
 ASSIGN>JOINT STATIC LOADS>FORCES
 Enter: Force Global Z= -1.0; then OK

Analyze: ANALYZE>SET OPTIONS
 Select available DOFs in the X and Y directions.
 Click on "Dynamic Analysis" and on "Set Dynamic Parameters"
 Enter: Numbers of Modes=5 then OK,OK
 ANALYZE>RUN
 Enter Filename: EXD14.6, then SAVE
 At the conclusion of the calculations,
 After checking for errors, press OK.

Print Input Tables
 FILES>PRINT INPUT TABLES
 Check: "Print to File". Then OK.
 Use NOTEPAD or another editor to retrieve, edit and print
 the file C:\SAP2000\EXD14.6.text

(The edited Input Table for Illustrative Example 14.6 is reproduced as Table 14.1)

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Table 14.1 Edited Input Table for Illustrative Example 14.6

STATIC LOAD CASES

STATIC LOAD1	CASE LIVE	SELF WT 0.0000
-----------------	--------------	-------------------

TIME HISTORY CASES

HISTORY CASE HIST1	HISTORY TYPE LINEAR	NUMBER OF TIME STEPS 80	TIME STEP INCREMENT 0.01000
--------------------------	---------------------------	-------------------------------	-----------------------------------

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-576.00000	0.00000	0.00000	1 1 1 0 0 0
2	-288.00000	0.00000	0.00000	0 0 1 0 0 0
3	0.00000	0.00000	0.00000	0 0 0 0 0 0
4	288.00000	0.00000	0.00000	0 0 1 0 0 0
5	576.00000	0.00000	0.00000	1 1 1 0 0 0
6	-288.00000	0.00000	384.00000	0 0 0 0 0 0
7	0.00000	0.00000	384.00000	0 0 0 0 0 0
8	288.00000	0.00000	384.00000	0 0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
1	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000
2	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000
3	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000
4	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000
5	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000
6	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000
7	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000
8	1.000E-02	1.000E-02	0.000	0.000	0.000	0.000

FRAME ELEMENT DATA

FRAME	JNT-1	JNT-2	SECTION	RELEASES	SEGMENTS	LENGTH
1	1	2	TRUSS143	0.000	4	288.000
2	2	3	TRUSS143	0.000	4	288.000
3	3	4	TRUSS143	0.000	4	288.000
4	4	5	TRUSS143	0.000	4	288.000
5	1	6	TRUSS143	0.000	2	480.000
6	6	7	TRUSS143	0.000	4	288.000
7	7	8	TRUSS143	0.000	4	288.000
8	8	5	TRUSS143	0.000	2	480.000
9	2	6	TRUSS143	0.000	2	384.000
10	6	3	TRUSS143	0.000	2	480.000
11	3	7	TRUSS143	0.000	2	384.000
12	8	4	TRUSS143	0.000	2	384.000
13	4	7	TRUSS143	0.000	2	480.000
14	3	8	TRUSS143	0.000	2	480.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
6	0.000	0.000	-1.000	0.000	0.000	0.000

Plot Displacement Function:

DISPLAY>SHOW TIME HISTORY TRACES

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 3

Vector Type = Displacement

Component = UZ.

Mode Number = Include All. Then OK.

Under List of Functions click once on Joint3,

Click on ADD to move Joint3 to Plot Functions column

Axis Labels

Horizontal = Time (sec)

Vertical = Z-Displacements Joint 3 (in)

Click "Display" to view on the screen the time-displacement plot for joint 3 in the Z direction. Then, to obtain a printed version, enter:

FILE>PRINT GRAPHICS

(Figure 14.7 reproduces the plot shown in the screen for the Displacement in Z direction of Joint 3 for the truss of Illustrative Example 14.6.)

Alternatively, a table of the displacement function may be obtained by entering:

FILE>PRINT TABLES TO FILE

The values in this table may be plotted using Excel or similar software.

(Table 14.2 contains a portion of the Displacement Function Values in the Z direction for joint 3 of the truss in Illustrative Example 14.6.)

Table 14.2 Displacement Function in the Z direction for Joint 3 of the truss of Illustrative Example 14.6.(in)

Time (sec)	Displ. (in)
0.00	0.00000
0.01	-0.00126
0.02	-0.00673
0.03	-0.01268
0.04	-0.01569
.....	
0.78	-0.00088
0.79	-0.00195

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Solution:

Begin: Open SAP2000 (already installed in the computer).

Hint: Maximize both windows for a full view of all windows.

Units: In the lower right-hand corner of the screen use the drop-down menu to select “kip-in”,

Model: From the Main Menu bar:

FILE>NEW MODEL

Number of Grid Spaces:

X – direction = 2

Y – direction = 2

Z – direction = 1

Grid Spacing:

X – direction = 48

Y – direction = 72

Z – direction = 144, then OK.

Edit: Default to the screen 3-D.

Then enter from the Main Menu bar:

VIEW>3D VIEW

Rotate plan view to 30 degrees by typing 30 in box of the plan view, to have a frontal view of the X axis. Then OK.

Draw points:

DRAW>ADD SPECIAL JOINT

Click once to draw points at coordinates (48,0,0), (0,72,0), (-48,0,0), (0,-72,0) and (0,0,144).

Note: as the mouse moves over the grid intersections the coordinates are shown in the lower right-corner of the screen. Press the Esc key to exit drawing points.

Label: From the Main Menu bar enter:

VIEW>SET ELEMENTS . . .

Under the Joints heading, click on labels to number the drawn points. Then OK.

Draw members: From the Main Menu bar enter:

DRAW>DRAW FRAME ELEMENT

Click on point 1 and drag mouse to move pointer to point 5 and click twice. Then press the Esc key. Now repeat successively the above command to draw frame elements from joints 2, 3, and 4, to joint 5.

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Then select Add Joint Disps/Forces. Then OK.

In Time History Joint Function enter Joint ID=5

In Vector Type select Displ and in Component select UX. Then OK

In Vector Type select Displ and in Component select UY. Then OK,OK.

In the List of Function column, highlight Joint 5 and click "Add" to move joint 5 (X displacement component) to the Plot Function column.

Highlight Joint5-1 to move Y- displacement component of joint 5 to the Plot

On the Axis Labels enter:

Horizontal: Time (sec)

Vertical: X, Y displ. components of Join 5 (in)

Then click on "Display"

To obtain a print of the plot shown on the screen, click:

File and select "Print Graphics"

(Fig 14.9. reproduces the plot showing the time displacement components of joint 5 in the X, Y-directions)

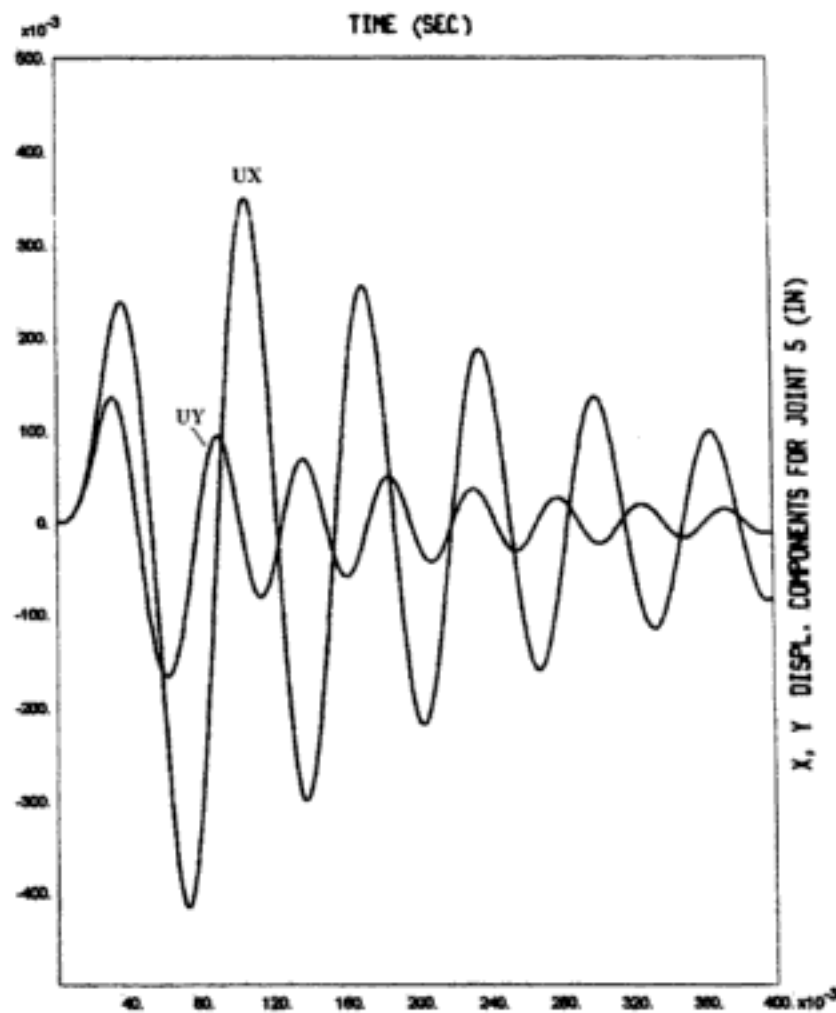


Figure 14.9 Displacement. components of Joint 5 in X,Y directions

Print input tables:

FILE>INPUT TABLES

Click "Print to File" and accept the default file name shown on the screen or enter a new name to the file to save the Input Tables.

Use Notepad or another word editor to retrieve and edit the file C:\SAP2000\EXD 14.7.text containing the Input tables.

(Tables 14.3 contains the edited Input tables for Illustrative Example 14.7)

Table 14.4 Edited input tables for Illustrative Example 14.7

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	LIVE	0.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	TIME STEP INCREMENT
--------------	--------------	----------------------	---------------------

HIST1	LINEAR	1200	0.00100
-------	--------	------	---------

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS	ANGLE-A	ANGLE-B	ANGLE-C
-------	----------	----------	----------	------------	---------	---------	---------

1	48.00000	0.00000	0.00000	1 1 1 0 0 0	0.000	0.000	0.000
2	0.00000	72.00000	0.00000	1 1 1 0 0 0	0.000	0.000	0.000
3	-48.00000	0.00000	0.00000	1 1 1 0 0 0	0.000	0.000	0.000
4	0.00000	-72.00000	0.00000	1 1 1 0 0 0	0.000	0.000	0.000
5	0.00000	0.00000	144.00000	0 0 0 0 0 0	0.000	0.000	0.000

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
5	1.000E-02	1.000E-02	1.000E-02	0.000	0.000	0.000

FRAME ELEMENT DATA

FRAME	JNT-1	JNT-2	SECTION	LENGTH
1	1	5	FSEC2	151.789
2	2	5	FSEC2	160.997
3	3	5	FSEC2	151.789
4	4	5	FSEC2	160.997

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
5	0.707	0.707	0.000	0.000	0.000	0.000

Natural Periods and Frequencies:

Use Notepad or other word editor to open the following file:

C:\SAP2000>EXD 14.7.OUT

The natural Period and Frequencies values may be found un this file.

(Table 14.4. contains the edited Natural Period and frequency values of Illustrative Example 14.7)

Table 14.4 Natural Periods frequencies for Illustrative Example 14.7.

Mode	Period (sec)	Frequency (c.p.s.)	Eigenvalue (rad/sec) ²
------	-----------------	-----------------------	--------------------------------------

MODAL PERIODS AND FREQUENCIES

MODE	PERIOD (TIME)	FREQUENCY (CYC/TIME)	FREQUENCY (RAD/TIME)	EIGENVALUE (RAD/TIME)**2
------	------------------	-------------------------	-------------------------	-----------------------------

1	0.064750	15.444111	97.038213	9416.415
2	0.047153	21.207515	133.250749	17755.762
3	0.015920	62.814966	394.678073	155770.781

Modes Shapes:

DISPLAY>SHOW MODE SHAPE

Modal Shape 1: Click "Wire Shadow" and "Cubic Curve"

Scale Factor =1. Then OK.

(To view Modes 2 and 3, repeat above commands)

Note: Numerical values for a mode may be read directly on the screen by right-clicking on a joint in the modal shape shown in the screen.

14.8 Summary

The dynamic analysis of trusses by the stiffness matrix method was presented in this chapter. As in the case of framed structures, discussed in the preceding chapters, the stiffness and mass matrices for a member of a truss were developed. The system matrices for a truss are assembled as explained for framed structures by the appropriate superposition of the coefficients in the matrices of the elements.

14.9 Problems**Problem 14.1**

For the plane truss shown in Fig. P14.1 determine the system stiffness and mass matrices corresponding to the two nodal coordinates indicated in the figure.

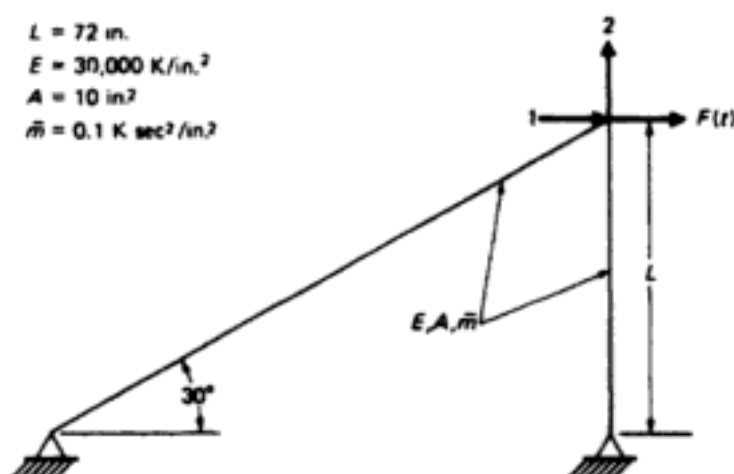


Fig. P14.1.

Problem 14.2

Determine the natural frequencies and corresponding normal modes for the truss shown in Fig. P14.1.

Problem 14.3

Determine the response of the truss shown in Fig. P14.1 when subjected to a force $F(t) = 10 \text{ Kip}$ suddenly applied for 2 sec at nodal coordinate 1. Use the results of Problem 14.2 to obtain the modal equations. Neglect damping in the system.

Problem 14.4

Solve Problem 14.3 assuming 10% damping in all the modes.

Determine the maximum response of the truss shown in Fig. P14.1 when subjected to a rectangular pulse of magnitude $F_0 = 10 \text{ Kip}$ and duration $t_d = 0.1 \text{ sec}$. Use the appropriate response spectrum to determine the maximum modal response (Fig. 4.4). Neglect damping in the system.

Problem 14.5

Determine the dynamic response of the frame shown in Fig. P14.1 when subjected to a harmonic force $F(t) = 10 \sin 10t$ (Kips) along nodal coordinate 1. Neglect damping in the system.

Problem 14.6

Repeat Problem 14.6 assuming that the damping in the system is proportional to the stiffness, $[C] = a_0 [K]$ where $a_0 = 0.1$.

Problem 14.7

Determine the response of the truss shown in Fig. P14.8 when acted upon by the forces $F_1(t) = 10t$ and $F_2(t) = 5t^2$ during 1 sec. Neglect damping.

Problem 14.8

Determine the response of the truss shown in Fig. P14.8 when acted upon by the forces $F_1(t) = 10t$ and $F_2(t) = 5t^2$ during 1 sec. Neglect damping.

Problem 14.9

Solve problem 14.8 assuming 10% modal damping in all the modes.

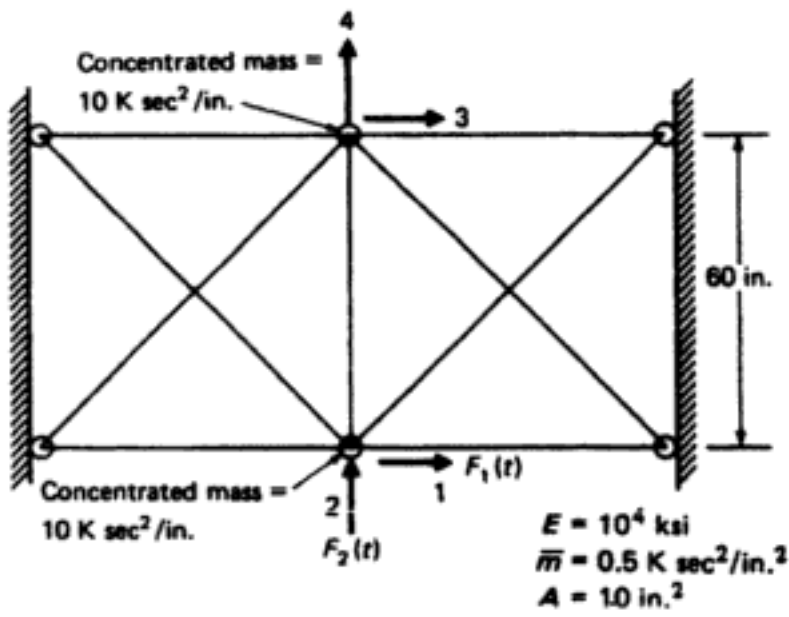


Fig. P14.8.

15 Dynamic Analysis of Structures Using the Finite Element Method

In the preceding chapters, we considered the dynamic analysis of structures modeled as beams, frames, or trusses. The elements of all these types of structures are described by a single coordinate along their longitudinal axis; that is, these are structures with unidirectional elements, called, "skeletal structures." They, in general, consist of individual members or elements connected at points designated as "nodal points" or "joints." For these types of structures, the behavior of each element is first considered independently through the calculation of the element stiffness matrix and the element mass matrix. These matrices are then assembled into the system stiffness matrix and the system mass matrix in such a way that the equilibrium of forces and the compatibility of displacements are satisfied at each nodal point. The analysis of such structures is commonly known as the Matrix Structural Method and could be applied equally to static and dynamic problems.



Fig. 15.1 Finite element modeling of thin plate with triangular elements.

The structures presented in this chapter are continuous structures which are conveniently idealized as consisting of two-dimensional elements connected only at the selected nodal points. For example, Fig.15.1 shows a thin plate idealized with plane triangular elements. The static or dynamic analysis of such idealized structures is known as the Finite Element Method (FEM). This is a powerful method for the analysis of structures with complex geometrical configurations, material properties or loading conditions. This method is entirely analogous to Matrix Structural Analysis for skeletal structures (beams, frames, and trusses) presented in the preceding chapters. The Finite Element Method differs from the Matrix Structural Method only in two respects: (1) the selection of elements and nodal points are not naturally or clearly established by the geometry as it is for skeletal structures and (2) the displacements at internal points of an element are expressed by approximate interpolating functions and not by an exact analytical relationship as it is in the Matrix Structural Method. Furthermore, for skeletal structures, the displacement of an interior point of an element is governed by an ordinary differential equation, while for a continuous two-dimensional element it is governed by a partial differential equation of much greater complexity.

15.1 Plane Elasticity Problems

Plane elasticity problems refer to plates that are loaded in their own planes. Out-of-plane displacements are induced when plates are loaded by normal forces that are perpendicular to the plane of the plate, such problems are generally referred to as plate bending. (Plate bending is considered in Section 15.2.) There are two different types of plane elasticity problems: (1) plane stress and (2) plane strain. In the plane stress problems, the plate is thin relative to the other dimensions and the stresses normal to the plane of the plate are not considered. Figure 15.2 shows a perforated strip-plate in tension as an example of a plane stress problem. For plane strain problems, the strain normal to the plane of loading is suppressed and assumed to be zero. Figure 15.3 shows a transverse slice of a retaining wall as an example of a plane strain problem.

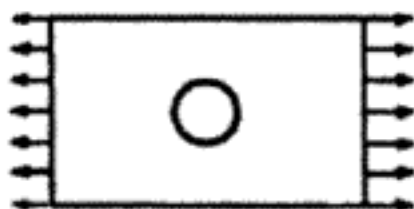


Fig.15.2 Perforated plate tension element as an example of a structural member load in plane stress.

In the analysis of plane elasticity problems, the continuous plate is idealized as finite elements interconnected at their nodal points. The displacements at these nodal points are the basic unknowns as are the displacements at the joints in the analysis of beams, frames or trusses. Consequently, the first step in the application of the FEM is to model the continuous system into discrete elements. The most common geometric elements used for plane elasticity problems are triangular, rectangular or quadrilateral, although other geometrical shapes could be used as well.

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$$\{q(x, y)\} = [g(x, y)]\{c\} \quad (15.3)$$

where $\{c\}$ is a vector containing the six coefficients c_i , $[g(x, y)]$ a matrix function of the position coordinates (x, y) and $\{q(x, y)\}$ a vector with the displacement components $u(x, y)$ and $v(x, y)$ at an interior point along x and y directions, respectively.

Step III: Displacements $\{q(x, y)\}$ at a point in the element are expressed in terms of the nodal displacements, $\{q\}_e$

The evaluation of eq. (15.3) for the displacements of the three nodal joints of the triangular element followed by the solution of the coefficients c_i ($i = 1, 2, \dots, 6$) results in

$$\{q\}_e = [A]\{c\} \quad (15.4)$$

and

$$\{c\} = [A]^{-1}\{q\}_e \quad (15.5)$$

The subsequent introduction of $\{c\}$ from eq. (15.5) into eq. (15.3) gives the displacements at any interior point of the element in terms of the nodal displacements $\{q\}_e$ as

$$\{q(x, y)\} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [g(x, y)][A]^{-1}\{q\}_e \quad (15.6)$$

or

$$\{q(x, y)\} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [f(x, y)]\{q\}_e \quad (15.7)$$

where the matrix $[f(x, y)] = [A]^{-1}[g(x, y)]$ is only a function of the coordinates x, y of a point in the triangular element. Because the displacement functions $u(x, y)$ and $v(x, y)$ in eq. (15.7) are both linear in x and y , displacement continuity is ensured along the interface adjoining elements for any value of nodal displacements.

Step IV: Relation between strain, $\{\varepsilon(x, y)\}$ at any point within the element to the displacements $\{q(x, y)\}$ and hence to the nodal displacements $\{q\}_e$

In Theory of Elasticity (S. Timoshenko and J.N. Goodier, 1970), it is shown that the linear strain vector, $\{\varepsilon(x, y)\}$, with axial strain components along the x, y directions, ε_x , ε_y and shearing strain γ_{xy} is given by differentiation of the displacement functions $u(x, y)$ and $v(x, y)$ as follows:

$$\{\varepsilon(x, y)\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \quad (15.8)$$

The strain components may then be expressed in terms of the nodal displacements $\{q\}_e$ by substituting from eq. (15.7) the derivatives of $u(x, y)$ and $v(x, y)$ into eq. (15.8):

$$\{\epsilon(x, y)\} = [B] \{q\}_e \quad (15.9)$$

in which the matrix $[B]$ is solely a function of the coordinates (x, y) at an interior point of the element.

Step V: Relationship between internal stresses $\{\sigma(x, y)\}$ to strains $\{\epsilon(x, y)\}$ and hence to the nodal displacements $\{q\}_e$.

For plane elasticity problems, the relationship between the normal stresses σ_x , σ_y and shearing stress τ_{xy} to the corresponding strains ϵ_x , ϵ_y , and γ_{xy} may be expressed, in general as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (15.10)$$

or in a short matrix notation as:

$$\{\sigma(x, y)\} = [D] \{\epsilon(x, y)\} \quad (15.11)$$

The substitution of $\{\epsilon(x, y)\}$ from eq. (15.9) into eq. (15.11) gives the desired relationship between stresses $\{\sigma(x, y)\}$ at an interior point in the element and the displacements $\{q\}_e$ at the nodes as:

$$\{\sigma(x, y)\} = [D] [B] \{q\}_e \quad (15.12)$$

The terms d_{ij} of the matrix D in eq. (15.10) have different expressions for plane stress problem than for plane strain problem. These expressions as given by the theory of elasticity are:

For plane stress problems:

$$\begin{aligned}d_{11} &= d_{22} = \frac{E}{1-\nu^2} \\d_{12} &= d_{21} = \frac{\nu E}{1-\nu^2} \\d_{33} &= \frac{E}{2(1+\nu)}\end{aligned}\tag{15.13}$$

For plane strain problems:

$$\begin{aligned}d_{11} &= d_{22} = \frac{E}{1-\nu^2} \\d_{12} &= d_{21} = \frac{\nu E}{1-\nu^2} \\d_{33} &= \frac{E}{2(1+\nu)}\end{aligned}\tag{15.14}$$

in which E is the modulus of elasticity and ν is the Poisson's ratio.

Step VI Element stiffness and mass matrices

Use is made of the Principle of Virtual Work to establish the expressions for the element stiffness matrix $[K]_e$ and the element mass matrix $[M]_e$. This principle states that for structures in dynamic equilibrium subjected to small, compatible virtual displacements, $\{\delta q\}$, the virtual work, δW_E , of the external forces is equal to the virtual work of internal stresses, δW_I , that is

$$\delta W_I = \delta W_E\tag{15.15}$$

In applying this principle to a finite element, we assume a vector of virtual displacements $\delta q\{x, y\}$. Hence, by eq. (15.7)

$$\delta\{q(x, y)\} = [f(x, y)] \delta\{q\}_e\tag{15.16}$$

Using the strain-displacement relationships, eq. (15.9), we obtain

$$\delta\{\varepsilon\} = [B] \delta\{q\}_e\tag{15.17}$$

The internal work over the volume of the element during this virtual displacement is then given by the product of the virtual strain and the stress integrated over the volume of the element:

$$\delta W_I = \int_V \delta\{\varepsilon\}^T \{\sigma(x, y)\} dV\tag{15.18}$$

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(Fig. 15.7 reproduces the modeled shown in the screen)

Boundary: Select joints 4, 21, and 22 by clicking on them, then enter:

ASSIGN>JOINT>RESTRAINTS

Select restraint only for Translation 2. Then OK.

Select joint 1 and enter:

ASSIGN>JOINT>RESTRAINTS

Select restraint in the directions 1 and 2. Then OK.

Section: DEFINE>SHELL SECTIONS

Click on Add New Section.

Select Material: STEEL

Enter thickness:

Membrane = 0.1

Bending = 0.1

Click on Type: Shell. Then OK, OK.

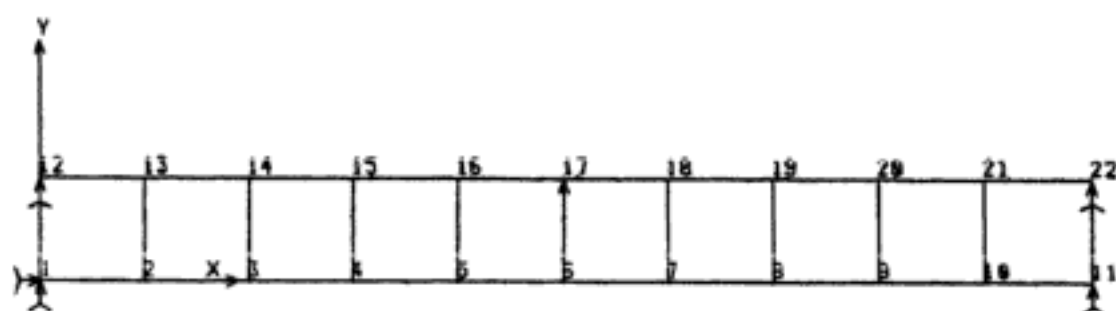


Fig. 15.7 Deep beam of Illustrative Example 15.1 modeled with shell elements

Assign: Select the 10 shell elements shown in the screen by clicking on their areas and enter:

ASSIGN>SHELL>SECTIONS

Select shell section: SSEC2 Then OIK.

Material: DEFINE>MATERIALS

Select: Steel

Click on Modify/Show Materials. Then OK.

Enter:

Modulus of Elasticity = 29E6

Poisson's Ratio = 0.3

Mass per unit of volume = 0.3. (This value corresponds to mass per unit of length of 0.03 lb.sec²/in² on a thickness of 0.1 in.) Then OK, OK.

Load: DEFINE>STATIC LOAD CASES

Select Type: Live Load and set Self Weight Multiplier = 0

Click on "Change Load." Then OK, OK.

Load History Function: DEFINE>TIME HISTORY FUNCTION

Enter: Add New Function

Click the Add button (to add the point 0, 0).

Type 0.1 in the Time box and 1000 in the Value box.

Click the Add button

Type 0.2 in the Time box and 0 in the Value box.

Then click the Add button.

Enter: 0.4 in the Time box and 0 in the Value box .

Click the Add button.

Then OK, Ok.

Load History Cases:

DEFINE>TIME HISTORY CASES

Click on "Add New History".

Accept the default HIST1.

Select: Analysis Type: Linear

Click the "Modify/Show values"

Enter 0.0 in the damping in all modes

Then OK.

Check the "Envelopes" button

Type 40 in the Number of Output Time Steps

And enter 0.01 for Output Time Step Size.

In "Load Assignment" Select LOAD1

For "Function" select FUNC1

Click the Add button

Then OK, OK.

Assign Load:

Click on joint 12 to select this joint.

ASSIGN>JOINT LOAD STATIC LOADS>FORCES

Enter: Force Global Y = 1.0. Then OK.

Options: ANALYZE>SET OPTIONS

Click UX, UY and RZ for available DOF's

Click on "Dynamic Analysis" and on "Set Dynamic Parameters".

Enter: Number of Modes = 5. Then OK, OK.

Analyze: ANALYZE>RUN

Enter: Filename "EXD 15.1"

Then press SAVE.

When the calculations are concluded enter OK.

Print Input Tables: FILE>PRINT INPUT TABLES

Click "print to File" and accept all other defaults values.

Then OK.

Use Notepad or other word editor to retrieve and edit the Input File "EXD 15.1.txt"
 (Input tables of Illustrative Example EXD 15.1 are reproduced in Table 15.1)

Table 15.1 Edited Input Tables for Illustrative Example 15.1**STATIC LOAD CASES**

STATIC CASE LOAD1	CASE TYPE LIVE	SELF WT FACTOR 0.0000
-------------------------	----------------------	-----------------------------

TIME HISTORY CASES

HISTORY CASE HIST1	HISTORY TYPE LINEAR	NUMBER OF TIME STEPS 40	TIME STEP INCREMENT 0.01000
--------------------------	---------------------------	-------------------------------	-----------------------------------

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-5.00000	-0.50000	0.00000	1 1 0 0 0 0
2	-4.00000	-0.50000	0.00000	0 0 0 0 0 0
3	-4.00000	0.50000	0.00000	0 0 0 0 0 0
4	-5.00000	0.50000	0.00000	0 1 0 0 0 0
5	-3.00000	-0.50000	0.00000	0 0 0 0 0 0
6	-3.00000	0.50000	0.00000	0 0 0 0 0 0
7	-2.00000	-0.50000	0.00000	0 0 0 0 0 0
8	-2.00000	0.50000	0.00000	0 0 0 0 0 0
9	-1.00000	-0.50000	0.00000	0 0 0 0 0 0
10	-1.00000	0.50000	0.00000	0 0 0 0 0 0
11	0.00000	-0.50000	0.00000	0 0 0 0 0 0
12	0.00000	0.50000	0.00000	0 0 0 0 0 0
13	1.00000	-0.50000	0.00000	0 0 0 0 0 0
14	1.00000	0.50000	0.00000	0 0 0 0 0 0
15	2.00000	-0.50000	0.00000	0 0 0 0 0 0
16	2.00000	0.50000	0.00000	0 0 0 0 0 0
17	3.00000	-0.50000	0.00000	0 0 0 0 0 0
18	3.00000	0.50000	0.00000	0 0 0 0 0 0
19	4.00000	-0.50000	0.00000	0 0 0 0 0 0
20	4.00000	0.50000	0.00000	0 0 0 0 0 0
21	5.00000	-0.50000	0.00000	0 1 0 0 0 0
22	5.00000	0.50000	0.00000	0 1 0 0 0 0

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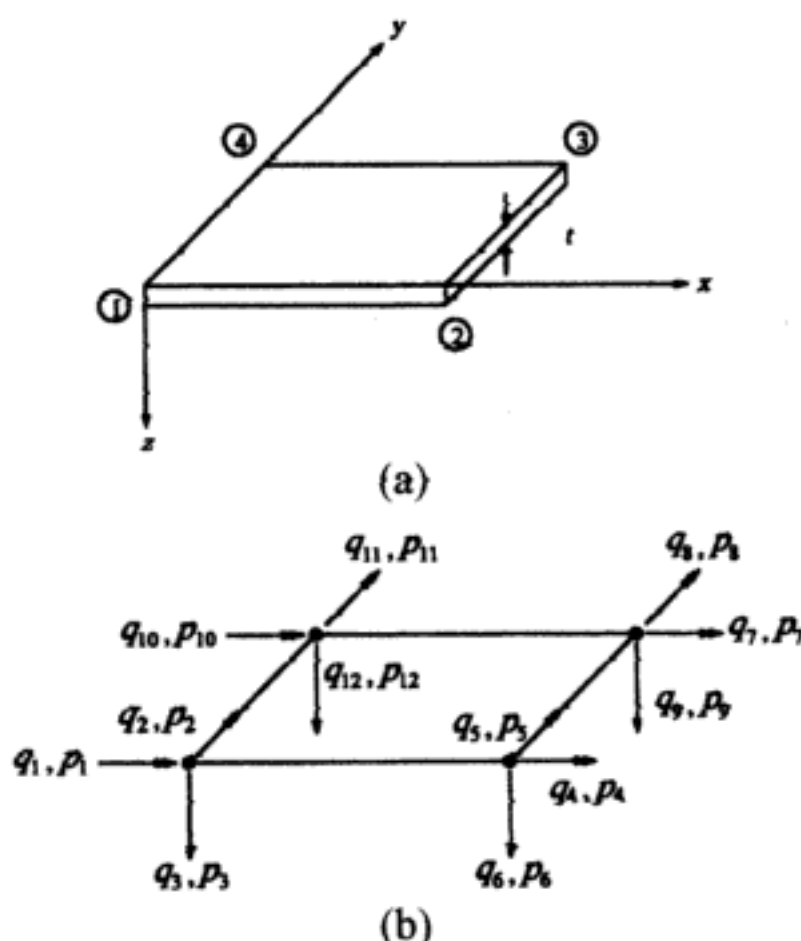


Fig. 15.9 Rectangular plate bending element. (a) Coordinate system and node numbering. (b) Nodal coordinates q_i and corresponding nodal forces p_i .

along the z -axis transverse to the plane of the plate. These nodal displacements are shown in their positive sense and labeled q_i ($i=1,2,\dots,12$) in Fig. 15.9(b)

Corresponding to the three nodal displacements at each node, two moments and a force which are labeled P_i , are also shown in this figure. These 12 element nodal displacements and 12 nodal forces are conveniently arranged in two vectors of 12 components, $\{q\}_e$ and $\{P\}_e$. Therefore, the stiffness matrix or the mass matrix, respectively, relating the nodal forces and the nodal displacements, or nodal forces and nodal accelerations for this rectangular plate bending element with four nodes are of dimension 12×12 . The angular displacements θ_x and θ_y at any point (x, y) of the plate element are related to the normal displacement w by the following expressions:

$$\theta_x = -\frac{\partial w}{\partial y} \quad \theta_y = \frac{\partial w}{\partial x} \quad (15.30)$$

The positive directions of θ_x and θ_y , are chosen to agree with the angular nodal displacement q_1, q_2, q_4, q_5 , etc. is selected in Fig. 15.9(b). Therefore, after a function, $w = w(x, y)$, is chosen for the lateral displacement w , the angular displacements are determined through the relations in eq. (15.30).

Step II: Selection of a suitable displacement function

Since the rectangular element in plate bending has twelve degrees of freedom, the polynomial expression chosen for the normal displacements w , must contain 12 constants. A suitable polynomial function is given by

$$w = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 + c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3 + c_{11} x^3 y + c_{12} xy^3 \quad (15.31)$$

The displacement function for the rotations θ_x and θ_y are then obtained from eqs. (15.30) and (15.31) as

$$\theta_x = -\frac{\partial w}{\partial y} = -(c_3 + c_5 x + 2c_6 y + c_8 x^2 + 2c_9 xy + 3c_{10} y^2 + c_{11} x^3 + 3c_{12} xy^2) \quad (15.32)$$

and

$$\theta_y = \frac{\partial w}{\partial x} = c_2 + 2c_4 x + c_5 y + 3c_7 x^2 + 2c_8 xy + c_9 y^2 + 3c_{11} x^2 y + c_{12} y^3 \quad (15.33)$$

By considering the displacements at the edge of one element, that is, in the boundary between adjacent elements, it may be demonstrated that there is continuity of normal lateral displacements and of the rotational displacement in the direction of the boundary line, but not in the direction transverse to this line as it is shown graphically in Fig. 15.10. The displacement function in eq. (15.31) is called "a non-conforming function" because it does not satisfy the condition of continuity at the boundaries between elements for all three displacements w , θ_x , and θ_y .

Step III: Displacements $\{q(x, y)\}$ at a point within the element are expressed in terms of the nodal displacements $\{q\}_e$.

Writing eq. (15.31), (15.32), and (15.33) in matrix notation, evaluating the displacements at the nodal coordinates and solving for the unknown constants results in

$$\{q(x, y)\} = \begin{Bmatrix} \theta_x \\ \theta_y \\ w \end{Bmatrix} = [g(x, y)] \{c\} \quad (15.34)$$

$$\{q\}_e = [A] \{c\} \quad (15.35)$$

and

$$\{c\} = [A]^{-1} \{q\}_e \quad (15.36)$$

where $[A]^{-1}$ is the inverse of the matrix $[A]$ in eq. (15.35), $[g(x, y)]$ is a function of the coordinates x, y at a point in the element, and $\{q\}_e$ is the vector of the 12 displacements at the nodal coordinates of the element.

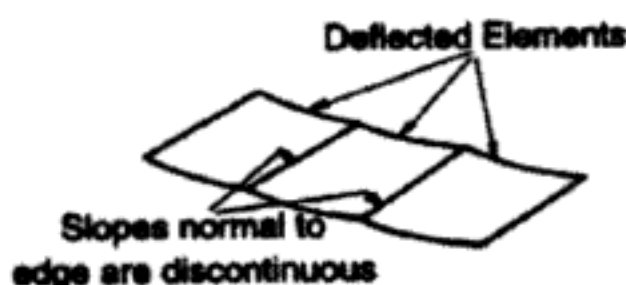


Fig. 15.10 Deflected continuous rectangular elements

The substitution of the vector of constants $\{c\}$ from eq. (15.36) into eq. (15.34) provides the required relationship for displacements $\{q(x, y)\}$ at an interior point in the rectangular element and the displacements $\{q\}_e$ at the nodes

$$\{q(x, y)\} = [g(x, y)] [A]^{-1} \{q\}_e \quad (15.37)$$

or using eq.(15.30):

$$\{q(x, y)\} = \left\{ \begin{array}{c} -\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} \\ w \end{array} \right\} = [f(x, y)] \{q\}_e \quad (15.38)$$

in which $[f(x, y)] = [g(x, y)] [A]^{-1}$ is solely a function of the coordinates x, y at a point within the element.

Step IV: Relationship between strains $(\epsilon(x, y))$ at any point within the element to displacements $\{q(x, y)\}$ and hence to the nodal displacements $\{q\}_e$

For plate bending, the state of strain at any point of the element may be represented by three components: the curvature in the x direction, the curvature in the y direction, and a component representing torsion in the plate. The curvature in the x direction is equal to the rate of change of the slope in that direction, that is, to the derivative of the slope,

$$-\frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) = -\frac{\partial^2 w}{\partial x^2} \quad (15.39)$$

Similarly, the curvature in the y direction is

$$-\frac{\partial}{\partial y} \left(\frac{\partial w}{\partial y} \right) = -\frac{\partial^2 w}{\partial y^2} \quad (15.40)$$

Finally, the torsional strain component is equal to the rate of change, with respect to y , of the slope in the x direction, that is

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$$M_x = -(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}) \quad (15.45)$$

$$M_y = -(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2})$$

$$M_{xy} = 2 D_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

These relations are written in general for an orthotropic plate, i.e. a plate which has different elastic properties in two perpendicular directions, in which D_x and D_y are flexural rigidities in the x and y directions, respectively, D_1 is a "coupling" rigidity coefficient representing the Poisson's ratio type of effect and D_{xy} is the torsional rigidity.

For an isotropic plate which has the same properties in all directions, the flexural and twisting rigidities are given by

$$D_x = D_y = D = \frac{Et^3}{12(1-\nu^2)} \quad (15.46)$$

$$D_1 = \nu D \quad D_{xy} = \frac{(1-\nu)}{2} D$$

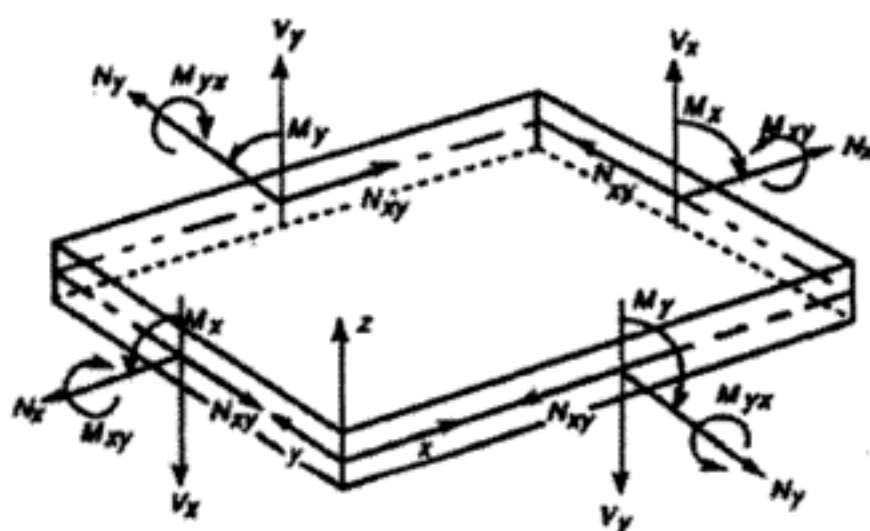


Fig. 15.11 Direction of force and moment per unit length as defined for thin shells.

Equations (15.45) may be written in matrix form as

$$\{\sigma(x, y)\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \quad (15.47)$$

or symbolically as

$$\{\sigma(x, y)\} = [D] \{\varepsilon(x, y)\} \quad (15.48)$$

where the matrix $[D]$ is defined in eq. (15.47). The substitution into eq. (15.48) of $\{\varepsilon(x, y)\}$ from eq. (15.43) results in the required relationship between element stresses and nodal displacements as

$$\{\sigma(x, y)\} = [D] [B] \{q\}_e \quad (15.49)$$

Step VI: Element stiffness and mass matrices

The stiffness matrix and the mass matrix for an element of plate bending obtained by applying the Principle of Virtual Work results in eqs. (15.22) and (15.23), and for the equivalent forces due to the applied body forces on the element in eq. (15.24). The matrices $[f(x, y)]$, $[B]$, and $[D]$ required in these equations are defined, respectively, in eqs. (15.38), (15.43), and (15.47). The calculation of these matrices and also of the integral indicated in eqs. (15.22), (15.23), and (15.24), is usually undertaken by numerical methods implemented in the coding of computer programs

Step VII: Assemblage of the system stiffness matrix $[K]$, the system mass matrix $[M]$, and the vector of the external forces $\{F\}$ at the system nodal coordinates $\{y\}$, which includes the equivalent forces for the body forces $\{P_b\}$ and for any other forces distributed over the structural element.

The system stiffness and mass matrices, as well as the nodal force vector, due to applied body forces, are assembled from the corresponding element matrices and vector as given in eqs. (15.25), (15.26), and (15.27).

Step VIII: Solution of the differential equations of motion

The system differential equation of motion is given by eq. (15.28) in which $[M]$ and $[K]$ are the assembled mass and stiffness matrices, respectively and $\{F\}$ is the equivalent vector of the external forces. The solution of eq. (15.28) thus provides the system vector of nodal displacements $\{y\}$.

Step IX: Determination of nodal stresses

The element nodal stresses, $\{\sigma(x_j, y_j)\}_e$, for node j of an element are given from eq. (15.49) as

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Model: From the Main Menu bar enter:

FILE>NEW MODEL

Number of Grid Spaces:

x - direction = 4

y - direction = 4

z - direction = 0

Grid Spacing:

x - direction = 10

y - direction = 10

z - direction = 10

Then OK.

Edit: Click on XY icon in the menu bar

Maximize the XY window and use the PAN icon in the tool bar to drag the plot into the center of the screen.

Draw: From the main menu enter:

DRAW>QUICK DRAW SHELL ELEMENT

Click on each of the square areas shown in the screen

Then press the Esc key

Label From the main menu enter:

VIEW>SET ELEMENTS

Click on Joint Labels. and on Shell Labels. Then OK.

Use the PAN icon on the tool bar to drag the figure to the center of the screen.

Boundary: Select all the joints on the boundary of the plate shown on the screen.

Then enter:

ASSIGN>JOINT>RESTRAINTS

Select restraints in all directions. Then OK.

Material: DEFINE>MATERIALS

Select: Steel

Click on Modify/Show Materials.

Enter:

Modulus of Elasticity = 29.5E6

Poisson's Ratio = 0.3

Mass per unit of volume $\rho=0.1$.

Then OK, OK.

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Load History Cases:

DEFINE>TIME HISTORY CASES

Click on "Add New History".

Accept the default HIST1.

Select: Analysis Type: Periodic

Click the "Modify/Show values"

Enter 0.0 for damping in all modes

Then OK.

Check the "Envelopes" button

Type 120 in the Number of Output Time Steps

And enter 0.01 for Output Time Step Size.

In the section "Load Assignment" Select LOAD1

and for "Function" select FUNC1

Click the Add button

Then OK, OK.

Table 15.3 Harmonic Function

t (sec)	$F(t) = \sin 5.3t$
0	0
0.0593	0.3090
0.1186	0.5878
0.1778	0.8090
0.2371	0.9511
0.2964	1
0.3557	0.9511
0.4149	0.8090
0.4742	0.5878
0.5335	0.3090
0.5928	0
0.6520	-0.3090
0.7113	-0.5878
0.7706	-0.8090
0.8299	-0.9511
0.8891	-1
0.9484	-0.9511
1.0077	-0.8090
1.0670	-0.5878
1.1262	-0.3090
1.1855	0

Assign Load:

Click on joint 15 and to select this joint.

ASSIGN>JOINT LOAD STATIC LOADS>FORCES

Enter: Force Global Z = -0.1 Then OK.

Options: ANALYZE>SET OPTIONS

Click UZ, RX and RY for available DOF's

Click on "Dynamic Analysis" and on "Set Dynamic Parameters".

Enter: Number of Modes = 5. Then OK, OK.

Analyze: ANALYZE>RUN

Enter: Filename "EXD 15.2"

Then press SAVE.

When the calculations are concluded enter OK.

Print Input Tables: FILE>PRINT INPUT TABLES

Click "print to File" and accept all other defaults values.

Then OK.

Use Notepad or other word editor to retrieve and edit the Input File stored in file:

"C:\SAP2000\EXD 15.2.txt"

(The edited input tables of Illustrative Example 15.2 are reproduced in Table 15.4)

Table 15.4 Edited Input Tables for Illustrative Example 15.2

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	LIVE	0.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	TIME STEP INCREMENT
HIST1	LINEAR	120	0.01000

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-20.00000	-20.00000	0.00000	1 1 1 1 1
2	20.00000	-20.00000	0.00000	1 1 1 1 1
3	20.00000	20.00000	0.00000	1 1 1 1 1
4	-20.00000	20.00000	0.00000	1 1 1 1 1
5	-10.00000	-20.00000	0.00000	1 1 1 1 1
6	-10.00000	-10.00000	0.00000	0 0 0 0 0
7	-20.00000	-10.00000	0.00000	1 1 1 1 1
8	-10.00000	0.00000	0.00000	0 0 0 0 0
9	-20.00000	0.00000	0.00000	1 1 1 1 1
10	-10.00000	10.00000	0.00000	0 0 0 0 0
11	-20.00000	10.00000	0.00000	1 1 1 1 1
12	-10.00000	20.00000	0.00000	1 1 1 1 1
13	0.00000	-20.00000	0.00000	1 1 1 1 1
14	0.00000	-10.00000	0.00000	0 0 0 0 0
15	0.00000	0.00000	0.00000	0 0 0 0 0
16	0.00000	10.00000	0.00000	0 0 0 0 0

Table 15.4 – Continued

17	0.00000	20.00000	0.00000	1 1 1 1 1
18	10.00000	-20.00000	0.00000	1 1 1 1 1
19	10.00000	-10.00000	0.00000	0 0 0 0 0
20	10.00000	0.00000	0.00000	0 0 0 0 0
21	10.00000	10.00000	0.00000	1 1 1 1 1
22	10.00000	20.00000	0.00000	1 1 1 1 1
23	20.00000	-10.00000	0.00000	1 1 1 1 1
24	20.00000	0.00000	0.00000	1 1 1 1 1
25	20.00000	10.00000	0.00000	1 1 1 1 1

SHELL ELEMENT DATA

SHELL JNT-1 JNT-2 JNT-3 JNT-4 SECTION ANGLE AREA

2	1	5	6	7	SSEC2	0.000	100.000
3	7	6	8	9	SSEC2	0.000	100.000
4	9	8	10	11	SSEC2	0.000	100.000
5	11	10	12	4	SSEC2	0.000	100.000
6	5	13	14	6	SSEC2	0.000	100.000
7	6	14	15	8	SSEC2	0.000	100.000
8	8	15	16	10	SSEC2	0.000	100.000
9	10	16	17	12	SSEC2	0.000	100.000
10	13	18	19	14	SSEC2	0.000	100.000
11	14	19	20	15	SSEC2	0.000	100.000
12	15	20	21	16	SSEC2	0.000	100.000
13	16	21	22	17	SSEC2	0.000	100.000
14	18	2	23	19	SSEC2	0.000	100.000
15	19	23	24	20	SSEC2	0.000	100.000
16	20	24	25	21	SSEC2	0.000	100.000
17	21	25	3	22	SSEC2	0.000	100.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
15	0.000	0.000	-0.100	0.000	0.000	0.000

Modal Periods and Natural Frequencies:

Using Notepad or other word editor, open the following file:

C:/SAP2000/EXD 15.2.OUT

The Natural Period and Frequency table is contained in this file.

MODAL PERIODS AND FREQUENCIES

MODE	PERIOD (TIME)	FREQUENCY (CYC/TIME)	FREQUENCY (RAD/TIME)	EIGENVALUE (RAD/TIME)**2
1	0.163827	6.104004	38.352588	1470.921
2	0.093938	10.645303	66.886414	4473.792
3	0.084367	11.852935	74.474189	5546.405
4	0.065660	15.229916	95.692386	9157.033
5	0.061039	16.382842	102.936430	10595.909

Plot Displacement

DISPLAY>SHOW TIME HISTORY TRACES

Click on "Define Functions" and on "Add Input Functions"

Select: "Add Joint Displs/Forces" from the drop down menu. Then OK.

In "Time History Joint Function" screen enter "15" for Joint ID

In Vector Type select Displ and in Component select UZ. Then OK, OK.

Highlight "Joint 15" in the "list of Functions" column

Click "ADD" to move joint 15 to the "Plot Functions" column.

Enter time range from 0 to 0.4.

Axis Labels:

Horizontal: TIME (sec)

Vertical: Z-DISPLACEMENT JOINT 15 (in)

Click on Display

The screen will show the displacement-time function for joint 15

Then press "Done" to exit

FILE>PRINT GRAPHICS

(Fig 15.14 reproduces the Z-Displacement-Time plot for joint 15)

To exit press "Done"

15.3 Summary

In this chapter we have presented an introduction to the Finite Element Method (FEM) for the analysis of problems in Structural Dynamics. The theory of FEM in structural dynamics was formulated through the following steps:

1. Modeling of the entire structure into one-dimensional, two-dimensional, or three-dimensional beam, rod, triangular, quadrilateral, rectangular, or other types of structural elements.
2. Identifying nodes and nodal coordinates at joints between structural elements.
3. Selecting an interpolating function, which usually is a polynomial to express the displacements at an interior point in the element in terms of the
 - a. displacements at its nodal coordinates.

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9. Solving of the system differential equation of motion, eq. (15.28), to obtain the system nodal displacements $\{u\}$.
10. Determining the element nodal stresses $\{\sigma(x_j, y_j)\}$ from the calculated element nodal displacements $\{q\}_e$.

In engineering practice, the solution of problems by the Finite Element Method is obtained with the computer and appropriate software, such as the program SAP2000 used in this text. In the implementation of the program SAP2000, items are selected from menus presented by the program and data is supplied in response at the prompts in the program.

15.4 Problems

Problem 15.1

The steel plate shown in Fig. P15.1 of dimensions 20 in \times 20 in and thickness 0.10 in with a circular hole of radius $r = 5$ in is subjected to a suddenly applied in-plane lateral compressive pressure along the edges AD and BC of magnitude $p = 100$ psi. Model the plate with elements PLANE2D on a 6×6 mesh in each quarter section of the plate. Determine the first five natural frequencies and plots of the response for displacements and stresses on the elements.

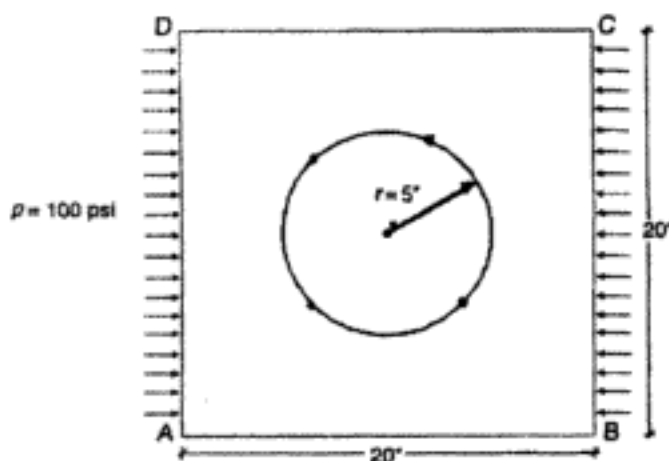


Fig. P15.1.

Problem 15.2

Solve Problem 15.1 taking advantage of the double symmetry of geometry and load introducing appropriate boundary conditions and analyzing only a quarter of the structure.

16 Time History Response of Multi-Degree-of-Freedom Systems

The nonlinear analysis of a single-degree-of-freedom system using the step-by-step linear acceleration method was presented in Chapter 6. The extension of this method with a modification known as the Wilson- θ method, for the solution of structures modeled as multidegree-of-freedom systems is developed in this chapter. The modification introduced in the method by Wilson et al. 1973 serves to assure the numerical stability of the solution process regardless of the magnitude selected for the time step; for this reason, such a method is said to be *unconditionally stable*. On the other hand, without Wilson's modification, the step-by-step linear acceleration method is only conditionally stable and for numerical stability of the solution it may require such an extremely small time step as to make the method impractical if not impossible. The development of the necessary algorithm for the linear and nonlinear multidegree-of-freedom systems by the step-by-step linear acceleration method parallels the presentation for the single-degree-of-freedom system in Chapter 6.

Another well-known method for step-by-step numerical integration of the equations of motion of a discrete system is the Newmark beta method. This method which also may be considered a generalization of the linear acceleration method is presented later in this chapter after discussing in detail the Wilson- θ method.

16.1 Incremental Equations of Motion

The basic assumption of the Wilson- θ method is that the acceleration varies linearly over the time interval from t to $t + \theta \Delta t$. Where $\theta \geq 1.0$. The value of the factor θ is determined to obtain optimum stability of the numerical process and accuracy of the solution. It has been shown by Wilson that, for $\theta \geq 1.38$, the method becomes unconditionally stable.

The equation of motion evaluated at time t_i for a multidegree-of-freedom system in matrix notation is given by

$$\mathbf{M}\ddot{\mathbf{u}}_i + \mathbf{C}(\dot{\mathbf{u}})\dot{\mathbf{u}}_i + \mathbf{K}(\mathbf{u})\mathbf{u}_i = \mathbf{F}_i(t) \quad (16.1)$$

where \mathbf{M} , \mathbf{C} and \mathbf{K} are, respectively the mass, damping and stiffness matrices of the system; \mathbf{u}_i , $\dot{\mathbf{u}}_i$, $\ddot{\mathbf{u}}_i$ the displacement, velocity and acceleration vectors; and $\mathbf{F}_i(t)$ the external force vector

Then the equations expressing the incremental equilibrium conditions for a multidegree-of-freedom system can be derived as the matrix¹ equivalent of the incremental equation of motion for the single degree-of-freedom system, eq. (6.6). Thus taking the difference between dynamic equilibrium conditions defined at time t_i and at $t_i + \tau$, where $\tau = \theta \Delta t$, we obtain the incremental equations of motion:

$$\mathbf{M}\hat{\Delta}\ddot{\mathbf{u}}_i + \mathbf{C}(\dot{\mathbf{u}})\hat{\Delta}\dot{\mathbf{u}}_i + \mathbf{K}(\mathbf{u})\hat{\Delta}\mathbf{u}_i = \hat{\Delta}\mathbf{F}_i \quad (16.2)$$

¹ Matrices and vectors are denoted with boldface lettering throughout this chapter

in which the circumflex over Δ indicates that the increments are associated with the extended time step $\tau = \theta \Delta t$. Thus

$$\hat{\Delta}u_i = u(t_i + \tau) - u(t_i) \quad (16.3)$$

$$\hat{\Delta}\dot{u}_i = \dot{u}(t_i + \tau) - \dot{u}(t_i) \quad \dots \quad (16.4)$$

$$\hat{\Delta}\ddot{u}_i = \ddot{u}(t_i + \tau) - \ddot{u}(t_i) \quad (16.5)$$

and

$$\hat{\Delta}F_i = F(t_i + \tau) - F(t_i) \quad (16.6)$$

In writing eq. (16.2), we assumed, as explained in Chapter 6 for single-degree-of-freedom systems, that the stiffness and damping are obtained for each time step as the initial values of the tangent to the corresponding curves as shown in Fig. 16.1 rather than the slope of the secant line which requires iteration. Hence the stiffness coefficient is defined as

$$k_{ij} = \frac{dF_{si}}{du_j} \quad (16.7)$$

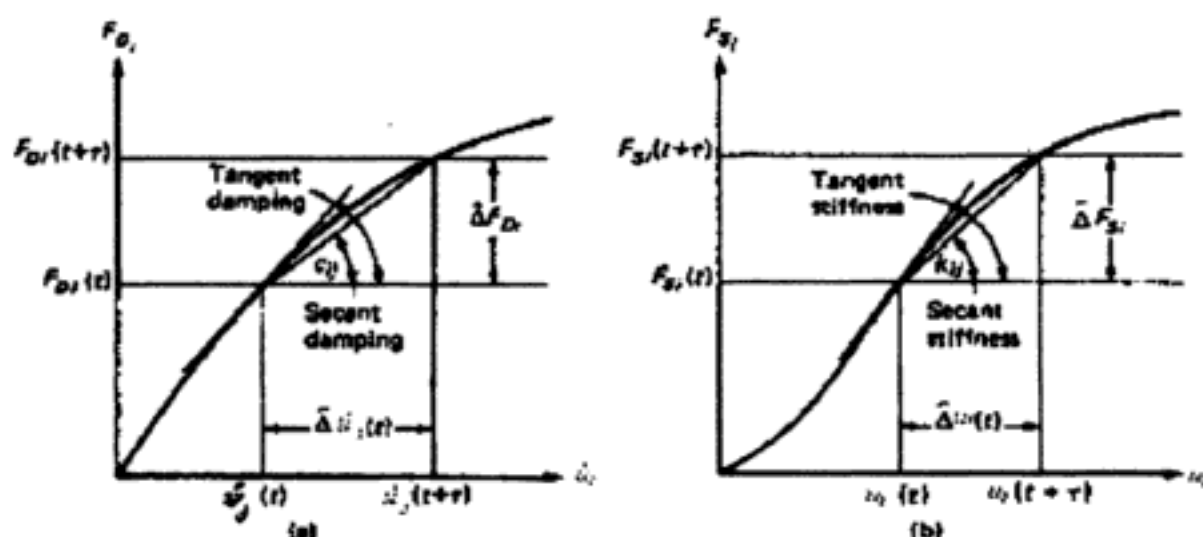


Fig. 16.1 Definition of influence coefficients. (a) Nonlinear viscous damping, c_{ij} . (b) Nonlinear stiffness, k_{ij} .

and the damping coefficient as

$$c_{ij} = \frac{dF_{Di}}{d\dot{u}_j} \quad (16.8)$$

in which F_{si} and F_{Di} are, respectively, the elastic and damping forces at nodal coordinate i and u_j and \dot{u}_j are, respectively, the displacement and velocity at nodal coordinate j .

16.2 The Wilson- θ Method

The integration of the nonlinear equations of motion by the step-by-step linear acceleration method with the extended time step introduced by Wilson is based, as has already been mentioned, on the assumption that the acceleration may be represented by a linear function during the time step $\tau = \theta \Delta t$ as shown in Fig. 16.2. From this figure we can write the linear expression for the acceleration during the extended time step as

$$\ddot{u}(t) = \ddot{u}_i + \frac{\hat{\Delta}\ddot{u}_i}{\tau}(t - t_i) \quad (16.9)$$

in which $\hat{\Delta}\ddot{u}_i$ is given by eq. (16.5). Integrating eq. (16.9) twice between limits t_i and t yields

$$\dot{u}(t) = \dot{u}_i + \ddot{u}_i(t - t_i) + \frac{1}{2} \frac{\hat{\Delta}\ddot{u}_i}{\tau}(t - t_i)^2 \quad (16.10)$$

and

$$u(t) = u_i + \dot{u}_i(t - t_i) + \frac{1}{2} \ddot{u}_i(t - t_i)^2 + \frac{1}{6} \frac{\hat{\Delta}\ddot{u}_i}{\tau}(t - t_i)^3 \quad (16.11)$$

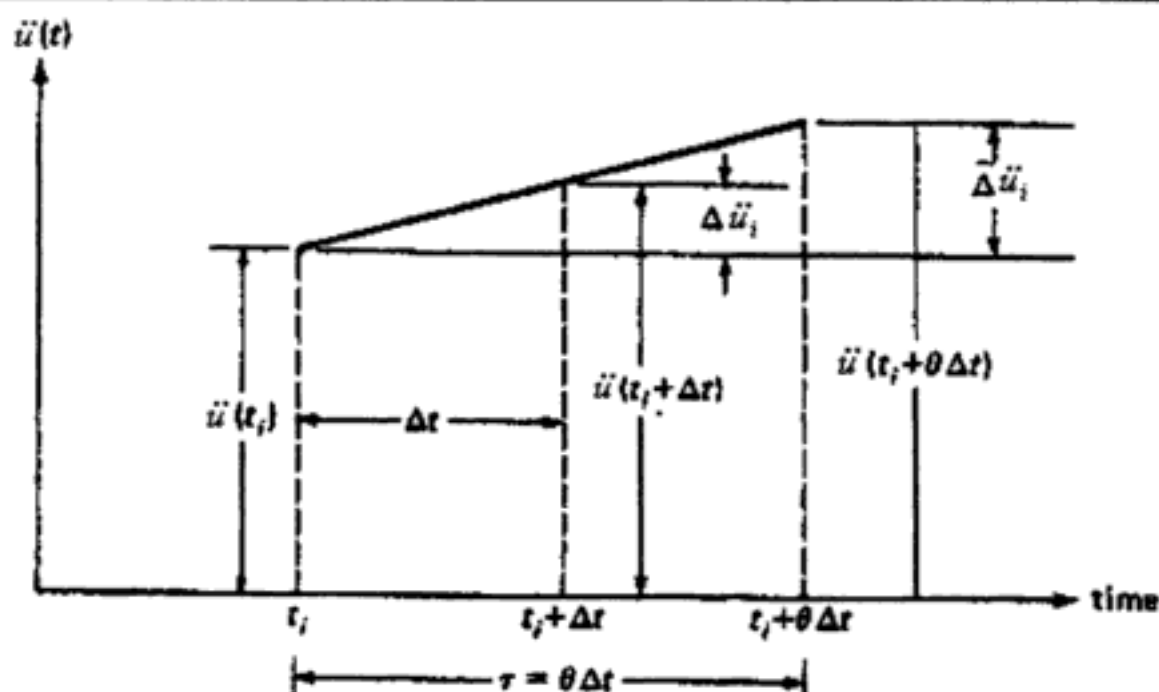


Fig. 16.2 Linear acceleration assumption in the extended time interval.

Evaluation of eqs. (16.10 and (16.11) at the end of the extended interval $t = t_i + \tau$ gives

$$\hat{\Delta \dot{u}}_i = \ddot{u}_i \tau + \frac{1}{2} \hat{\Delta \ddot{u}}_i \tau^2 \quad (16.12)$$

and

$$\hat{\Delta u}_i = \dot{u}_i \tau + \frac{1}{2} \ddot{u}_i \tau^2 + \frac{1}{6} \hat{\Delta \ddot{u}}_i \tau^3 \quad (16.13)$$

in which $\hat{\Delta u}_i$ and $\hat{\Delta \dot{u}}_i$ are defined by eqs. (16.3) and (16.4), respectively.

Now eq. (16.13) is solved for the incremental acceleration $\hat{\Delta \ddot{u}}_i$ and substituted in eq. (16.12) to obtain:

$$\hat{\Delta \ddot{u}}_i = \frac{6}{\tau^2} \hat{\Delta u}_i - \frac{6}{\tau} \dot{u}_i - 3\ddot{u}_i \quad (16.14)$$

and

$$\hat{\Delta \dot{u}}_i = \frac{3}{\tau} \hat{\Delta u}_i - 3\dot{u}_i - \frac{\tau}{2} \ddot{u}_i \quad (16.15)$$

Finally, substituting eqs. (16.14) and (16.15) into the incremental equation of motion, eq. (16.2), results in an equation for the incremental displacement $\hat{\Delta u}_i$, which may be conveniently written as

$$\overline{K}_i \hat{\Delta u}_i = \hat{\Delta F}_i \quad (16.16)$$

where

$$\overline{K}_i = K_i + \frac{6}{\tau^2} M + \frac{3}{\tau} C_i \quad (16.17)$$

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+ Δt , the outlined procedure is repeated to calculate these quantities at the next time step $t_{i+2} = t_{i+1} + \Delta t$ and the process is continued to any desired final time.

The step-by-step linear acceleration, as indicated in the discussion for the single degree-of-freedom system, involves two basic approximations: (1) the acceleration is assumed to vary linearly during the time step, and (2) the damping and stiffness characteristics of the structure are evaluated at the initiation of the time step and are assumed to remain constant during this time interval. The algorithm for the integration process of a linear system by the Wilson- θ method is outlined in the next section. The application of this method to linear structures is then developed in the following section.

16.3 Algorithm for Step-by-Step Solution of a Linear System Using the Wilson- θ Method

16.3.1 Initialization

- (1) Assemble the system stiffness matrix K , mass matrix M , and damping matrix C .
- (2) Set initial values for displacement u_0 , velocity \dot{u}_0 , and forces F_0 .
- (3) Calculate initial acceleration \ddot{u}_0 from eq.16.1) as

$$M\ddot{u}_0 = F_0 - C\dot{u}_0 - Ku_0$$

- (4) Select a time step Δt , the factor θ (usually taken as 1.4), and calculate the constants τ , a_1 , a_2 , a_3 , and a_4 from the relationships:

$$\tau = \theta\Delta t, \quad a_1 = \frac{3}{\tau}, \quad a_2 = \frac{6}{\tau}, \quad a_3 = \frac{\tau}{2}, \quad a_4 = \frac{6}{\tau^2},$$

- (5) Form the effective stiffness matrix \bar{K} [eq. (16.17)], namely,

$$\bar{K} = K + a_4M + a_1C$$

16.3.2 For Each Time Step

- (1) Calculate by linear interpolation the incremental load $\hat{\Delta F}_i$ for the time interval t_i to $t_i + \tau$, from the relationship

$$\hat{\Delta F}_i = F_{i+1} + (F_{i+2} - F_{i+1})(\theta - 1) - F_i$$

- (2) Calculate the effective incremental load $\overline{\hat{\Delta F}_i}$; for the time interval t_i to $t_i + \tau$, from eq. (16.18) as

$$\overline{\hat{\Delta F}_i} = \hat{\Delta F}_i + (a_2M + 3C)\dot{u}_i + (3M + a_3C)\ddot{u}_i$$

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$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

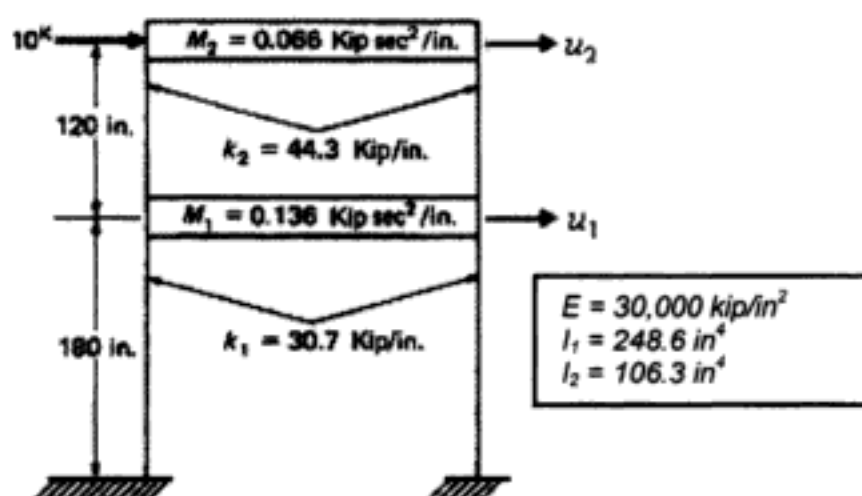


Fig.16.3 Two-story shear building for Illustrative Examples 16.1 and 16.2

which, for free vibration, become

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substitution of $y_i = a_i \sin \omega t$ results in the eigenproblem:

$$\begin{bmatrix} 75.0 - 0.136\omega^2 & -44.3 \\ -44.3 & 44.3 - 0.066\omega^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which requires for a nontrivial solution that

$$\begin{vmatrix} 75.0 - 0.136\omega^2 & -44.3 \\ -44.3 & 44.3 - 0.066\omega^2 \end{vmatrix} = 0$$

Expansion of this determinant yields

$$\omega^4 - 1222.68\omega^2 + 151516 = 0$$

which has the roots

$$\omega_1^2 = 139.94 \quad \text{and} \quad \omega_2^2 = 1082.0$$

Hence, the natural frequencies are

$$\omega_1 = 11.83 \text{ rad/sec}, \quad \text{and} \quad \omega_2 = 32.90 \text{ rad/sec},$$

or

$$f_1 = 1.883 \text{ cps}, \quad \text{and} \quad f_2 = 5.237 \text{ cps}$$

and the natural periods

$$T_1 = 0.531 \text{ sec}, \quad \text{and} \quad T_2 = 0.191 \text{ sec}$$

The initial acceleration at the nodal coordinates is calculated from eq.(16.1) after setting the initial displacement and velocity equal to zero. Thus we obtain:

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{10} \\ \ddot{u}_{20} \end{Bmatrix} \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

giving

$$\begin{aligned} \ddot{u}_{10} &= 0 \\ \ddot{u}_{20} &= 151.51 \text{ in/sec}^2 \end{aligned}$$

Conveniently, we select $\Delta t = 0.02 \text{ sec}$ and $\theta = 1.4$, so $\tau = \theta \Delta t = 0.028$, and calculate the constants to obtain:

$$\begin{aligned} a_1 &= \frac{3}{\tau} = 107.14, & a_2 &= \frac{6}{\tau} = 0.014 \\ a_3 &= \frac{\tau}{2} = 214.28, & a_4 &= \frac{6}{\tau^2} = 7653 \end{aligned}$$

The effective stiffness is then

$$\overline{K} = K + a_1 M + a_1 C \quad (C = 0 \text{ for an undamped system})$$

$$\overline{K} = \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} + 7653 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix}$$

$$\overline{K} = \begin{bmatrix} 1115.8 & -44.3 \\ -44.3 & 549.4 \end{bmatrix}$$

and the effective force

$$\overline{\Delta F}_i = \hat{\Delta F}_i + (a_2 M + 3C)\dot{u}_i + (3M + a_3 C)\ddot{u}_i$$

$$\overline{\Delta F}_i = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 21428 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 3 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix}$$

$$\overline{\hat{\Delta F}_i} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}$$

Solving for $\hat{\Delta u}$ from $\overline{K} \hat{\Delta u} = \overline{\hat{\Delta F}}$ yields

$$\begin{bmatrix} 1115.8 & -44.3 \\ -44.3 & 549.4 \end{bmatrix} \begin{Bmatrix} \hat{\Delta u}_1 \\ \hat{\Delta u}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}, \quad \hat{\Delta u} = \begin{Bmatrix} 0.002175 \\ 0.054780 \end{Bmatrix}$$

Solving for $\hat{\Delta \ddot{u}}$ from eq. (16.14) we obtain

$$\hat{\Delta \ddot{u}} = 7653 \begin{Bmatrix} 0.002175 \\ 0.054780 \end{Bmatrix} - 214.28 \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - 3 \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix}, \quad \hat{\Delta \ddot{u}} = \begin{Bmatrix} 16.645 \\ -35.299 \end{Bmatrix}$$

Then

$$\Delta \ddot{u} = \frac{\hat{\Delta \ddot{u}}}{\theta} = \frac{1}{1.4} \begin{Bmatrix} 16.647 \\ -35.299 \end{Bmatrix} = \begin{Bmatrix} 11.891 \\ -25.21 \end{Bmatrix}$$

From eq. (16.20), it follows that

$$\Delta \dot{u} = \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix} (0.02) + \frac{0.02}{2} \begin{Bmatrix} 11.891 \\ -25.21 \end{Bmatrix} = \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix}$$

From eq. (16.21),

$$\Delta u = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} (0.02) + \frac{(0.02)^2}{2} \begin{Bmatrix} 0 \\ 151.61 \end{Bmatrix} + \frac{(0.02)^2}{6} \begin{Bmatrix} 11.891 \\ -25.21 \end{Bmatrix} = \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix}$$

From eq. (16.21) and (16.23)

$$\{u\}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix} = \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix} \quad (a)$$

and

$$\{\dot{u}\}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix} = \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix} \quad (b)$$

From eq. (16.23),

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \{\ddot{u}\}_1 = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} - \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix}$$

Which gives

$$\{\ddot{u}\}_1 = \begin{Bmatrix} 8.875 \\ 132.85 \end{Bmatrix} \quad (c)$$

The results given in eqs. (a), (b), and (c) for the displacement, velocity, and acceleration, respectively, at time $t_1 = t_0 + \Delta t$ complete a first cycle of the integration process. The continuation in determining the response for this structure is given in Illustrative Example 16.2 with the use of the computer program described in the next section.

16.4 Program 19—Response by Step Integration

Program 19 performs the step-by-step integration of the equations of motion for a linear system using the linear acceleration method with the Wilson- θ modification. The program requires previous modeling of the structure to determine the stiffness matrix and the mass matrix of the system. Input data include the time step Δt , the maximum time computation, TMAX, and a table of the time-force values for each load applied at the nodal coordinates of the structure.

The program performs a linear interpolation between the load data points, which result in a table giving the magnitude of the applied forces at each nodal coordinate calculated at increments of time equal to the time step Δt . The output consists of a table giving the response for each nodal coordinate in terms of displacement, velocity, and acceleration at increments of time Δt up to the maximum time specified by the duration of the forces including, if desired, an extension with forces set to zero.

Illustrative Example 16.2

Use Program 19 to determine the response of the two-story shear building shown in Fig. 16.3. The first cycle of the integration process for this structure has been hand calculated in Illustrative Example 16.1.

Solution:

Program Input Data and Output Results

PROGRAM 19 RESPONSE BY STEP INTEGRATION DATA FILE=D19

RESPONSE BY LINEAR ACCELERATION METHOD

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM
NUMBER OF POINTS DEFINING THE EXCITATION

ND = 2
NEX = 2

COORD. AT EXCITATION (FOR SUPPORT MOTION=0)
TIME STEP OF INTEGRATION

NCX = 2
H = .02

SYSTEM STIFFNESS MATRIK

0.7500E+02 -0.4430E+02
-0.4430E+02 0.4430E+02

SYSTEM MASS MATRIX

0.1360E+00 0.0000E+00
0.0000E+00 0.6600E-01

SYSTEM DAMPING MATRIX

0.0000E+00 0.0000E+00
0.0000E+00 0.0000E+00

EXCITATION FUNCTION

TIME	EXCITATION
0.00	10.00
0.40	10.00

PRINT TIME HISTORY TABLE (Y/N) ? N
OUTPUT RESULTS

MAXIMUM RESPONSE

COORD.	MAX. DISPL.	MAX. VELOC.	MAX. ACC.
1	0.692	5.875	89.763
2	1.087	7.273	147.611

16.5 The Newmark Beta Method

The Newmark beta method may be considered a generalization of the linear acceleration method. It uses a numerical parameter designated as β . The method, as originally proposed by Newmark (1959), contained in addition to β , a second parameter γ . These parameters replace the numerical coefficients $\frac{1}{2}$ and $\frac{1}{6}$ of the terms containing the incremental acceleration $\Delta\ddot{u}_i$ in eqs. (16.20) and (16.21), respectively. Thus, replacing by γ the coefficient $\frac{1}{2}$ of $\Delta\ddot{u}_i$ in eq. (16.20) and by β the coefficient $\frac{1}{6}$ also of $\Delta\ddot{u}_i$ in eq. (16.20), we have

$$\Delta\dot{u}_i = \ddot{u}_i\Delta t + \gamma\Delta\ddot{u}_i\Delta t \quad (16.25)$$

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the interval. In this last case, that is, $\beta = \frac{1}{4}$, the Newmark beta method is unconditionally stable and it provides the satisfactory accuracy.

16.6 Elastoplastic Behavior of Framed-Structures

The dynamic analysis of beams and frames having linear elastic behavior was presented in the preceding chapters. To extend this analysis to structures whose members may be strained beyond the yield point of the material, it is necessary to develop the member stiffness matrix for the assumed elastoplastic behavior. The analysis is then carried out by a step-by-step numerical integration of the differential equations of motion. Within each short time interval Δt , the structure is assumed to behave in a linear elastic manner, but the elastic properties of the structure are changed from one interval to another as dictated by the response. Consequently, the nonlinear response is obtained as a sequence of linear responses of different elastic systems. For each successive interval, the stiffness of the structure is evaluated based on the moments in the members at the beginning of the time increment.

The changes in displacements of the linear system are computed by integration of the differential equations of motion over the finite interval and the total displacements by addition of the incremental displacement to the displacements calculated in the previous time step. The incremental displacements are also used to calculate the increment in member end forces and moments from the member stiffness equation. The magnitude of these end moments relative to the yield conditions (plastic moments) determines the characteristics of the stiffness and mass matrices to be used in the next time step.

16.7 Member Stiffness Matrix

If only bending deformations are considered, the force-displacement relationship for a uniform beam segment (Fig. 16.4) with elastic behavior (no hinges) is given by eq. (10.20). This equation may be written in incremental quantities as follows:

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \Delta \delta_1 \\ \Delta \delta_2 \\ \Delta \delta_3 \\ \Delta \delta_4 \end{Bmatrix}$$

(16.33)

in which ΔP_i and $\Delta \delta_i$ are, respectively, the incremental forces and the incremental displacements at the nodal coordinates of the beam segment. When the moment at one end of the beam reaches the value of the plastic moment M_p , a hinge is formed at that end. Under the assumption of an elastoplastic relation-between the bending moment and the angular displacement as depicted in Fig. 16.5, the section that has been transformed into a hinge cannot support a moment higher than the plastic moment M_p but it may continue to deform plastically at a constant moment M_p . The relationship reverses to an elastic behavior when the angular displacement begins to decrease as shown in Fig. 16.5. We note the complete similarity for the behavior between an

elastoplastic spring (Fig. 6.5) in a single degree-of-freedom system and an elastoplastic section of a beam (Fig. 16.5).

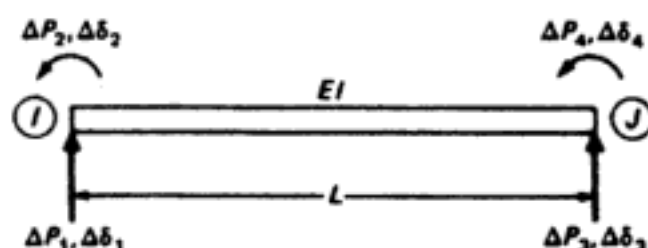


Fig. 16.4 Beam segment indicating incremental end forces and corresponding incremental displacements.

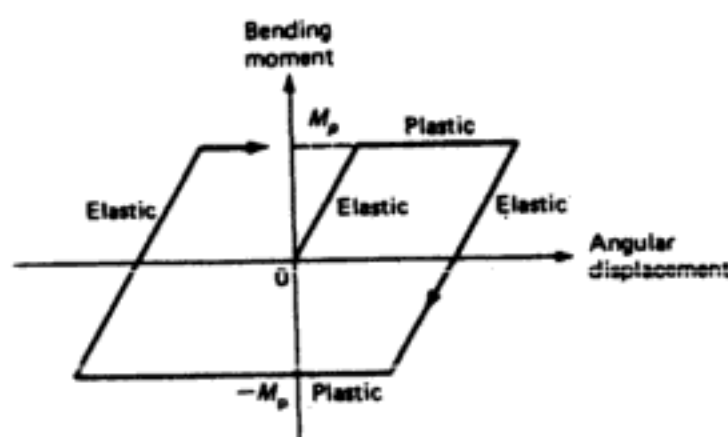


Fig. 16.5 Elastoplastic relationship between bending moment and angular displacement at a section of a beam.

The stiffness matrix for a beam segment with a hinge at one end (Fig. 16.6) may be obtained by application of eq. (10.16) which is repeated here for convenience, namely

$$k_{ij} = \int_0^L EI N_i^*(x) N_j^*(x) dx \quad (16.34)$$

Where $\psi_i(x)$ and $\psi_j(x)$ are displacement functions. For a uniform beam in which the formation of the plastic hinge takes place at end Θ as shown in Fig. 16.6, the deflection functions corresponding to unit displacement at one of the nodal coordinates δ_1 , δ_2 , δ_3 , or δ_4 are given respectively by

$$N_1(x) = 1 - \frac{3x}{2L} + \frac{x^3}{2L^3} \quad (16.35a)$$

$$N_2(x) = 0 \quad (16.35b)$$

$$N_3(x) = \frac{3x}{2L} + \frac{x^3}{2L^3} \quad (16.35c)$$

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$$\begin{aligned} N_3(x) &= \frac{x}{L} \\ N_4(x) &= 0 \end{aligned} \quad (16.45)$$

and the corresponding relationship in incremental form by

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta \ddot{\delta}_1 \\ \Delta \ddot{\delta}_2 \\ \Delta \ddot{\delta}_3 \\ \Delta \ddot{\delta}_4 \end{Bmatrix} \quad (16.46)$$

16.9 Rotation of Plastic Hinges

In the solution process, at the end of each step interval it is necessary to calculate the end moments of every beam segment to check whether or not a plastic hinge has been formed. The calculation is done using the element incremental moment-displacement relationship. It is also necessary to check if the plastic deformation associated with a hinge is compatible with the sign of the moment. The plastic hinge is free to rotate in one direction only, and in the other direction the section returns to an elastic behavior. The assumed moment rotation characteristics of the member are of the type illustrated in Fig. 16.5. The conditions implied by this model are: (1) the moment cannot exceed the plastic moment; (2) if the moment is less than the plastic moment, the hinge cannot rotate; (3) if the moment is equal to the plastic moment, then the hinge may rotate in the direction consistent with the sign of the moment; and (4) if the hinge starts to rotate in a direction inconsistent with the sign of the moment, the hinge is removed.

The incremental rotation of a plastic hinge is given by the difference between the incremental joint rotation of the frame and the increase in rotation of the end of the member at that joint. For example, with a hinge at end "i" only (Fig. 20.6), the incremental joint rotation is $\Delta \delta_2$ and the increase in rotation of this end due to rotation $\Delta \delta_4$ is $-\Delta \delta_4 / 2$ and that due to the displacements $\Delta \delta_1$ and $\Delta \delta_3$ is $1.5 (\Delta \delta_3 - \Delta \delta_1) / L$. Hence the increment in rotation $\Delta \rho_i$ of a hinge formed at end i is given by

$$\Delta \rho_i = \Delta \delta_2 + \frac{1}{2} \Delta \delta_4 - 1.5 \frac{\Delta \delta_3 - \Delta \delta_1}{L} \quad (16.47)$$

Similarly, with a hinge formed at end "j" only (Fig. 16.7), the increment in rotation of this hinge is given by

$$\Delta \rho_j = \Delta \delta_4 + \frac{1}{2} \Delta \delta_2 - 1.5 \frac{\Delta \delta_3 - \Delta \delta_1}{L} \quad (16.48)$$

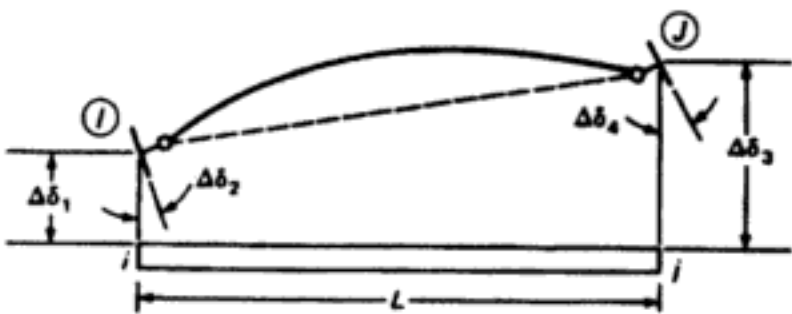


Fig. 16.8 Beam geometry with plastic hinges at both ends.

Finally, with hinges formed at both ends of a beam segment (Fig. 16.8), the rotations of the hinges are given by

$$\Delta\rho_i = \Delta\delta_1 - \frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.49}$$

$$\Delta\rho_j = \Delta\delta_2 - \frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.50}$$

16.10 Calculation of Member Ductility Ratio

Nonlinear beam deformation are expressed in terms of the member ductility ratio, which is defined as the maximum total end rotation of the member to the end rotation at the elastic limit. The elastic limit rotation is the angle developed when the member is subjected to antisymmetric yield moments M_x as shown in Fig. 16.9. In this case the relationship between the end rotation and end moment is given by

$$\phi_y = \frac{M_y L}{6EI} \tag{16.51}$$

The member ductility ratio μ is then defined as

$$\mu = \frac{\phi_y + \rho_{\max}}{\phi_y} \tag{16.52}$$

which from eq. (16.50) becomes

$$\mu = 1 + \frac{6EI}{M_y L} \rho_{\max} \tag{16.53}$$

where ρ_{\max} is the maximum rotation of the plastic hinge.

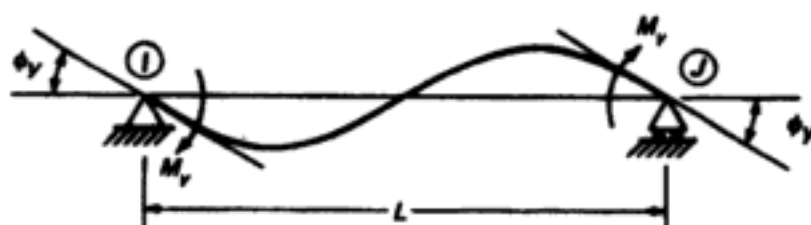


Fig. 16.9 Definition of yield rotation for beam segment.

16.11 Time-History Response of Multidegree-of-Freedom Systems Using SAP2000

The computer program SAP2000 includes the step-by-step Wilson- θ Method, as well as, several other methods for the analysis of structures with linear or nonlinear behavior (material or large displacements nonlinearities). The following example illustrates the use of the program SAP2000 to obtain the response of a structure with linear or nonlinear behavior:

Illustrative Example 16.3.

Use SAP2000 to calculate the natural frequencies and the response of the two-story shear building of Illustrative Example 16.3.(Fig. 16.10)

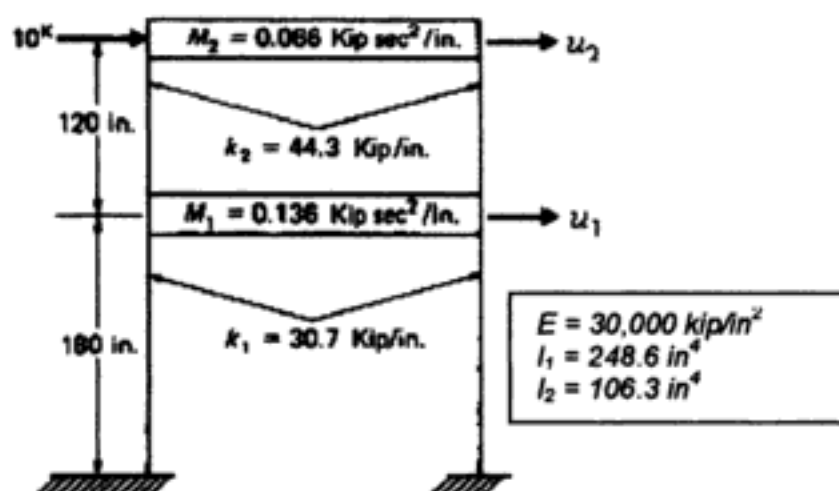


Fig. 16.10 Shear Building model for Illustrative Example 16.3

Solution:

The following commands are implemented in SAP2000:

Begin: Open SAP2000 (already installed)

Hint: Maximize both windows for a full view of the screen.

Select: In the lower right-hand corner of the screen use the drop-down menu to select “kip-in”.

Select: From the Main Menu bar:
 FILE>NEW MODEL FROM TEMPLATE
 Click on Portal Frame
 Enter:
 Number of Stories = 2
 Number of Bays = 1
 Story Height = 180
 Bay width = 288. Then OK.

Edit: Click on XZ icon in the Menu Bar
 Maximize the XZ window and use the PAN icon in the toolbar to drag the figure to the center of the screen.

Label: From the Main Menu enter:
 VIEW>SET ELEMENTS
 Click on Joint and Frame Labels. Then OK.
 Use the PAN icon on the toolbar to center the figure on the screen.

Draw: DRAW>EDIT GRID
 Click on “Glue Joints to Grid”
 Axis “Z”
 Click on 360
 Type “300”
 Click “Move Grid Line”
 Then OK

Define: DEFINE>FRAME SECTIONS>ADD I/WIDE FLANGE
 Select “Add General”
 Set Moment of Inertia about Axis 3 = 248.6
 Set Shear Area in Direction 2 = 0
 Accept the other default values
 Then OK

Select “Add General”
 Select Moment of Inertia about Axis 3 = 106.3
 Set Shear Area in Direction 2 = 0
 Then OK

Select “General Section”
 Set Area A = 1E6
 Set Moment of Inertia = 1E6
 Then OK, OK

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Load Case: DEFINE>TIME HISTORY CASES

Enter: Add New History

Enter: Analysis Type = Linear

Number of Output Time Steps = 40

Time Steps = 0.01.

Click on "Envelopes"

Load Assignments:

Enter: Load = LOAD1

Function = FUNC1. Then press ADD

Then OK, OK.

Assign Loads: Click on Joint 3 and enter:

ASSIGN>JOINT STATIC LOADS>FORCES

Select LOAD1 from the drop down menu

Enter: Load, Force Global X = 1. Then OK.

Then OK.

Analyze: ANALYZE>SET OPTIONS

Check only UX in Available DOF's (Degrees of Freedom)

Check Dynamic Analysis and Set Dynamic Parameters

Enter: Number of Modes = 2.

Then OK, OK.

ANALYZE>RUN

Enter filename: EXD16.3. Then SAVE.

At the conclusion of the calculations, after checking for errors, press OK.

Select X-Z view.

Output Table Mode: DISPLAY>SET OUTPUT TABLE MODE

Select LOAD1 and HIST1 History (hold down CTRL key to select all) then OK.

Print Input Tables: FILE>PRINT INPUT TABLES

CLICK "Print to File" and accept all other default values.

Then OK.

(The edited Input Tables of Illustrative Example 16.3 are given in Table 16.1)

Table 16.1 Edited Input Data for Illustrative Example 16.3.

STATIC LOAD CASES

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	LIVE	0.0000

TIME HISTORY CASES

HISTORY CASE	HISTORY TYPE	NUMBER OF TIME STEPS	INCREMENT
HIST1	LINEAR	40	0.01000

JOINT DATA

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS
1	-144.00000	0.00000	0.00000	1 1 1 1 1
2	-144.00000	0.00000	180.00000	0 0 0 0 0
3	-144.00000	0.00000	300.00000	0 0 0 0 0
4	144.00000	0.00000	0.00000	1 1 1 1 1
5	144.00000	0.00000	180.00000	0 0 0 0 0
6	144.00000	0.00000	300.00000	0 0 0 0 0

JOINT MASS DATA

JOINT	M-U1	M-U2	M-U3	M-R1	M-R2	M-R3
5	0.136	0.000	0.000	0.000	0.000	0.000
6	6.600E-02	0.000	0.000	0.000	0.000	0.000

FRAME ELEMENT DATA

FRM	JNT1	JNT2	SEC	SEGMENTS	LENGTH
1	1	2	FSEC2	2	180.000
2	2	3	FSEC3	2	120.000
3	4	5	FSEC2	2	180.000
4	5	6	FSEC3	2	120.000
5	2	5	FSEC4	4	288.000
6	3	6	FSEC4	4	288.000

JOINT FORCES Load Case LOAD1

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	GLOBAL-XX	GLOBAL-YY	GLOBAL-ZZ
3	1.000	0.000	0.000	0.000	0.000	0.000

Plot Displacement Function:

DISPLAY>SHOW TIME HISTORY TRACES

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 2

Vector Type = Displacement

Component = UX.

Mode Number = Include All. Then OK.

Click On Define Functions

Click to "Add Joint Disps/Forces"

Joint ID = 3

Vector Type = Displacement

Component = UX.

Mode Number = Include All. Then OK., OK.

Under List of Functions click once on Joint2,
Click on ADD to move Joint2 to Plot Functions column
Repeat these two steps for joint 3
(These two functions will then appear on the Time History Traces Plot)
Axis Labels
 Horizontal = Time (sec)
 Vertical = Displacements Joints 2 & 3 (in)
Click "Display" to view Displacement functions
Enter: FILE>PRINT GRAPHICS to obtain a printed version of these plots.

(Figure 16.11 displays the plot of the Displacement Function Values for Joints 2 and 3 of Illustrative Example 16.3.)

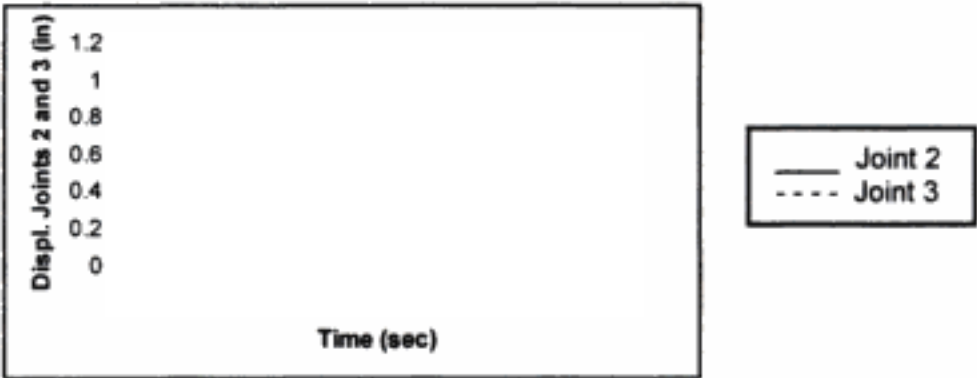


Fig. 16.11 Displacements for Joints 2 and 3 of Illustrative Example 16.3.

Alternatively, a table of the displacement function may be obtained by entering:

FILE>PRINT TABLES TO FILE

The values in this table may be plotted using Excel or similar software.

(Table 16.2 contains the Displacement Function Values for Joints 2 and 3 of Illustrative Example 16.3)

Table 16.2 Displacement Function Values of Joints 2 and 3 of Illustrative Example 16.3

TIME (sec)	Joint 2 (in)	Joint 3 (in)
0	0	0
0.01	2.05E-05	0.00753
0.02	3.24E-04	0.02963
0.03	0.00161	0.06485
0.04	0.00493	0.11093
0.05	0.0116	0.16501
-----	-----	-----
0.39	0.40966	0.52246

Natural Periods and Frequencies:

Using Notepad or other word editor, open the following file:

C:/SAP2000/EXD16.3.OUT

The Natural Period and Frequency table may be found in this file.

(Table 16.3 contains the edited Natural Period and Frequency values of Illustrative Example 16.3)

Table 16.3 Natural Periods and Natural Frequencies for Illustrative Example 16.3.

MODAL PERIODS AND FREQUENCIES				
MODE	PERIOD (TIME)	FREQUENCY (CYC/TIME)	FREQUENCY (RAD/TIME)	EIGENVALUE (RAD/TIME)**2
1	0.531	1.882	11.827	139.898
2	0.190	5.236	32.901	1082.532

Mode Shapes: DISPLAY>SHOW MODE SHAPE

Modal Shape 1 : Click "Wire Shadow" and "Cubic Curve"

Scale Factor = 1. Then OK.

(To view Mode 2 repeat previous entries)

* Numerical values for a mode shape in the X direction axis may be read on the screen by right-clicking on a node in the modal shape shown on the screen. Values for Modes 1 and 2 are given in Table 16.4

Table 16.4 Mode Shapes for the shear building of Illustrative Example 16.3.

Joint	Mode 1	Mode 2
	($T_1=0.5312$ sec)	($T_2=0.081$ sec)
1	0	0
2	0.23555	1.79149
3	2.57166	-2.922

16.12 Summary

The determination of the nonlinear response of multidegree-of-freedom structures requires the numerical integration of the governing equations of motion. There are many methods available for the solution of these equations. The step-by-step linear acceleration method with a modification known as the Wilson- θ method was presented in this chapter. This method is unconditionally stable, that is, numerical errors do not tend to accumulate during the integration process regardless of the magnitude selected for the time step. The basic assumption of the Wilson- θ method is that the acceleration varies linearly over the extended interval $\tau = \theta \Delta t$ in which $\theta \geq 1.38$ for unconditional stability.

In the final sections of this chapter, stiffness and mass matrices for elastoplastic behavior of framed structures are presented. Formulas to determine the plastic rotation of hinges and to calculate the corresponding ductility ratios are also presented in this chapter.

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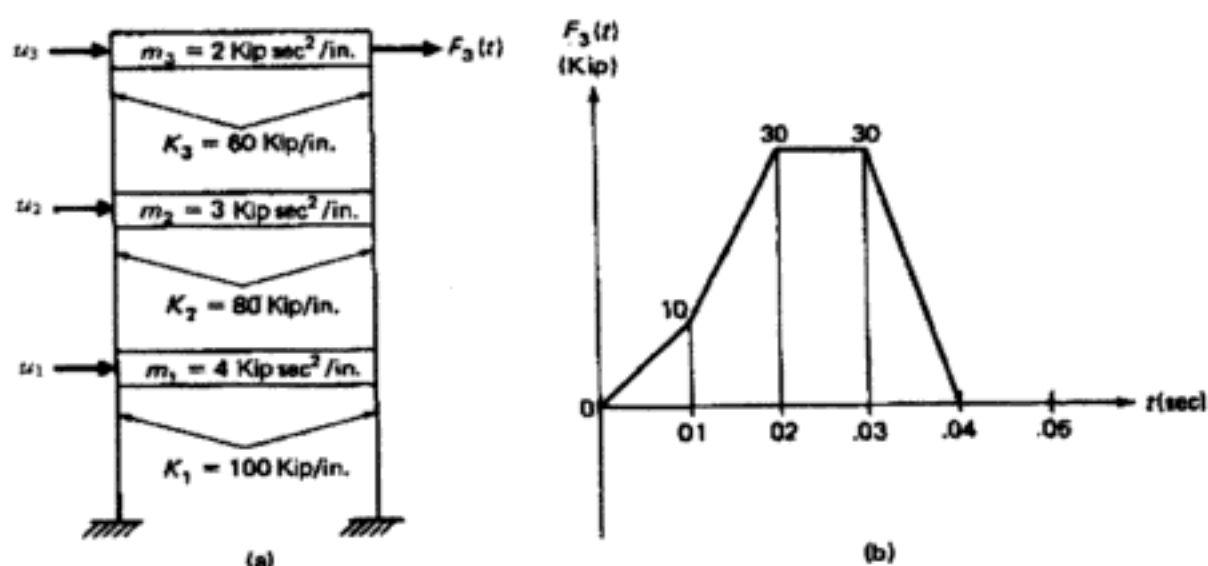


Fig. P16.3

Problem 16.4

Solve Problem 16.3 considering damping in the system of 10% in all the modes.

Problem 16.5

Use Program 19 to obtain the response in the elastic range for the structure of Problem 16.1 subjected to an acceleration at its foundation given by the function $f(t)$ shown in Fig. P16.1.

Problem 16.6

Solve Problem 16.5 considering damping in the system as indicated in Problem 16.2.

Problem 16.7

Use Program 19 to obtain the response in the elastic range of the shear building shown in Fig. P16.3 when subjected to an acceleration of its foundation given by the function $f(t)$ depicted in Fig. P16.1. Neglect damping in the system.

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17 Dynamic Analysis of Systems with Distributed Properties

The dynamic analysis of structures, modeled as lumped parameter systems with discrete coordinates, was presented in Part I for single-degree-of-freedom systems and in Parts II and III for multidegree-of-freedom systems. Modeling structures with discrete coordinates provides a practical approach for the analysis of structures subjected to dynamic loads. However, the results obtained from these discrete models can only give approximate solutions to the actual behavior of dynamic systems which have continuous distributed properties and, consequently, an infinite number of degrees of freedom.

The present chapter considers the dynamic theory of beams and rods having distributed mass and elasticity for which the governing equations of motion are partial differential equations. The integration of these equations is in general more complicated than the solution of ordinary differential equations governing discrete dynamic systems. Due to this mathematical complexity, the dynamic analysis of structures as continuous systems has limited use in practice. Nevertheless, the analysis, as continuous systems, of some simple structures provides, without much effort, results which are of great importance in assessing approximate methods based on discrete models.

17.1 Flexural Vibration of Uniform Beams

The treatment of beam flexure developed in this section is based on the simple bending theory as it is commonly used for engineering purposes. The method of analysis is known as the Bernoulli-Euler theory which assumes that a plane cross section of a beam remains a plane during flexure.

Consider in Fig. 17.1 the free body diagram of a short segment of a beam. It is of length dx and is bounded by plane faces that are perpendicular to its axis. The forces and moments which act on the element are also shown in the figure: they are the shear forces V and $V + (\partial V / \partial x) dx$; the bending moments M and $M + (\partial M / \partial x) dx$; the lateral load $p dx$; and the inertia force $(\bar{m} dx) \partial^2 y / \partial t^2$. In this notation \bar{m} is the mass per unit length and $p = p(x, t)$ is the load per unit length. Partial derivatives are used to express acceleration and variations of shear and moment because these quantities are functions of two variables, position x along the beam and time t . If the deflection of the beam is small, as the theory presupposes, the inclination of the beam segment from the unloaded position is also small. Under these conditions, the equation of motion perpendicular to the x axis of the deflected beam obtained by equating to zero the sum of the forces in the free body diagram of Fig. 17.1(b) is

$$V - (V + \frac{\partial V}{\partial x} dx) + p(x, t) dx - \bar{m} dx \frac{\partial^2 y}{\partial t^2} = 0$$

which, upon simplification, becomes

$$\frac{\partial V}{\partial x} + \bar{m} \frac{\partial^2 y}{\partial t^2} = p(x, t) \quad (17.1)$$

From simple bending theory, we have the relationships

$$M = EI \frac{\partial^2 y}{\partial x^2} \quad (17.2)$$

and

$$V = \frac{\partial M}{\partial x} \quad (17.3)$$

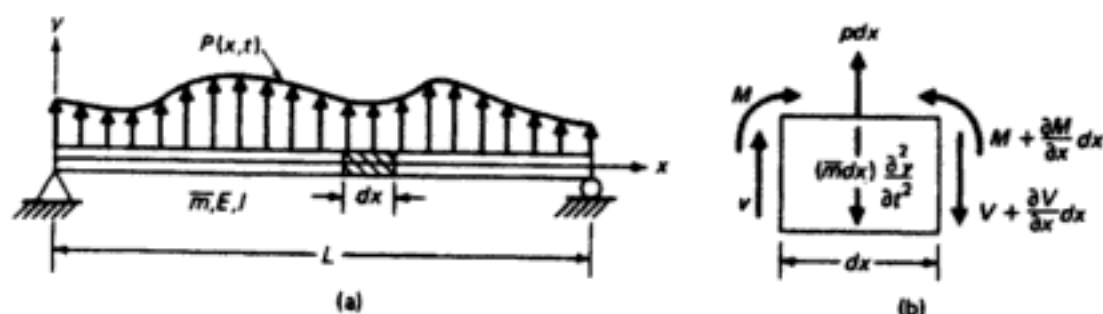


Fig. 17.1 Simple beam with distributed mass and load.

where E is Young's modulus of elasticity and I is the moment of inertia of the cross-sectional area with respect to the neutral axis through the centroid. For a uniform beam, the combination of eqs. (17.1), (17.2), and (17.3) results in

$$V = EI \frac{\partial^3 y}{\partial x^3} \quad (17.4)$$

and

$$EI \frac{\partial^4 y}{\partial x^4} + \bar{m} \frac{\partial^2 y}{\partial t^2} = p(x, t) \quad (17.5)$$

It is seen that eq. (17.5) is a partial differential equation of fourth order. It is an approximate equation. Only lateral flexural deflections were considered while the deflections due to shear forces and the inertial forces caused by the rotation of the cross section (rotary inertia) were neglected. The inclusion of shear deformations and rotary inertia in the differential equation of motion considerably increases its complexity. The equation taking into consideration shear deformation and rotary inertia is known as Timoshenko's equation. The differential equation (17.5) also does not include the flexural effects due to the presence of forces which may be applied axially to the beam. The axial effects will be discussed in Chapter 22.

17.2 Solution of The Equation of Motion in Free Vibration

For free vibration [$p(x, t) = 0$], eq. (17.5) reduces to the homogeneous differential equation

$$EI \frac{\partial^4 y}{\partial x^4} + \bar{m} \frac{\partial^2 y}{\partial t^2} = 0 \quad (17.6)$$

The solution of eq. (17.6) can be found by the method of separation of variables. In this method, it is assumed that the solution may be expressed as the product of a function of position $\Phi(x)$ and a function of time $f(t)$, that is,

$$y(x, t) = \Phi(x)f(t) \quad (17.7)$$

The substitution of eq. (17.7) in the differential equation (17.6) leads to

$$EI f(t) \frac{d^4 \Phi(x)}{dx^4} + \bar{m} \Phi(x) \frac{d^2 f(t)}{dt^2} = 0 \quad (17.8)$$

This last equation may be written as

$$\frac{EI}{\bar{m}} \frac{\Phi^{IV}(x)}{\Phi(x)} = -\frac{\ddot{f}(t)}{f(t)} \quad (17.9)$$

In this notation Roman indices indicate derivatives with respect to x and overdots indicate derivatives with respect to time. Since the left-hand side of eq. (17.9) is a function only of x while the right-hand side is a function only of t , each side of the equation must equal the same constant value; otherwise, the identity of eq. (17.9) cannot exist. We designate the constant by ω^2 which equated separately to each side of eq. (17.9) results in the two following differential equations:

$$\Phi^{IV}(x) - \omega^4 \Phi(x) = 0 \quad (17.10)$$

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Substitution of these relationships into eq. (17.18) yields

$$\Phi(x) = A \sin ax + B \cos ax + C \sinh ax + D \cosh ax \quad (17.20)$$

where A , B , C , and D are new constants of integration. These four constants of integration define the shape and the amplitude of the beam in free vibration; they are evaluated by considering the boundary conditions at the ends of the beam as illustrated in the examples presented in the following section.

17.3 Natural Frequencies and Mode Shapes for Uniform Beams

17.3.1 Both Ends Simply Supported

In this case the displacements and bending moments must be zero at both ends of the beam; hence the boundary conditions for the simply supported beams are

$$y(0, t) = 0, \quad M(0, t) = 0$$

$$y(L, t) = 0, \quad M(L, t) = 0$$

In view of eqs. (17.2) and (17.7), these boundary conditions imply the following conditions on the shape function $\Phi(x)$.

At $x = 0$,

$$\Phi(0) = 0, \Phi'(0) = 0 \quad (17.21)$$

At $x = L$,

$$\Phi(L) = 0, \Phi'(L) = 0 \quad (17.22)$$

The substitution of the first two of these boundary conditions into eq. (17.20) yields

$$\Phi(0) = A0 + B1 + C0 + D1 = 0$$

$$\Phi'(0) = a^2(-A0 - B1 + C0 + D1) = 0$$

which reduce to

$$B + D = 0$$

$$-B + D = 0$$

Hence

$$B = D = 0$$

Similarly, substituting the last two boundary conditions into eq. (17.20) and setting $B = D = 0$ leads to

$$\Phi(L) = A \sin aL + C \sinh aL = 0$$

$$\Phi''(L) = a^2(-A \sin aL + C \sinh aL) = 0 \quad (17.23)$$

which, when added, give

$$2C \sinh aL = 0$$

From this last relation, $C = 0$ since the hyperbolic sine function cannot vanish except for a zero argument. Thus eqs. (17.23) reduce to

$$A \sin aL = 0 \quad (17.24)$$

Excluding the trivial solution ($A = 0$), we obtain the frequency equation

$$\sin aL = 0 \quad (17.25)$$

which will be satisfied for

$$a_n L = n\pi, \quad n = 0, 1, 2, \dots \quad (17.26)$$

Substitution of the roots, eq. (17.26), into eq. (17.13) yields

$$\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{mL^4}} \quad (17.27)$$

where the subscript n serves to indicate the order of the natural frequencies.

Since $B = C = D = 0$, it follows that eq. (17.20) reduces to

$$\Phi_n(x) = A \sin \frac{n\pi x}{L}$$

or simply

$$\Phi_n(x) = \sin \frac{n\pi x}{L} \quad (17.28)$$

We note that in eq. (17.28) the constant A is absorbed by the other constants in the modal response given below by eq. (17.29).

From eq. (17.7) a modal shape or normal mode of vibration is given by

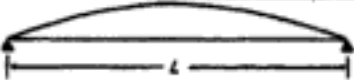




$$y_n(x, t) = \Phi_n(x) f_n(t)$$

or from eqs. (17.14) and (17.28) by

$$y_n(x, t) = \sin \frac{n\pi x}{L} [A_n \cos \omega_n t + B_n \sin \omega_n t] \quad (17.29)$$

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TABLE 17.1 Natural Frequencies and Normal Modes for Simply Supported Beams

Natural Frequencies			Normal Modes
$\omega_n = C_n \sqrt{\frac{EI}{mL^4}}$			$\Phi_n = \sin \frac{n\pi x}{L}$
n	C_n	I_n^*	Shape
1	π^2	$4/\pi$	
2	$4\pi^2$	0	
3	$9\pi^2$	$4/3\pi$	
4	$16\pi^2$	0	
5	$25\pi^2$	$4/5\pi$	

* $I_n = \int_0^L \Phi_n(x) dx / \int_0^L \Phi_n^2(x) dx$

17.3.2 Both Ends Free (Free Beam)

The boundary conditions for a beam with both ends free are as follows.

At $x = 0$,

$(0, t) = 0$ or $\Phi''(0) = 0$ M

$V(0, t) = 0$ or $\Phi'''(0) = 0$ (17.32)

At $x=L$,

$M(L, t) = 0$ or $\Phi''(L) = 0$

$V(L, t) = 0$ or $\Phi'''(L) = 0$ (17.33)

The substitutions of these conditions in eq. (17.20) yield

$\Phi''(0) = a^2(-B + D) = 0$

$\Phi'''(0) = a^3(-A + C) = 0$

and

$\Phi''(L) = a^2(-A \sin aL - B \cos aL + C \sinh aL + D \cosh aL) = 0$

$\Phi'''(L) = a^3(-A \cos aL + B \sin aL + C \cosh aL + D \sinh aL) = 0$

From the first two equations we obtain

$$D = B, \quad C = A \quad (17.34)$$

which, substituted into the last two equations, result in

$$\begin{aligned} (\sinh aL - \sin aL) A + (\cosh aL - \cos aL) B &= 0 \\ (\cosh aL - \cos aL) A + (\sinh aL + \sin aL) B &= 0 \end{aligned} \quad (17.35)$$

For nontrivial solution of eqs. (17.35), it is required that the determinant of the unknown coefficients A and B be equal to zero; hence

$$\begin{vmatrix} \sinh aL - \sin aL & \cosh aL - \cos aL \\ \cosh aL - \cos aL & \sinh aL + \sin aL \end{vmatrix} = 0 \quad (17.36)$$

The expansion of this determinant provides the frequency equation for the free beam, namely

$$\cos aL \cdot \cosh aL - 1 = 0 \quad (17.37)$$

The first five natural frequencies which are obtained by substituting the roots of eq. (17.37) into eq. (17.13) are presented in Table 17.2. The corresponding normal modes are obtained by letting $A = 1$ (normal modes are determined only to a relative magnitude), substituting in eqs. (17.35) the roots of a_n of eq. (17.37), solving one of these equations for B , and finally introducing into eq. (17.20) the constants C, D from eq. (17.34) together with B . performing these operations, we obtain

$$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n (\sinh a_n x + \sin a_n x) \quad (17.38)$$

where

$$\sigma_n = \frac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L} \quad (17.39)$$

17.3.3 Both Ends Fixed

The boundary conditions for a beam with both ends fixed are as follows:

At $x = 0$,

$$y(0, t) = 0 \quad \text{or} \quad \Phi(0) = 0$$

$$y'(0, t) = 0 \quad \text{or} \quad \Phi'(0) = 0 \quad (17.40)$$

At $x = L$,

$$y(L, t) = 0 \quad \text{or} \quad \Phi(L) = 0$$

$$y'(L, t) = 0 \quad \text{or} \quad \Phi'(L) = 0 \quad (17.41)$$

Table 17.2 Natural Frequencies and Normal Modes for Free Beams

Natural Frequencies				Normal Modes
				$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n (\sinh a_n x + \sin a_n x)$
$\omega_n = C_n \sqrt{\frac{EI}{\bar{m}L^4}}$		$\sigma_n = \frac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L}$		
n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.000000	0.2323	

* $I_n = \int_0^L \Phi_n(x) dx / \int_0^L \Phi_n^2(x) dx$

The use of the boundary conditions, eqs. (17.40), into eq. (17.20) gives

$$B + D = 0 \quad \text{and} \quad A - C = 0$$

while conditions, eqs. (17.41), yield the homogeneous system

$$\begin{aligned} (\cos AL - \cosh aL) B + (\sin aL - \sinh aL) A &= 0 \\ -(\sin aL + \sinh aL) B + (\cos aL - \cosh aL) A &= 0 \end{aligned} \tag{17.42}$$

Equating to zero the determinant of the coefficients of this system results in the frequency equation

$$\cos a_n L \cosh a_n L - 1 = 0 \tag{17.43}$$

From the first of eqs. (17.42), it follows that

$$A = - \frac{\cos aL - \cosh aL}{\sin aL - \sinh aL} B \tag{17.44}$$

where B is arbitrary. To each value of the natural frequency

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{mL^4}} \quad (17.45)$$

obtained by the substitution of the roots of eq. (17.43) into eq. (17.13), there corresponds a normal mode

$$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n (\sinh a_n x - \sin a_n x) \quad (17.46)$$

$$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L} \quad (17.47)$$

The first five natural frequencies calculated from eqs. (17.43) and (17.45) and the corresponding normal modes obtained from eq. (17.46) are presented in Table 17.3.

TABLE 17.3 Natural Frequencies and Normal Modes for Fixed Beams

Natural Frequencies				Normal Modes
$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n (\sinh a_n x - \sin a_n x)$				
$\omega_n = C_n \sqrt{\frac{EI}{mL^4}}$	$C_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$			
n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.000000	0.2323	

$$* I_n = \int_0^L \Phi_n(x) dx / \int_0^L \Phi_n^2(x) dx$$

17.3.4 One End Fixed and the Other End Free (Cantilever Beam)

At the fixed end ($x = 0$) of the cantilever beam, the deflection and the slope must be zero, and at the free end ($x = L$) the bending moment and the shear force must be zero. Hence the boundary conditions for this beam are as follows.

At $x = 0$,

$$y(0, t) = 0 \quad \text{or} \quad \Phi(0) = 0$$

$$y'(0, t) = 0 \quad \text{or} \quad \Phi'(0) = 0 \quad (17.48)$$

At $x = L$,

$$M(L, t) = 0 \quad \text{or} \quad \Phi''(L) = 0$$

$$V(L, t) = 0 \quad \text{or} \quad \Phi'''(L) = 0 \quad (17.49)$$

These boundary conditions when substituted into the shape equation (17.20) lead to the frequency equation

$$\cos a_n L \cdot \cosh a_n L + 1 = 0 \quad (17.50)$$

To each root of eq. (17.50) corresponds a natural frequency

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{mL}} \quad (17.51)$$

and a normal shape

$$\Phi_n(x) = (\cosh a_n x - \cos a_n x) - \sigma_n (\sinh a_n x - \sin a_n x) \quad (17.52)$$

where

$$\sigma_n = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L} \quad (17.53)$$

The first five natural frequencies and the corresponding mode shapes for cantilever beams are presented in Table 17.4.

17.3.5 One End Fixed and the Other Simply Supported

The boundary conditions for a beam with one end fixed and the other simply supported are as follows.

At $x = 0$,

$$y(0, t) = 0 \quad \text{or} \quad \Phi(0) = 0$$

$$y'(0, t) = 0 \quad \text{or} \quad \Phi'(0) = 0 \quad (17.54)$$

At $x = L$,

$$y(L, t) = 0 \quad \text{or} \quad \Phi(L) = 0$$

$$M(L, t) = 0 \quad \text{or} \quad \Phi''(L) = 0 \quad (17.55)$$

The substitution of these boundary conditions into the shape equation (17.20) results in the frequency equation

$$\tan a_n L - \tanh a_n L = 0 \quad (17.56)$$

TABLE 17.4 Natural Frequencies and Normal Modes for Cantilever Beams

Natural Frequencies				Normal Modes
$\Phi_n = (\cosh a_n x - \cos a_n x) - \sigma_n (\sinh a_n x - \sin a_n x)$				
$\omega_n = C_n \sqrt{\frac{EI}{\bar{m}L^4}}$		$\sigma = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$		
n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	3.5160	0.734096	0.7830	
2	22.0345	1.018466	0.4340	
3	61.6972	0.999225	0.2589	
4	120.0902	1.000033	0.0017	
5	199.8600	1.000000	0.0707	

$* I_n = \int_0^L \Phi_n(x) dx / \int_0^L \Phi_n^2(x) dx$

$\Phi_n(x) = (\cosh a_n x - \cos a_n x) + \sigma_n (\sinh a_n x - \sin a_n x)$ (17.58)

where

$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$ (17.59)

The first five natural frequencies for the fixed simply supported beam and corresponding mode shapes are presented in Table 17.5.

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$$F_n(t) = \int_0^L \Phi_n(x) p(x, t) dx \quad (17.72)$$

is the modal force.

The equation of motion for the n th normal mode, eq. (17.70), is completely analogous to the modal equation, eq. (12.9), for discrete systems. Modal damping could certainly be introduced by simply adding the damping term in eq. (17.70); hence we would obtain

$$M_n \ddot{z}_n(t) + C_n \dot{z}_n(t) + K_n z_n(t) = F_n(t) \quad (17.73)$$

which, upon dividing by M_n gives

$$\ddot{z}_n(t) + 2\xi_n \omega_n \dot{z}_n(t) + \omega_n^2 z_n(t) = \frac{F_n(t)}{M_n} \quad (17.74)$$

where $\xi_n = c_n/c_{n,cr}$ is the modal damping ratio and $K_n = M_n \omega_n^2$ is the modal stiffness. The total response is then obtained from eq. (17.65) as the superposition of the solution of the modal equation (17.74) for as many modes as desired. Though the summation in eq. (17.65) is over an infinite number of terms, in most structural problems only the first few modes have any significant contribution to the total response and in some cases the response is given essentially by the contribution of the first mode alone.

The modal equation (17.74) is completely general and applies to beams with any type of load distribution. If the loads are concentrated rather than distributed, the integral in eq. (17.72) merely becomes a summation having one term for each load. The computation of the integral in eqs. (17.71) and (17.72) becomes tedious except for the simply supported beam because the normal shapes are rather complicated functions. Values of the ratios of these integrals needed for problems with uniform distributed load are presented in the last columns of Tables 17.1 through 17.5 for some common types of beams.

Illustrative Example 17.1.

Consider in Fig. 17.3 a simply supported uniform beam subjected to a concentrated constant force suddenly applied at a section XI units from the left support. Determine the response using modal analysis. Neglect damping.

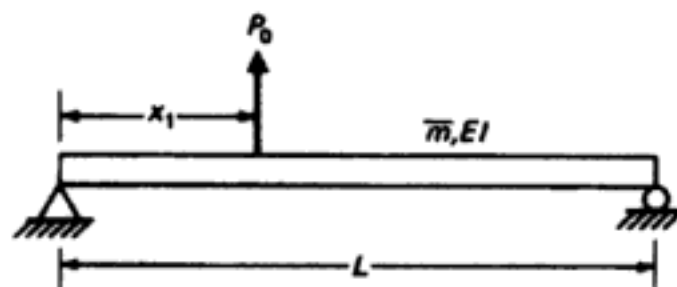


Fig. 17.3 Simply support beam subjected to a suddenly applied force.

Solution:

The modal shapes of a simply supported beam by eq. (17.28) are

$$\Phi_n = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3 \dots \quad (a)$$

and the modal force by eq.(17.72)

$$F_n(t) = \int_0^L \Phi_n(x) p(x, t) dx$$

In this problem $p(x, t) = P_0$ at $x = x_1$; otherwise, $p(x, t) = 0$. Hence

$$F_n(t) = P_0 \Phi_n(x_1)$$

or using eq. (a), we obtain

$$F_n(t) = P_0 \sin \frac{n\pi x_1}{L} \quad (b)$$

The modal mass by eq. (17.71) is

$$\begin{aligned} M_n &= \int_0^L \bar{m} \Phi^2(x) dx \\ &= \int_0^L \bar{m} \sin^2 \frac{n\pi x}{L} dx = \frac{\bar{m}L}{2} \end{aligned} \quad (c)$$

Substituting the modal force, eq. (b), and the modal mass, eq. (c), into the modal equation (17.70) results in

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{P_0 \sin(n\pi x_1 / L)}{\bar{m}L / 2} \quad (d)$$

For initial conditions of zero displacement and zero velocity, the solution of eq. (d) from eqs.. (4.5) is

$$z_n = (z_{st})_n (1 - \cos \omega_n t) \quad (e)$$

in which

$$(z_{st})_n = \frac{2P_0 \sin(n\pi x_1 / L)}{\omega_n^2 \bar{m}L} \quad (f)$$

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Illustrative Example 17.2.

Determine the maximum deflection at the midpoint of the fixed beam shown in Fig. 17.4 subjected to a harmonic load $p(x, t)=p_0 \sin 300t$ lb/in uniformly distributed along the span. Consider in the analysis the first three modes contributing to the response.

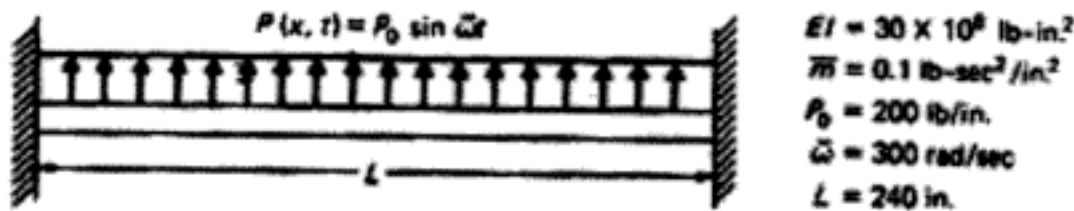


Fig. 17.4 Fixed beam with uniform harmonic load.

Solution:

The natural frequencies for uniform beams are given by eq.(17.13) as

$$\omega_n = C_n \sqrt{\frac{EI}{\overline{m}L^4}}$$

or, substituting numerical values for this example, we get

$$\omega_n = C_n \sqrt{\frac{30 \times 10^8}{0.1(240)^4}} \tag{a}$$

where the values of C_n are given for the first five modes in Table 17.3. The deflection of the beam is given by eq. (17.65) as

$$y(x, t) = \sum_{n=1}^{\infty} \Phi_n(x) z_n(t) \tag{b}$$

in which $\Phi_n(x)$ is the modal shape defined for a fixed beam by eq. (17.46) and $z_n(t)$ is the modal response.

The modal equation by eq. (17.70) (neglecting damping) may be written as

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{\int_0^L p(x, t) \phi_n(x) dx}{\int_0^L \overline{m} \phi_n^2(x) dx}$$

Then, substituting numerical values to this example, we obtain

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{200 \int_0^L \phi_n(x) dx}{0.1 \int_0^L \phi_n^2(x) dx}$$

or

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = 2000 I_n \sin 300t \quad (c)$$

TABLE 17.6 Modal Response at Midspan for the Beam in Fig. 17.4

Mode	$\omega_n \left(\frac{\text{rad}}{\text{sec}} \right)$	$a_n L$	I_n	$z_n = \frac{2000 I_n}{\omega_n^2 - \bar{\omega}^2} (\text{in})$	$\Phi_n \left(x = \frac{L}{2} \right)$
1	67.28	4.730	0.8380	-0.0194	1.588
2	185.45	7.853	0	0	0
3	363.56	10.996	0.3640	0.0173	-1.410
4	600.98	14.137	0	0	0
5	897.76	17.279	0.2323	0.00065	1.414

in which

$$I_n = \frac{\int_0^L \phi_n(x) dx}{\int_0^L \phi_n^2(x) dx}$$

is given for the first five modes in Table 17.3. The modal steady-state response is

$$z_n(t) = \frac{2000 I_n}{\omega_n^2 - (300)^2} \sin 300t \quad (d)$$

The numerical calculations are conveniently presented in Table 17.6.

The deflections at midspan of the beam are then calculated from eq. (b) and values in Table 17.6 as

$$y\left(\frac{L}{2}, t\right) = [(1.588)(-0.0194) + (-1.410)(0.0173) + (1.414)(0.00065)] \sin 300t$$

$$y\left(\frac{L}{2}, t\right) = -0.0541 \sin 300t \text{ (in)}$$

17.6 Dynamic Stresses in Beams

To determine stresses in beams, we apply the following well-known relationships for bending moment M and shear force V , namely

$$M = EI \frac{\partial^2 y}{\partial x^2}$$

$$V = \frac{\partial M}{\partial x} = EI \frac{\partial^3 y}{\partial x^3}$$

Therefore, the calculation of the bending moment or the shear force requires only differentiation of the deflection function $y = y(x, t)$ with respect to x . For example, in the case of the simple supported beam with a concentrated load suddenly applied at its center, differentiation of the deflection function, eq. (17.75), gives

$$M = -\frac{2\pi^2 P_0 EI}{\bar{m} L^3} \sum_n \left[\frac{n^2}{\omega_n^2} \sin \frac{n\pi}{2} (1 - \cos \omega_n t) \sin \frac{n\pi x}{L} \right] \quad (17.76)$$

$$V = -\frac{2\pi^3 P_0 EI}{\bar{m} L^4} \sum_n \left[\frac{n^3}{\omega_n^2} \sin \frac{n\pi}{2} (1 - \cos \omega_n t) \sin \frac{n\pi x}{L} \right] \quad (17.77)$$

We note that the higher modes are increasingly more important for moments than for deflections and even more so for shear force, as indicated by the factors 1, n^2 , and n^3 , respectively, in eqs. (17.75), (17.76), and (17.77).

To illustrate, we compare the amplitudes for the first and third modes at their maximum values. Noting that ω_n^2 is proportional to n^4 [eq. (17.27)], we obtain from eqs. (17.75), (17.76), and (17.77) the following ratios:

$$\frac{y_1}{y_3} = 3^4 = 81$$

$$\frac{M_1}{M_3} = 3^2 = 9$$

$$\frac{V_1}{V_3} = 3$$

This tendency in which higher modes have increasing importance in moment and shear calculation is generally true of beam response.

In those cases in which the first mode dominates the response, it is possible to obtain approximate deflections and stresses from static values of these quantities amplified by the dynamic load factor. For example, the maximum deflection of a simple supported beam with a concentrated force at midspan may be closely approximated by

$$y(x = \frac{L}{2}) = \frac{P_0 L^3}{48EI} (1 - \cos \omega_1 t)$$

If we consider only the first mode, the corresponding value given by eq. (17.75) is

$$y(x = \frac{L}{2}) = \frac{2P_0}{\bar{m}L\omega_1^2} (1 - \cos \omega_1 t) b$$

Since, by eq. (17.12), $\omega_1^2 = \pi^2 EI / \bar{m} L^4$, it follows that

$$\begin{aligned} y(x = \frac{L}{2}) &= \frac{2P_0 L^3}{\pi^4 EI} (1 - \cos \omega_1 t) b \\ &= \frac{P_0 L^3}{48.7 EI} (1 - \cos \omega_1 t) \end{aligned}$$

The close agreement between these two computations is due to the fact that static deflections can also be expressed in terms of modal components, and for a beam supporting a concentrated load at mid-span the first mode dominates both static and dynamic response.

17.7 Summary

The dynamic analysis of single-span beams with distributed properties (mass and elasticity) and subjected to flexural loading was presented in this chapter. The extension of this analysis to multi-span or continuous beams and other structures, though possible, becomes increasingly complex and impractical. The results obtained from these single-span beams are particularly important in evaluating approximate methods based on discrete models, as those presented in preceding chapters. From such evaluation, it has been found that the stiffness method of dynamic analysis in conjunction with the consistent mass formulation provides in general satisfactory results even with a rather coarse discretization of the structure.

The natural frequencies and corresponding normal modes of single-span beams with different supports are determined by solving the differential equation of motion and imposing the corresponding boundary conditions. The normal modes satisfy the orthogonality condition between any two modes m and n , namely,

$$\int_0^L \phi_m(x) \phi_n(x) \bar{m} dx = 0 \quad (m \neq n)$$

The response of a continuous system may be determined as the superposition of modal contributions, that is

$$y(x, t) = \sum_n \phi_n(x) z_n(t)$$

where $z_n(t)$ is the solution of n modal equation

Illustrative Example 17.2

in which

$$F_n(t) = \int_0^L \phi_n(x) p(x, t) dx$$

and

$$M_n = \int_0^L \bar{m} \phi_n^2(x) dx$$

The bending moment M and the shear for V at any section of a beam are calculated from the well-known relations

$$M = EI \frac{\partial^2 y}{\partial x^2}$$

$$V = EI \frac{\partial^3 y}{\partial x^3}$$

17.8 Problems**Problem 17.1**

Determine the first three natural frequencies and corresponding nodal shapes of a simply supported reinforced concrete beam having a cross section 10 in wide by 24 in deep with a span of 36 ft. Assume the flexural stiffness of the beam, $EI = 3.5 \times 10^9 \text{ lb}\cdot\text{in}^2$ and weight per unit volume $W = 150 \text{ lb/ft}^3$. (Neglect shear distortion and rotary inertia.)

Problem 17.2

Solve Problem 17.1 for the beam with its two ends fixed.

Problem 17.3

Solve Problem 17.1 for the beam with one end fixed and the other simply supported.

Problem 17.4

Determine the maximum deflection at the center of the simply supported beam of Problem 17.1 when a constant force of 2000 lb is suddenly applied at 9 ft from the left support.

Problem 17.5

A simply supported beam is prismatic and has the following properties: $\bar{m} = 0.3 \text{ lb}\cdot\text{sec}^2/\text{in}$ per inch of span, $EI = 10^6 \text{ lb}\cdot\text{in}^2$, and $L = 150 \text{ in}$. The beam is subjected to a uniform distributed static load p_0 which is suddenly removed. Write the series expression for the resulting free vibration and determine the amplitude of the first mode in terms of p_0 .

Problem 17.6

The beam of Problem 17.5 is acted upon by a concentrated force given by $P(t) = 1000 \sin 500t$ lb applied at its midspan. Determine the amplitude of the steady-state motion at a quarter point from the left support in each of the first two modes. Neglect dampin

Problem 17.7

Solve Problem 17.6 assuming 10% of critical damping in each mode. Also determine the steady-state motion at the quarter point considering the first n modes.

Problem 17.8

The cantilever beam shown in Fig. P17.8 is prismatic and has the following properties: $\bar{m} = 0.5 \text{ lb} \cdot \text{sec}^2 / \text{in}$ per inch of span, $E = 30 \times 10^6$ psi, $L = 100$ in, and $I = 120 \text{ in}^4$. Considering only the first mode, compute the maximum deflection and the maximum dynamic bending moment in the beam due to the load time function of Fig. P17.8(b). (Chart in Fig. 4.5 may be used.)

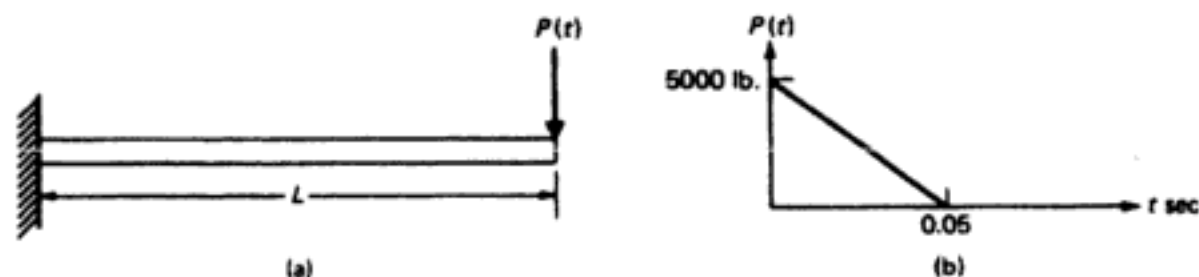


Fig. P17.8.

Problem 17.9

A prismatic simply supported beam of the following properties: $L = 120$ in, $EI = 10^7 \text{ lb} \cdot \text{in}^2$, and $\bar{m} = 0.5 \text{ lb} \cdot \text{sec}^2 / \text{in}$ per inch of span is loaded as shown in Fig. P17.9. Write the series expression for the deflection at the midsection of the beam.

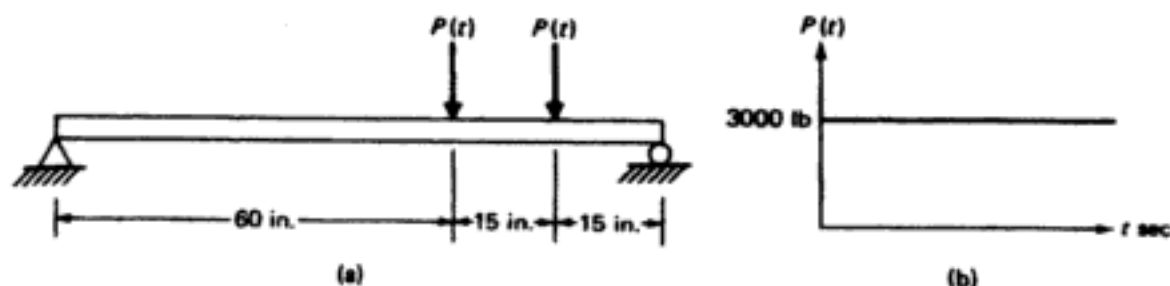


Fig. P17.9.

Problem 17.10

Assuming that the forces on the beam of Problem 17.9 are applied for only a time duration $t_d = 0.1$ sec. and considering only the first mode, determine the maximum deflection at each of the load points of the beam. (Chart in Fig. 4.4 may be used.)

Problem 17.11

A prismatic beam with its two ends fixed has the following properties: $L = 180$ in, $EI = 30 \times 10^8$ lb*in², $\overline{m} = 1$ lb*sec²/in per inch of span. The beam is acted upon by a uniformly distributed impulsive force $p(x, t) = 2000 \sin 400t$ lb during a time interval equal to half of the period of the sinusoidal load function ($t_d = \pi/400$ sec). Considering only the first mode, determine the maximum deflection at the midsection. (Chart in Fig. 5.3 may be used.)

Problem 17.12

Solve Problem 17.11 considering the first two modes.

18 Discretization of Continuous Systems

The modal superposition method of analysis was applied in the preceding chapter to some simple structures having distributed properties. The determination of the response by this method requires the evaluation of several natural frequencies and corresponding mode shapes. The calculation of these dynamic properties is rather laborious, as we have seen, even for simple structures such as one-span uniform beams. The problem becomes increasingly more complicated and unmanageable as this method of solution is applied to more complex structures. However, the analysis of such structures becomes relatively simple if for each segment or element of the structure the properties are expressed in terms of dynamic coefficients much in the same manner as done previously when static deflection functions were used as an approximation to dynamic deflections in determining stiffness, mass, and other coefficients.

In this chapter the dynamic coefficients relating harmonic forces and displacements at the nodal coordinates of a beam segment are obtained from dynamic deflection functions. These coefficients can then be used to assemble the dynamic matrix for the whole structure by the direct method as shown in the preceding chapters for assembling the system stiffness and mass matrices. Also, in the present chapter, the mathematical relationship between the dynamic coefficients based on dynamic displacement functions and the coefficients of the stiffness and consistent mass matrices derived from static displacement functions is established.

18.1 Dynamic Matrix for Flexural Effects

As in the case of static influence coefficients (stiffness coefficients, for example), the dynamic influence coefficients also relate forces and displacements at the nodal coordinates of a beam element. The difference between the dynamic and static coefficients is that the dynamic coefficients refer to nodal forces and displacements that vary harmonically while the static coefficients relate static forces and displacements at the nodal coordinates. The dynamic influence coefficient S_{ij} is then defined as the harmonic force of frequency ω at nodal coordinates i , due to a harmonic displacement of unit amplitude and of the same frequency at nodal coordinate j .

To determine the expressions for the various dynamic coefficients for a uniform beam element as shown in Fig. 18.1, we refer to the differential equation of motion, eq. (17.5), which in the absence of external loads in the span, that is, $p(x, t) = 0$, is

$$EI \frac{\partial^4 y}{\partial x^4} + \bar{m} \frac{\partial^2 y}{\partial t^2} = 0 \quad (18.1)$$

For harmonic boundary displacements of frequency ω , we introduce in eq. (18.1) the trial solution

$$y(x, t) = \Phi(x) \sin \bar{\omega} t \quad (18.2)$$

Substitution of eq. (18.2) into eq. (18.1) yields

$$\Phi^{IV}(x) - \bar{a}^4 \Phi(x) = 0 \quad (18.3)$$

where

$$\bar{a}^4 = \frac{\bar{m} \bar{\omega}^2}{EI} \quad (18.4)$$

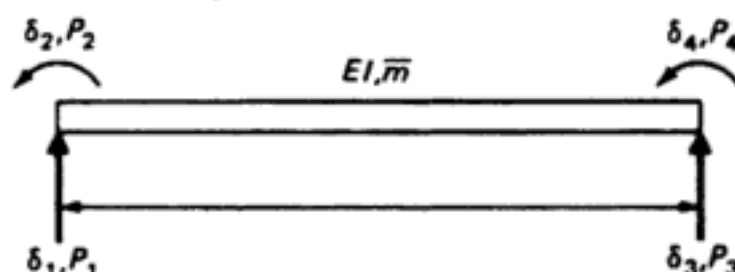


Fig. 18.1 Nodal coordinates of a flexural beam segment.

We note that eq. (18.3) is equivalent to eq. (17.10), which is the differential equation for the shape function of a beam segment in free vibration. The difference between these two equations is that eq. (18.3) is a function of the parameter \bar{a} which, in turn, is a function of the forcing frequency ω , while " a " in (17.10) depends on the natural frequency ω . The solution of eq. (18.3) is of the same form as the solution of eq. (17.10). Thus by analogy with eq. (17.20), we can write

$$\Phi(x) = C_1 \sin \bar{a}x + C_2 \cos \bar{a}x + C_3 \sinh \bar{a}x + C_4 \cosh \bar{a}x \quad (18.5)$$

Now, to obtain the dynamic coefficient for the beam segment, boundary conditions indicated by eqs. (18.6) and (18.7) are imposed:

$$\begin{aligned}\Phi(0) &= \delta_1, & \Phi(L) &= \delta_3 \\ \Phi'(0) &= \delta_2, & \Phi'(L) &= \delta_4\end{aligned}\quad (18.6)$$

Also

$$\begin{aligned}\Phi''(0) &= \frac{P_1}{EI}, & \Phi''(L) &= -\frac{P_3}{EI} \\ \Phi'''(0) &= -\frac{P_2}{EI}, & \Phi'''(L) &= \frac{P_4}{EI}\end{aligned}\quad (18.7)$$

In eqs. (18.6), δ_1 , δ_2 , δ_3 , and δ_4 are amplitudes of linear and angular harmonic displacements at the nodal coordinates while in eqs. (18.7) P_1 , P_2 , P_3 , and P_4 are the corresponding harmonic forces and moments as shown in Fig. 18.1. The substitution of the boundary conditions, eqs. (18.6) and (18.7), into eq. (18.5) results in

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \bar{a} & 0 & \bar{a} & 0 \\ s & c & S & C \\ \bar{a}c & -\bar{a}s & \bar{a}C & \bar{a}S \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}\quad (18.8)$$

and

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = EI \begin{bmatrix} -\bar{a}^3 & 0 & \bar{a}^3 & 0 \\ 0 & \bar{a}^2 & 0 & -\bar{a}^2 \\ \bar{a}^3c & -\bar{a}^3s & -\bar{a}^3C & -\bar{a}^3S \\ \bar{a}^2s & -\bar{a}^2c & \bar{a}^2S & \bar{a}^2C \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}\quad (18.9)$$

in which

$$\begin{aligned}s &= \sin \bar{a}L, & S &= \sinh \bar{a}L \\ c &= \cos \bar{a}L, & C &= \cosh \bar{a}L\end{aligned}\quad (18.10)$$

Next, eq. (18.8) is solved for the constants of integration C_1 , C_2 , C_3 , and C_4 , which are subsequently substituted into eq. (18.9). We thus obtain the dynamic matrix relating harmonic displacements and harmonic forces at the nodal coordinate of the beam element, namely

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = B \begin{bmatrix} \bar{a}^2(cS+sC) & \text{Symmetric} \\ \bar{a}sS & sC-cS \\ -\bar{a}^2(s+S) & \bar{a}(c-C) & \bar{a}^2(cS+sC) \\ \bar{a}(C-c) & S-s & -\bar{a}sS & sC-cS \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}\quad (18.11)$$

where

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$$\frac{\partial^2 u}{\partial x^2} - \frac{\bar{m}}{AE} \frac{\partial^2 u}{\partial t^2} = 0 \quad (18.17)$$

A solution of eq. (18.17) of the form

$$u(x, t) = U(x) \sin \bar{\omega} t \quad (18.18)$$

will result in a harmonic motion of amplitude

$$U(x) = C_1 \sin bx + C_2 \cos bx \quad (18.19)$$

where

$$b = \sqrt{\frac{\bar{m} \bar{\omega}^2}{AE}} \quad (18.20)$$

and C_1 and C_2 are constants of integration.

To obtain the dynamic matrix for the axially vibrating beam segment, boundary conditions indicated by eqs. (18.21) and (18.22) are imposed, namely.

$$U(0) = \delta_1, \quad U(L) = \delta_2 \quad (18.21)$$

$$U'(0) = -\frac{P_1}{AE}, \quad U'(L) = \frac{P_2}{AE} \quad (18.22)$$

where δ_1 and δ_2 are the displacements and P_1 and P_2 are the forces at the nodal coordinates of the beam segment as shown in Fig. 18.2.

Substitution of the boundary conditions, eqs. (18.21) and (18.22), into eq. (18.19) results in

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin bL & \cos bL \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad (18.23)$$

and

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = AEb \begin{bmatrix} -1 & 0 \\ \cos bL & -\sin bL \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad (18.24)$$

Then, solving eq. (18.23) for the constants of integration, we obtain

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\cot bL & \operatorname{cosec} bL \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \quad (18.25)$$

subject to the condition

$$\sin bL \neq 0 \quad (18.26)$$

Finally, the substitution of eq. (18.25) into eq. (18.24) results in eq. (18.27) relating harmonic forces and displacement at the nodal coordinates through the dynamic matrix for an axially vibrating beam segment. Thus we have

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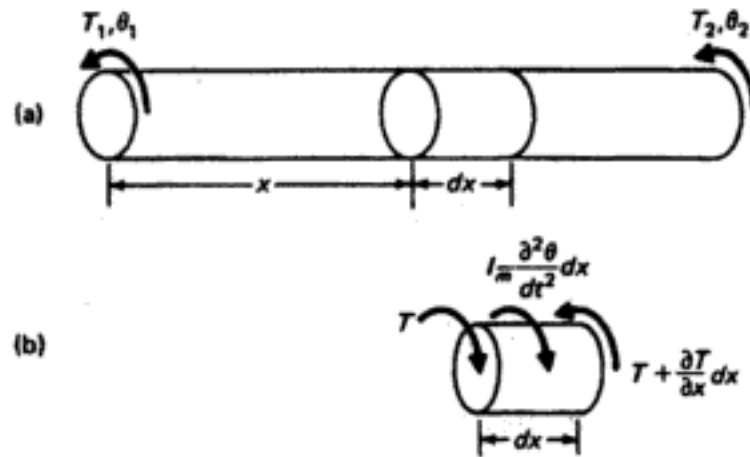


Fig. 18.3 Torsional effects on a beam. (a) Nodal torsional coordinates. (b) Moments acting on a differential element.

in which I_0 is the polar moment of inertia of the cross-sectional area A .

We seek a solution of eq. (18.30) in the form

$$\theta(x, t) = \theta(x) \sin \bar{\omega} t$$

which, upon substitution into eq. (18.30), results in a harmonic torsional motion of amplitude

$$\theta(x) = C_1 \sin cx + C_2 \cos cx \quad (18.32)$$

in which

$$c = \sqrt{\frac{I_m \bar{\omega}^2}{J_T G}} \quad (18.33)$$

For a circular section, the torsional constant J_T is equal to the polar moment of inertia I_0 . Thus eq. (18.33) reduces to

$$c = \sqrt{\frac{m \bar{\omega}^2}{AG}} \quad (18.34)$$

since $I_m = I_0 m / A$ as indicated by eq. (18.31).

We note that eq. (18.30) for torsional vibration is analogous to eq. (18.17) for axial vibration of beam segments. It follows that by analogy to eq. (18.27) we can write the dynamic relation between torsional moments and rotations in a beam segment. Hence

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = J_T G c \begin{bmatrix} \cot cL & -\operatorname{cosec} cL \\ -\operatorname{cosec} cL & \cot cL \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad (18.35)$$

18.4 Beam Flexure Including Axial-Force Effect

When a beam is subjected to a force along its longitudinal axis in addition to lateral loading, the dynamic equilibrium equation for a differential element of the beam is affected by the presence of this force. Consider the beam shown in Fig. 18.4 in which the axial force is assumed to remain constant during flexure with respect to both magnitude and direction. The dynamic equilibrium for a differential element dx of the beam [Fig. 18.4(b)] is established by equating to zero both the sum of the forces and the sum of the moments.

Summing forces in the y direction, we obtain

$$V + p(x,t) dx - \left(V + \frac{\partial V}{\partial x} dx \right) - (\bar{m} dx) \frac{\partial^2 y}{\partial t^2} = 0 \quad (18.36)$$

which, upon reduction, yields

$$\frac{\partial V}{\partial x} + \bar{m} \frac{\partial^2 y}{\partial t^2} = p(x,t) \quad (18.37)$$

The summation of moments about point 0 gives

$$M + V dx - \left(M + \frac{\partial M}{\partial x} dx \right) + \frac{1}{2} \left(p(x,t) - \bar{m} \frac{\partial^2 y}{\partial t^2} \right) dx^2 - N \frac{\partial y}{\partial x} dx = 0 \quad (18.38)$$

Discarding higher order terms, we obtain for the shear force the expression

$$V = N \frac{\partial y}{\partial x} + \frac{\partial M}{\partial x} \quad (18.39)$$

Then using the familiar relationship from bending theory,

$$M = EI \frac{\partial^2 y}{\partial x^2} \quad (18.40)$$

and combining eqs. (18.37), (18.39), and (18.40), we obtain the equation of motion of a beam segment including the effect of the axial forces, that is,

$$EI \frac{\partial^4 y}{\partial x^4} + N \frac{\partial^2 y}{\partial x^2} + \bar{m} \frac{\partial^2 y}{\partial t^2} = p(x,t) \quad (18.41)$$

A comparison of eqs. (18.41) and (17.5) reveals that the presence of the axial force gives rise to an additional transverse force acting on the beam. As indicated previously in Section 17.1, in the derivation of eq. (18.41) it has been assumed that the deflections are small and that the deflections due to shear forces or rotary inertia are negligible.

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$$p_2 = \sqrt{\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \beta}} \quad (18.46)$$

$$\alpha = \frac{N}{EI} \quad (18.47)$$

$$\beta = \frac{\bar{m} \bar{\omega}^2}{EI} \quad (18.48)$$

To obtain the dynamic matrix (which in this case includes the effect of axial forces) for the transverse vibration of the beam element, the boundary conditions, eqs. (18.49), are imposed, namely

$$\begin{aligned} \Phi(0) &= \delta_1, & \Phi(L) &= \delta_3 \\ \frac{d\Phi(0)}{dx} &= \delta_2, & \frac{d\Phi(L)}{dx} &= \delta_4 \\ \frac{d^3\Phi(0)}{dx^3} &= \frac{P_1}{EI} - \frac{N}{EI} \delta_2, & \frac{d^3\Phi(L)}{dx^3} &= -\frac{P_3}{EI} - \frac{N}{EI} \delta_4 \\ \frac{d^3\Phi(0)}{dx^3} &= -\frac{P_2}{EI}, & \frac{d^2\Phi(L)}{dx^2} &= \frac{P_4}{EI} \end{aligned} \quad (18.49)$$

In eqs. (18.49) δ_1 , δ_3 and δ_2 , δ_4 are, respectively, the transverse and angular displacements at the ends of the beam, while P_1 , P_3 and P_2 , P_4 are corresponding forces and moments at these nodal coordinates. The substitution into eq. (18.45) of the boundary conditions given by eqs. (18.49) results in a system of eight algebraic equations which upon elimination of the four constants of integration A , B , C , and D yields the dynamic matrix (including the effect of axial forces) relating harmonic forces and displacements at the nodal coordinates of a beam segment. The final result is

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} S_{11} & & & \\ S_{21} & S_{22} & & \\ S_{31} & S_{32} & S_{33} & \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \quad (18.50)$$

Symmetric

where

$$\begin{aligned} S_{11} &= S_{33} = B[(p_1^2 p_2^3 + p_1^4 p_2) c S + (p_1 p_2^4 + p_1^3 p_2^2) s C] \\ S_{21} &= -S_{43} = B[(p_1 p_2^3 - p_1^3 p_2) + (p_1^3 p_2 - p_1 p_2^3) c C + 2 p_2^2 p_2^2 s S] \\ S_{22} &= S_{44} = B[(p_2^2 p_1 + p_1^3) s C - (p_2^3 + p_1^2 p_2) c S] \\ S_{41} &= -S_{32} = B[(p_1 p_2^3 + p_1^3 p_2)(C - c)] \\ S_{31} &= B[(-p_1^2 p_2^3 - p_1^4 p_2) S - (p_1^3 p_2^2 + p_1 p_2^4) s] \\ S_{42} &= B[(p_1^2 p_2 + p_2^3) S - (p_1 p_2^2 + p_1^3) s] \end{aligned} \quad (18.51)$$

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where $x = \bar{a}L$. Substitution of these series equations in the dynamic coefficient, eq. (18.54), yields

$$S_{21} = \frac{\bar{a}^2 EI \sin \bar{a}L \sinh \bar{a}L}{1 - \cos \bar{a}L \cosh \bar{a}L} = \frac{6EI}{L^2} - \frac{11\bar{m}L^2\bar{\omega}^2}{210} - \frac{223\bar{m}^2L^6\bar{\omega}^4}{2,910,600EI} \quad (18.55)$$

The first term on the right-hand side of eq. (18.55) is the stiffness coefficient k_{21} in the stiffness matrix, eq. (10.20), and the second term, the consistent mass coefficient m_{21} in the mass matrix, eq. (10.34). The series expansion, eq. (18.55), is convergent in the positive real field for

$$0 < \bar{a}L < 4.73 \quad (18.56)$$

or from eq. (18.4)

$$0 < \bar{\omega} < (4.73)^2 \sqrt{\frac{EI}{\bar{m}L^4}} \quad (18.57)$$

In eq. (18.56) the numerical value 4.73 is an approximation of the closest singularity to the origin of the functions in the quotient expanded in eq. (18.54).

The series expansions for all the coefficients in the dynamic matrix, eq. (18.11), are obtained by the method explained in obtaining the expansion of the coefficient S_{21} . These series expansions are:

$$\begin{aligned} S_{33} = S_{11} &= \frac{12EI}{L^3} - \frac{13L\bar{m}\bar{\omega}^2}{35} - \frac{59L^5\bar{m}^2\bar{\omega}^4}{161,700EI} - \dots \\ S_{21} = -S_{43} &= \frac{6EI}{L^2} - \frac{11L^2\bar{m}\bar{\omega}^2}{210} - \frac{223L^6\bar{m}^2\bar{\omega}^4}{2,910,600EI} - \dots \\ S_{41} = -S_{32} &= \frac{6EI}{L^2} - \frac{13L^2\bar{m}\bar{\omega}^2}{420} + \frac{1681L^6\bar{m}^2\bar{\omega}^4}{23,284,800EI} - \dots \\ S_{22} = -S_{44} &= \frac{4EI}{L} - \frac{L^3\bar{m}\bar{\omega}^2}{105} - \frac{71L^7\bar{m}^2\bar{\omega}^4}{4,365,800EI} - \dots \\ S_{31} &= -\frac{12EI}{L^3} - \frac{9L\bar{m}\bar{\omega}^2}{70} - \frac{1279L^5\bar{m}^2\bar{\omega}^4}{3,880,800EI} - \dots \\ S_{42} &= \frac{2EI}{L} - \frac{L^3\bar{m}\bar{\omega}^2}{140} - \frac{1097L^7\bar{m}^2\bar{\omega}^4}{69,854,400EI} - \dots \end{aligned} \quad (18.58)$$

18.6 Power Series Expansion of the Dynamic Matrix for Axial and for Torsional Effects

Proceeding in a manner entirely analogous to expansion of the dynamic coefficients for flexural effects, we can also expand the dynamic coefficients for axial and for torsional effects. The Taylor's series expansions, up to three terms, of the coefficients of the dynamic matrix in eq. (18.27) (axial effects) are

¹ Knopp, K., *Theory and Application of Infinite Series*, Blackie, London, 1963.

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$$[A_1] = \frac{\bar{m} L^3}{EI} \begin{bmatrix} \frac{1}{3150} & & & & \\ \frac{L}{1360} & \frac{L^2}{3150} & & & \\ -\frac{1}{3150} & \frac{L}{1680} & \frac{1}{3150} & & \\ -\frac{L}{1680} & \frac{L^2}{3600} & -\frac{L}{1260} & \frac{L^2}{3150} & \\ \frac{1}{1260} & -\frac{L}{3150} & -\frac{L^2}{3600} & \frac{1}{1360} & \end{bmatrix} \quad \text{Symmetric}$$

The second-order geometrical matrix:

$$[G_1] = \frac{1}{EI} \begin{bmatrix} \frac{L}{700} & & & & \\ \frac{L^2}{1400} & \frac{11L^2}{6300} & & & \\ -\frac{L}{700} & -\frac{L^2}{1400} & \frac{L}{700} & & \\ -\frac{L^2}{1400} & -\frac{13L^2}{12600} & -\frac{L^2}{1400} & \frac{11L^3}{6300} & \\ \frac{L}{700} & \frac{L^2}{1400} & \frac{L}{700} & -\frac{11L^2}{6300} & \end{bmatrix} \quad \text{Symmetric}$$

The second-order mass matrix:

$$[M_1] = \frac{\bar{m}^2 L^3}{1000 EI} \begin{bmatrix} \frac{59}{161.7} & & & & \\ \frac{223L}{2910.6} & \frac{71L^2}{4365.9} & & & \\ \frac{1279}{3880.8} & \frac{1681L}{23284.8} & \frac{59}{161.7} & & \\ -\frac{1681L}{23284.8} & -\frac{1097L^2}{69854.4} & -\frac{223L}{2910.6} & \frac{71L^2}{4365.9} & \\ \frac{1279}{3880.8} & \frac{1681L}{23284.8} & \frac{59}{161.7} & -\frac{71L^2}{4365.9} & \end{bmatrix} \quad \text{Symmetric}$$

18.8 Summary

The dynamic coefficients relating harmonic forces and displacements at the nodal coordinates of a beam segment were obtained from dynamic deflection equations. These coefficients can then be used in assembling the dynamic matrix for the entire structure by the same procedure (direct method) employed in assembling the stiffness and mass matrices for discrete systems.

In this chapter it has been demonstrated that the stiffness, consistent mass, and other influence coefficients may be obtained by expanding the dynamic influence coefficients in Taylor's series. This mathematical approach also provides higher order influence coefficients and the determination of the radius of convergence of the series expansion.

PART V

Special Topics:

Fourier Analysis
Absolute Damping
Generalized Coordinates

19 Fourier Analysis and Response in the Frequency Domain

This chapter presents the application of Fourier series to determine: (1) the response of a system to periodic forces, and (2) the response of a system to nonperiodic forces in the frequency domain as an alternate approach to the usual analysis in the time domain. In either case, the calculations require the evaluation of integrals that, except for some relatively simple loading functions, employ numerical methods for their computation. Thus, in general, to make practical use of the Fourier method, it is necessary to replace the integrations with finite sums.

19.1 Fourier Analysis

The subject of Fourier series and Fourier analysis has extensive ramifications in its application to many fields of science and mathematics. We begin by considering a single-degree-of-freedom system under the action of a periodic loading, that is, a forcing function that repeats itself at equal intervals of time, T (the period of the function). Fourier has shown that a periodic function may be expressed as the summation of an infinite number of sine and cosine terms. Such a sum is known as a Fourier series.

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Similarly, the response to any cosine term is

$$u_n(t) = \frac{a_n / k}{1 - r_n^2} \cos n \omega t \quad (19.5)$$

The total response of an undamped, single-degree-of-freedom system may then be expressed as the superposition of the responses to all the force terms of the series, including the response a_0/k (steady-state response) to the constant force a_0 . Hence we have

$$u(t) = \frac{a_0}{k} + \sum \frac{1}{1 - r_n^2} \left(\frac{a_n}{k} \cos n \omega t + \frac{b_n}{k} \sin n \omega t \right) \quad (19.6)$$

When the damping in the system is considered, the steady-state response for the general sine term of the series is given from eq. (3.20) as

$$u_n(t) = \frac{b_n / k \sin(n \omega t - \theta)}{\sqrt{(1 - r_n^2)^2 + (2r_n\xi)^2}} \quad (19.7)$$

or

$$u_n(t) = \frac{b_n}{k} \cdot \frac{\sin n \omega t \cos \theta - \cos n \omega t \sin \theta}{\sqrt{(1 - r_n^2)^2 + (2r_n\xi)^2}}$$

The substitution of $\sin \theta$ and $\cos \theta$ from eq. (3.21) gives

$$u_s(t) = \frac{b_n}{k} \frac{(1 - r_n^2) \sin n \omega t + 2r_n\xi \cos n \omega t}{(1 - r_n^2)^2 + (2r_n\xi)^2} \quad (19.8)$$

Similarly, for a cosine term of the series, we obtain

$$u_s(t) = \frac{a_n}{k} \frac{(1 - r_n^2) \sin n \omega t + 2r_n\xi \cos n \omega t}{(1 - r_n^2)^2 + (2r_n\xi)^2} \quad (19.9)$$

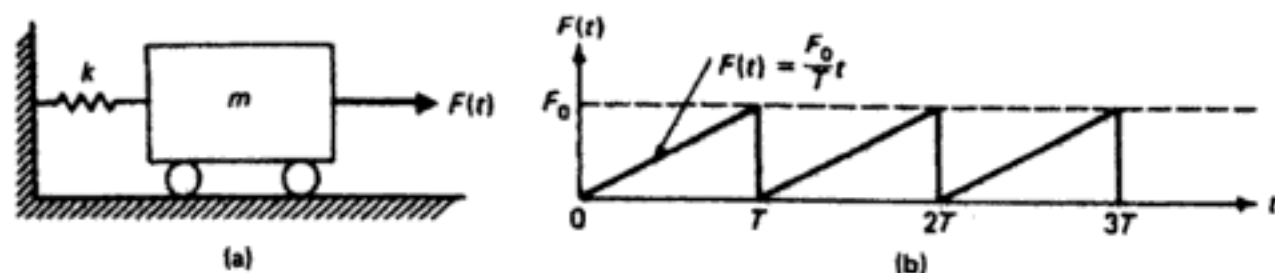


Fig. 19.2 Undamped oscillator acted upon by a periodic force.

Finally, the total response is then given by the superposition of the terms expressed by eqs. (19.8) and (19.9) in addition to the response to the constant term of the series. Therefore, the total response of a damped single-degree-of freedom system may be expressed as

$$u(t) = \frac{a_0}{k} + \frac{1}{k} \sum_{n=1}^{\infty} \left\{ \frac{a_n 2r_n \xi + b_n (1 - r_n^2)}{(1 - r_n^2)^2 + (2r_n \xi)^2} \sin n\omega t + \frac{a_n (1 - r_n^2) - b_n 2r_n \xi}{(1 - r_n^2)^2 + (2r_n \xi)^2} \cos n\omega t \right\} \quad (19.10)$$

Illustrative Example 19.1

As an application of the use of Fourier series in determining the response of a system to a periodic loading, consider the undamped simple oscillator in Fig. 19.2(a) which is acted upon by the periodic force shown in Fig. 19.2(b).

Solution:

The first step is to determine the Fourier series expansion of $F(t)$. The corresponding coefficients are determined from eqs. (19.3) as follows:

$$a_0 = \frac{1}{T} \int_0^T \frac{F_0}{T} t \, dt = \frac{F_0}{2}$$

$$a_n = \frac{2}{T} \int_0^T \frac{F_0}{T} t \cos n\omega t \, dt = 0$$

$$b_n = \frac{2}{T} \int_0^T \frac{F_0}{T} t \sin n\omega t \, dt = -\frac{F_0}{n\pi}$$

The response of the undamped system is then given from eq. (19.6) as

$$U(t) = \frac{F_0}{2k} - \sum_{n=1}^{\infty} \frac{F_0 \sin n\omega t}{n\pi k (1 - r_n^2)}$$

or in expanded form as

$$U(t) = \frac{F_0}{2k} - \frac{F_0 \sin \omega t}{\pi k (1 - r_1^2)} - \frac{F_0 \sin 2\omega t}{\pi k (1 - 4r_1^2)} - \frac{F_0 \sin 3\omega t}{\pi k (1 - 9r_1^2)} - \dots$$

where

$$r_1 = \varpi / \omega, \omega = \sqrt{k/m}, \text{ and } \varpi = 2\pi / T$$

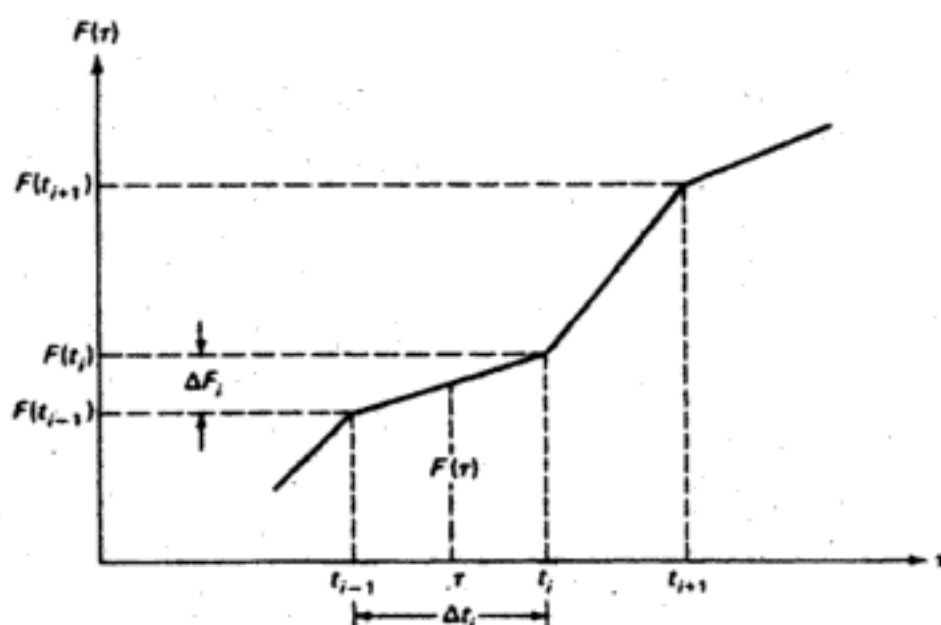


Fig. 19.3 Piecewise linear forcing function.

19.3 Fourier Coefficients for Piecewise Linear Functions

Proceeding as before in the evaluation of Duhamel's integral, we can represent the forcing function by piecewise linear function as shown in Fig. 19.3. The calculation of Fourier coefficients, eq. (19.3), is then obtained as a summation of the integrals evaluated for each linear segment of the forcing function, that is, as

$$a_0 = \frac{1}{T} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} F(t) dt \quad (19.11)$$

$$a_n = \frac{2}{T} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} F(t) \cos n\varpi t dt \quad (19.12)$$

$$b_n = \frac{2}{T} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} F(t) \sin n\varpi t dt \quad (19.13)$$

where N is the number of segments of the piecewise forcing function. The forcing function in any interval $t_{i-1} \leq t \leq t_i$ is expressed by eq. (4.20) as

$$F(t) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} (t - t_{i-1}) \quad (19.14)$$

in which $\Delta F_i = F(t_i) - F(t_{i-1})$ and $\Delta t_i = t_i - t_{i-1}$. The integrals required in the expressions of a_n and b_n have been evaluated in eqs. (4.21) and (4.22) and designated as $A(t_i)$ and $B(t_i)$ in the recurrent expressions (4.18) and (4.19). The use of eqs. (4.18) through (4.22) to evaluate the coefficients a_n and b_n yields

$$a_n = \frac{2}{T} \sum_{i=1}^N \left\{ \frac{1}{n\omega} \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) (\sin n\omega t_i - \sin n\omega t_{i-1}) \right. \\ \left. + \frac{\Delta F_i}{n^2 \omega^2 \Delta t_i} ((\cos n\omega t_i - \cos n\omega t_{i-1}) + n\omega (t_i \sin n\omega t_i - t_{i-1} \sin n\omega t_{i-1})) \right\} \quad (19.15)$$

$$b_n = \frac{2}{T} \sum_{i=1}^N \left\{ \frac{1}{n\omega} \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) (\cos n\omega t_{i-1} - \cos n\omega t_i) \right. \\ \left. + \frac{\Delta F_i}{n^2 \omega^2 \Delta t_i} ((\sin n\omega t_i - \sin n\omega t_{i-1}) - n\omega (t_i \cos n\omega t_i - t_{i-1} \cos n\omega t_{i-1})) \right\} \quad (19.16)$$

The integral appearing in the coefficient a_0 of eq. (19.3) is readily evaluated after substituting $F(t)$ from eq. (19.14) into eq. (19.11). This evaluation yields

$$a_0 = \frac{1}{T} \sum_{i=1}^N \left\{ \Delta t_i (F_i + F_{i-1}) / 2 \right\} \quad (19.17)$$

19.4 Exponential Form of Fourier Series

The Fourier series expression given by eq. (19.2) may also be written in exponential form by substituting the trigonometric functions using Euler's relationships:

$$\sin n\omega t = \frac{e^{in\omega t} - e^{-in\omega t}}{2i} \\ \cos n\omega t = \frac{e^{in\omega t} + e^{-in\omega t}}{2} \quad (19.18)$$

The result of this substitution may be written as

$$F(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega t} \quad (19.19)$$

where

$$C_n = \frac{1}{T} \int_0^T F(t) e^{-in\omega t} dt \quad (19.20)$$

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$$C_n = \frac{1}{N} \sum_{j=0}^{N-1} F(t_j) e^{-2\pi i(nj/N)}, \quad n = 0, 1, 2, \dots \quad (19.26)$$

Equation (19.26) may be considered as an approximate formula for calculating the complex Fourier coefficients in eq. (19.20). The discrete coefficients given by eq. (19.26) do not provide sufficient information to obtain a continuous function for $F(t)$; however, it is a most important fact that it does allow to obtain all the discrete values of the series $\{F(t_j)\}$ exactly [Newland, D.E., 1984]. This fact leads to the formal definition of the discrete Fourier transform of the series $\{F(t_j)\}$, $j = 0, 1, 2, \dots, N-1$, given by

$$C_n = \frac{1}{N} \sum_{j=0}^{N-1} F(t_j) e^{-2\pi i(nj/N)}, \quad n = 0, 1, 2, \dots, (N-1) \quad (19.27)$$

and its inverse discrete Fourier transform by

$$F(t_j) = \sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)}, \quad j = 0, 1, 2, \dots, (N-1) \quad (19.28)$$

The range of the summation in eq. (19.28) has been limited from 0 to $(N-1)$ in order to maintain the symmetry of transform pair eqs. (19.27) and (19.28). It is important to realize that in the calculation of the summation indicated in eq. (19.28), the frequencies increase with increasing index n up to $n = N/2$. It will be shown very shortly that, for $n > N/2$, the corresponding frequencies are equal to the negative of frequencies of order $N-n$. This fact restricts the harmonic components that may be represented in the series to a maximum of $N/2$. The frequency corresponding to this maximum order $\omega_{N/2} = (N/2)\omega$ is known as the Nyquist frequency or sometimes as the *folding frequency*. Moreover, if there are harmonic components above $\omega_{N/2}$ in the original function, these higher components will introduce distortions in the lower harmonic components of the series. This phenomenon is called *aliasing* [Newland, D.E., 1984, p. 118]. In view of this fact, it is recommended that the number of intervals or sampled points N should be at least twice the highest harmonic component present in the function.

The Nyquist frequency ω_u is given in radians per second by

$$\omega_u = \frac{2\pi N/2}{T} = \frac{2\pi N/2}{N\Delta t} = \frac{\pi}{\Delta t} \left(\frac{\text{rad}}{\text{sec}} \right) \quad (19.29)$$

and in cycles per second by

$$f_u = \frac{\omega_u}{2\pi} = \frac{1}{2\Delta t} (\text{cps}) \quad (19.30)$$

As a matter of interest, Example 19.4 is presented later in this chapter to illustrate the importance of choosing the number of sampling points N for the excitation function sufficiently large to avoid spurious results due to aliasing.

Having represented an arbitrary discrete function by a finite sum, we may then also obtain as a discrete function the response of a simple oscillator excited by the harmonic components of the loading function. Again, only the steady-state response will be considered. The introduction of the unit exponential forcing function $E_n = e^{i\omega_n t}$ into the equation of motion, eq. (3.13), leads to

$$m\ddot{u} + c\dot{u} + ku = e^{i\omega_n t} \quad (19.31)$$

which has a steady-state solution of the form

$$u(t) = H(\omega_n)e^{i\omega_n t} \quad (19.32)$$

When eq. (19.32) is introduced into eq. (19.31), it is found that the function $H(\omega_n)$, which will be designated as the *complex frequency response* function, takes the form

$$H(\omega_n) = \frac{1}{k - m\omega_n^2 + ic\omega_n} \quad (19.33)$$

Upon introducing the frequency ratio

$$r_n = \frac{\omega_n}{\omega}$$

and the damping ratio

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}$$

eq. (19.33) becomes

$$H(\omega_n) = \frac{1}{k(1 - r_n^2 + 2ir_n\xi)}$$

Therefore, the response $y_n(t_j)$ at time $t_j = j\Delta t$ to a harmonic force component of amplitude C_n indicated in eq. (19.28) is given by

$$u_n(t_j) = \frac{C_n e^{2\pi i(nj/N)}}{k(1 - r_n^2 + 2ir_n\xi)} \quad (19.34)$$

and the total response due to the N harmonic force components by

$$u_n(t_j) = \sum_{n=0}^{N-1} \frac{C_n e^{2\pi i(nj/N)}}{k(1 - r_n^2 + 2ir_n\xi)} \quad (19.35)$$

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value of unity in the first factor of the summation. The reduction in computational time that results from this formulation is significant when the time interval is divided into a large number of increments. The comparative times required for computing the Fourier series by a conventional program and by the fast Fourier transform algorithm are illustrated in Fig. 19.4. It is seen here how, for large values of N , one can rapidly consume so much computer time as to make the conventional method unfeasible.

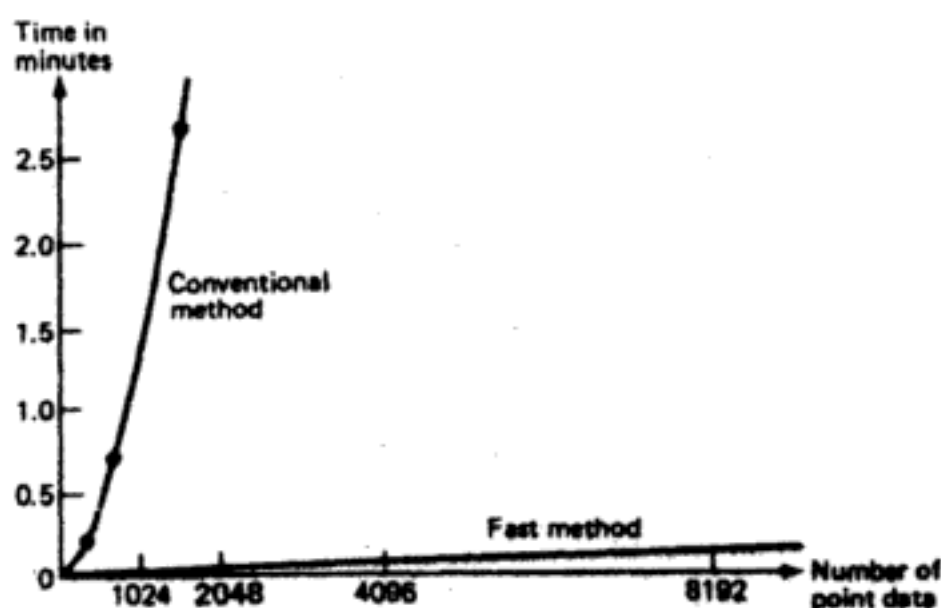


Fig. 19.4 Time required for Fourier transform using conventional and fast method.[from Cooley, J. W., Lewis, P. A. W., and Welch, P. D. (1969), IEEE Trans. Education, E-12 (1).]

19.7 Program 4-Response in the Frequency Domain

The computer program presented in this chapter calculates the response in the frequency domain for a damped single-degree-of-freedom system. The excitation is input as a discrete function of time. The computer output consists of two tables: (1) a table giving the first N complex Fourier coefficients of the series expansions of the excitation and of the response and (2) a table giving the displacement history of the steady-state motion of the response. This second table also gives the excitation function as calculated by eq. (19.28), thus providing a check of the computations. The main body of this program performs the tasks of calculating, using the FFT algorithm, the coefficients C_n in eq. (19.27), and the function $F(t_j)$ in eq. (19.28), and the response $u(t_j)$ in eq. (19.35).

Illustrative Example 19.2.

Determine the response of the tower shown in Fig. 19.5(a) subjected to the impulsive load of duration 0.64 sec as shown in Fig. 19.5(b).

Assume damping equal to 10% of the critical damping.

Solution:

Problem Data:

Mass: $m = 38,600/386 = 100 \text{ (lb} \cdot \text{sec}^2/\text{in)}$

Spring constant: $k = 100,000 \text{ (lb/in)}$

Damping coefficient: $c = 2 \xi \sqrt{km} = 632 \text{ (lb} \cdot \text{sec/in)}$

Select M such that $2^M = 8$: $M = 3$

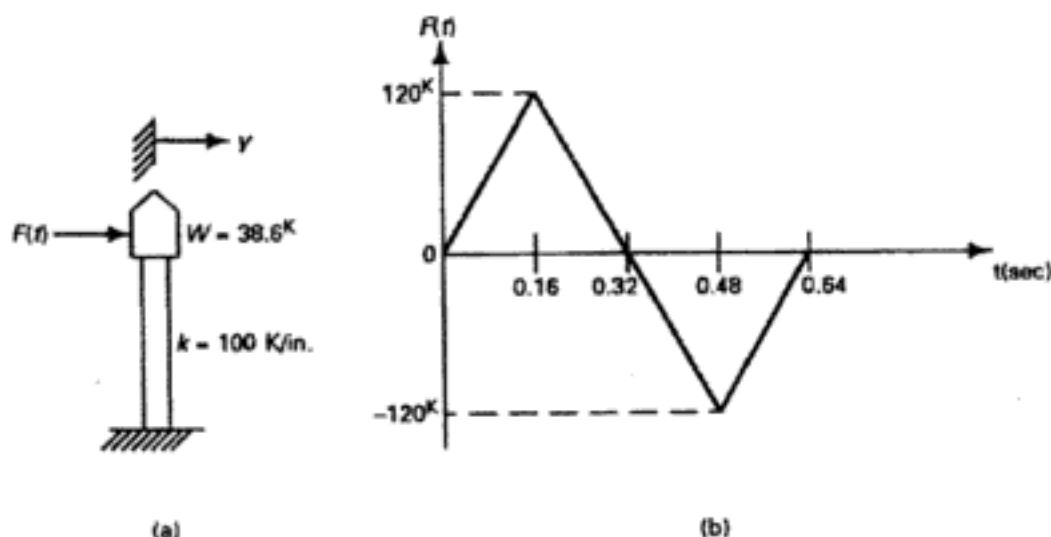


Fig. 19.5 Idealized structure and loading for Illustrative Example 19.2.

Excitation function:	Time (sec)	Force (lb)
	0.00	0
	0.16	120,000
	0.48	-120,000
	0.64	0

Input Data and Output Results

PROGRAM 4: --RESPONSE IN THE FREQUENCY--
INPUT DATA:

FILE: D4

NUMBER OF POINTS DEFINING THE EXCITATION
MASS
DAMPING COEFFICIENT
EXPONENT OF 2^M FOR FFT
INDEX (GRAVITY OR ZERO)

NE = 4
AM = 100
C = 632
M = 3
G = 0

TIME	EXCITATION	TIME	EXCITATION	TIME	EXCITATION	TIME	EXCITATION
0.000	0.00	0.160	120000.00	0.480	-120000.00	0.640	0.00

OUTPUT RESULTS:

FORCE FOURIER COEFFICIENTS			RESPONSE FOURIER COEFFICIENTS	
N	REAL	IMAG	REAL	IMAG
0	0.0015	0.0000	0.0000	0.0000
1	-0.0003	-51213.2100	-0.0387	-0.5641
2	0.0000	0.0015	0.0000	0.0000
3	-0.0006	8786.7950	0.3132	0.2230
4	-0.0015	0.0000	0.0000	0.0000
5	-0.0023	-8786.7970	0.3132	-0.2230
6	0.0000	-0.0015	0.0000	-0.0000
7	0.0033	51213.2100	-0.0387	0.5641

TIME	DISPL. REAL	DISPL. IMAG.	FORCE REAL	FORCE IMAG.
0.000	0.5490	-0.0000	0.0000	0.0000
0.080	-0.0154	0.0000	60000.0100	-0.0020
0.160	1.5743	0.0000	120000.0000	-0.0052
0.240	0.9800	-0.0000	60000.0200	-0.0020
0.320	-0.5490	0.0000	0.0000	0.0000
0.400	0.0154	-0.0000	-60000.0100	0.0020
0.480	-1.5743	-0.0000	-120000.0000	0.0052
0.560	-0.9800	0.0000	-60000.0000	0.0020

Illustrative Example 19.3.

Determine the response of the simple oscillator shown in Fig. 19.6(a) when subjected to the forcing function depicted in Fig. 19.6(b). Use $M = 4$ for the exponent in $N = 2M$. Assume 15% of the critical damping.

Solution:

Problem Data:

Mass: $m = 100 / 386 = 0.259 \text{ (Kip. sec}^2 \text{ /in)}$

Spring constant: $k = 200 \text{ Kip/in}$

Damping coefficient: $c = 2 \xi \sqrt{km}$
 $c = 2 \times 0.15 \sqrt{200 \times 0.259} = 2.159 \text{ (Kip .sec/in)}$

Exponent of $N = 2^M$ $M = 4$

Gravitational index $G = 0$ (force on the mass)

Excitation function:

Time (sec)	Force (Kip)
0.00	0
0.10	- 10
0.20	- 8
0.40	0
0.45	6
0.60	0
1.00	0

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TIME	DISPL. REAL	DISPL. IMAG.	FORCE REAL	FORCE IMAG.
0.0000	0.0052	0.0000	0.0000	0.0000
0.0625	-0.0096	-0.0000	-6.2500	-0.0000
0.1250	-0.0571	0.0000	-9.5000	-0.0000
0.1875	-0.0638	-0.0000	-8.2500	0.0000
0.2500	-0.0254	0.0000	-6.0000	-0.0000
0.3125	-0.0092	-0.0000	-3.5000	0.0000
0.3750	-0.0144	0.0000	-1.0000	0.0000
0.4375	0.0056	-0.0000	4.5000	0.0000
0.5000	0.0430	0.0000	4.0000	0.0000
0.5625	0.0210	-0.0000	1.5000	-0.0000
0.6250	-0.0167	0.0000	0.0000	0.0000
0.6875	-0.0076	-0.0000	0.0000	0.0000
0.7500	0.0116	0.0000	0.0000	0.0000
0.8125	0.0018	-0.0000	0.0000	0.0000
0.8750	-0.0071	0.0000	0.0000	-0.0000
0.9375	0.0001	-0.0000	-0.0000	-0.0000

Illustrative Example 19.4.

Consider a single-degree-of-freedom undamped system in which $k = 200 \text{ lb/in}$, $m = 100 \text{ lb} \cdot \text{sec}^2/\text{in}$ subjected to a force expressed as

$$P(t) = \sum_{n=1}^{16} 100 \cos 2\pi n t \quad (\text{a})$$

Determine the steady-state response of the system using Program 4 with $M = 3, 4, 5$, and 6 corresponding to $N = 8, 16, 32$, and 64 sampled points. Then discuss the results in relation to the limitations imposed by the Nyquist frequency.

Solution:

The fundamental frequency of the excitation function, eq. (a), is $\omega_1 = 2\pi$ and its period $T = \omega_1 / 2\pi = 1 \text{ sec}$. Since the highest component in eq. (a) is of order $\omega_{16} = 16\omega_1$, to avoid aliasing, the number of sampled points should be at least twice that order, that is, the minimum number of sampled points should be $N = 32$.

With a simple modification of Program 4, the applied force is calculated in the program, instead of being supplied through a numerical table as normally required by the program. The results given by the computer for this example are conveniently arranged in two tables: Table 19.1, giving the displacement response to the excitation having all 16 harmonic components as prescribed for this problem; and Table 19.2, showing the displacement response to a reduced number of harmonic terms in the excitation function.

For this example, in which the exciting force is supplied in 16 harmonic components, the response given in Table 19.1 corresponding to $N=32$ or $N=64$ may be considered the exact solution. A comparison of the response shown for sample points $N=8$ or $N=16$ with the exact solution ($N=32$) dramatically demonstrates the risk of not choosing N sufficiently large enough so that none of the frequencies of the

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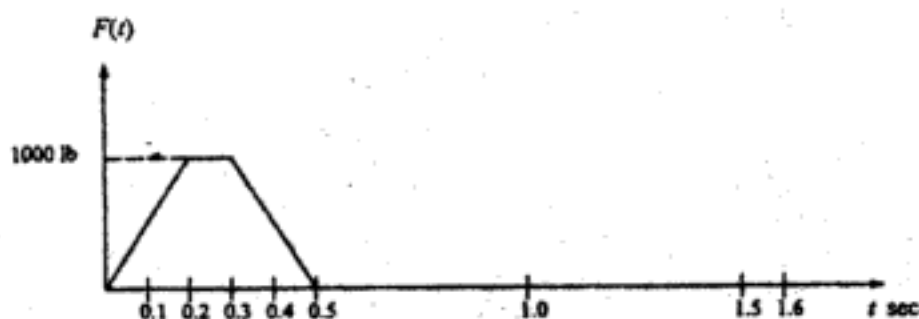


Fig. P19.16.

Problem 19.17

Solve Problem 19.16 assuming a 10% of the critical damping in the system.

Problem 19.18

The water tower shown in Fig. P19.8(a) is subjected to impulsive acceleration of its base that varies as half the sine function shown in Fig. P19.18(b). Use Program 4 to determine: (a) the discrete Fourier coefficients for the excitation and for the response, (b) the relative displacement of the tower with respect to the ground displacement, and (c) the excitation obtained by the inverse discrete transform. Use an extended excitation of total duration 1.6 sec and time step $\Delta t = 0.1$ sec. Neglect damping.

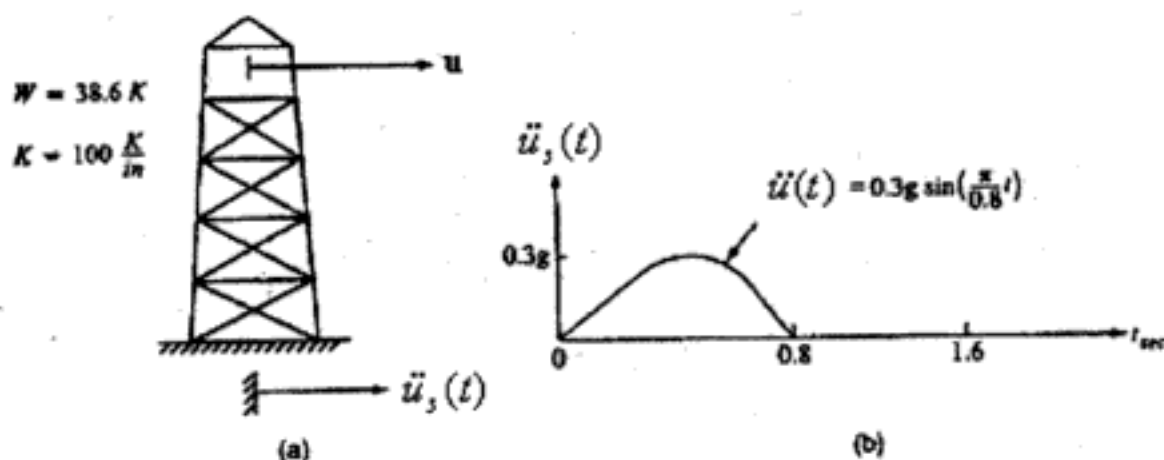


Fig. P19.18.

Problem 19.19

Repeat Problem 19.18 assuming that the damping in the system is 5% of the critical.

Problem 19.20

Solve Problem 19.18 for an acceleration at the base of tower that varies as a symmetrical triangular load as shown in Fig. P19.19.

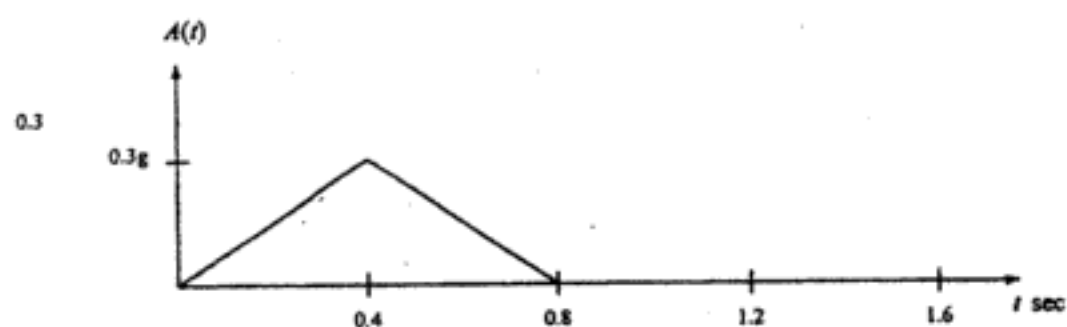


Fig.P19.20.

20 Evaluation of Absolute Damping from Modal Damping Ratios

In the previous chapters, we determined the natural frequencies and modal shapes for undamped structures when modeled as shear buildings. We also determined the response of these structures using the modal superposition method. In this method, as we have seen, the differential equations of motion are uncoupled by means of a transformation of coordinates that incorporates the orthogonality property of the modal shapes.

The consideration of damping in the dynamic analysis of structures complicates the problem. Not only will the differential equations of motion have additional terms due to damping forces, but the uncoupling of the equations will be possible only by imposing some restrictions or conditions on the functional expression for the damping coefficients.

The damping normally present in structures is relatively small and practically does not affect the calculation of natural frequencies and modal shapes of the system. Hence, the effect of damping is neglected in determining the natural frequencies and modal shapes of the structural systems. Therefore, in practice, the eigenproblem for the damped structure is solved by using the same methods employed for undamped structures.

20.1 Equations for Damped Shear Building

For a viscously damped shear building, such as the three-story building shown in Fig. 20.1, the equations of motion obtained by summing forces in the corresponding free body diagrams are

$$\begin{aligned} m_1 \ddot{u}_1 + c_1 \dot{u}_1 + k_1 u_1 - c_2 (\dot{u}_2 - \dot{u}_1) - k_2 (u_2 - u_1) &= F_1(t) \\ m_2 \ddot{u}_2 + c_2 (\dot{u}_2 - \dot{u}_1) + k_2 (u_2 - u_1) - c_3 (\dot{u}_3 - \dot{u}_2) - k_3 (u_3 - u_2) &= F_2(t) \\ m_3 \ddot{u}_3 + c_3 (\dot{u}_3 - \dot{u}_2) + k_3 (u_3 - u_2) &= F_3(t) \end{aligned} \quad (20.1)$$

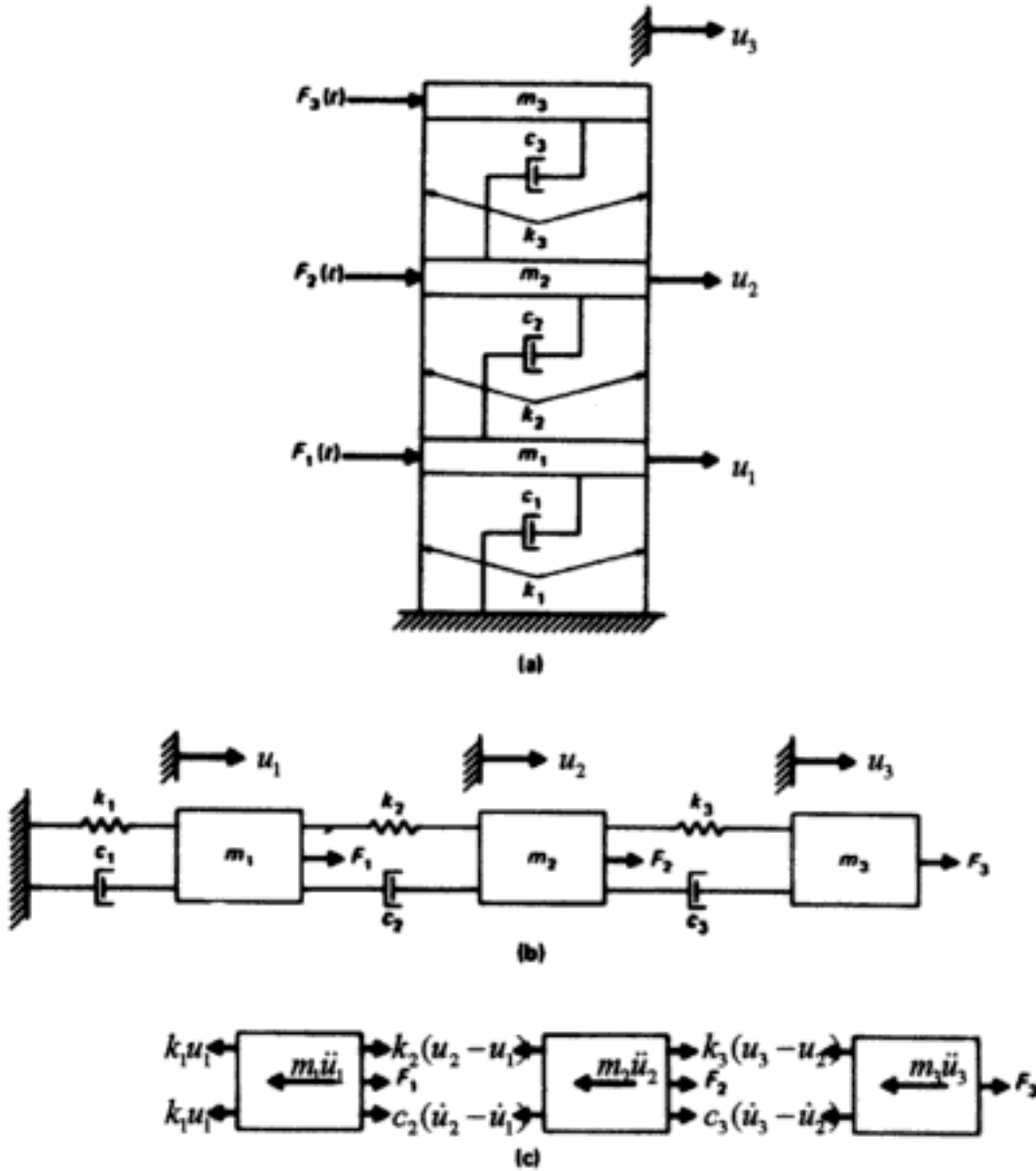


Fig. 20.1 (a) Damped shear building. (b) Mathematical model. Free body diagram.

These equations may be conveniently written in matrix notation as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\} \quad (20.2)$$

where the matrices and vectors are as previously defined, except for the damping matrix $[C]$, which is given by

$$[C] = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \quad (20.3)$$

In the next section, we shall establish the conditions under the damped equations of motion may be transformed to an uncoupled set of independent equations.

20.2 Uncoupled Damped Equations

To solve the differential equations of motion, eq. (20.2), we seek to uncouple these equations. We, therefore, introduce the transformation of coordinates

$$\{u\} = [\Phi]\{z\} \quad (20.4)$$

where $[\Phi]$ is the modal matrix obtained in the solution of the undamped free-vibration system. The substitution of eq. (20.4) and its derivatives into eq. (20.2) leads to

$$[M][\Phi]\{\ddot{z}\} + [C][\Phi]\{\dot{z}\} + [K][\Phi]\{z\} = \{F(t)\} \quad (20.5)$$

Premultiplying eq. (20.5) by the transpose of the n th modal vector $\{\phi\}_n^T$ yield

$$\{\phi\}_n^T [M][\Phi]\{\ddot{z}\} + \{\phi\}_n^T [C][\Phi]\{\dot{z}\} + \{\phi\}_n^T [K][\Phi]\{z\} = \{\phi\}_n^T \{F(t)\} \quad (20.6)$$

It is noted that the orthogonality property of the modal shapes,

$$\begin{aligned} \{\phi\}_n^T [M]\{\phi\}_m &= 0, \quad m \neq n \\ \{\phi\}_n^T [K]\{\phi\}_m &= 0 \end{aligned} \quad (20.7)$$

causes all components except the n th mode in the first and third terms of eq. (20.6) to vanish. A similar reduction is *assumed* to apply to the damping term in eq. (20.6), that is, if it is assumed that

$$\{\phi\}_n^T [C]\{\phi\}_m = 0, \quad n \neq m \quad (20.8)$$

then the coefficient of the damping term in eq. (20.6) will reduce to

$$\{\phi\}_n^T [C]\{\phi\}_n$$

In this case eq. (20.6) may be written as

$$M_n \ddot{z}_n + C_n \dot{z}_n + K_n z_n = F_n(t)$$

or alternatively as

$$\ddot{z}_n + 2\xi_n \omega_n \dot{z}_n + \omega_n^2 z_n = F_n(t)/M_n \quad (20.9)$$

in which case

$$M_n = \{\phi\}_n^T [M]\{\phi\}_n \quad (20.10a)$$

$$K_n = \{\phi\}_n^T [K]\{\phi\}_n = \omega_n^2 M_n \quad (20.10b)$$

$$C_n = \{\phi\}_n^T [C] \{\phi\}_n = 2\xi_n \omega_n M_n \quad (20.10c)$$

$$F_n(t) = \{\phi\}_n^T \{F(t)\} \quad (20.10d)$$

The normalization discussed previously (section 7.2)

$$\{\phi\}_n^T [M] \{\phi\}_n = 1 \quad (20.11)$$

will result in

$$M_n = 1$$

so that eq. (20.9) reduces to

$$\ddot{z}_n + 2\xi_n \omega_n \dot{z}_n + \omega_n^2 z_n = F_n(t) \quad (20.12)$$

which is a set of N uncoupled differential equations ($n = 1, 2, \dots, N$).

20.3 Conditions for Damping Uncoupling

In the derivation of the uncoupled damped equation (20.12), it has been assumed that the normal coordinate transformation, eq. (20.4), that serves to uncouple the inertial and elastic forces also uncouples the damping forces. It is of interest to consider the conditions under which this uncoupling will occur, that is, the form of the damping matrix $[C]$ to which eq. (20.8) applies.

When the damping matrix is of the form

$$[C] = a_0 [M] + a_1 [K] \quad (20.13)$$

in which a_0 and a_1 are arbitrary proportionality factors, the orthogonality condition will be satisfied. This may be demonstrated by applying the orthogonality condition to eq. (20.13), that is, premultiplying both sides of this equation by the transpose of the n th mode $\{\phi\}_n^T$ and postmultiplying by the modal matrix $[\Phi]$. We obtain

$$\{\phi\}_n^T [C] [\Phi] = a_0 \{\phi\}_n^T [M] [\Phi] + a_1 \{\phi\}_n^T [K] [\Phi] \quad (20.14)$$

The orthogonality conditions, eqs. (20.7), then reduce eq. (20.14) to

$$\{\phi\}_n^T [C] [\Phi] = a_0 \{\phi\}_n^T [M] \{\phi\}_n + a_1 \{\phi\}_n^T [K] \{\phi\}_n$$

or, by eqs. (20.10), to

$$\begin{aligned} \{\phi\}_n^T [C] [\Phi]_n &= a_0 M_n + a_1 M_n \omega_n^2 \\ \{\phi\}_n^T [C] [\Phi]_n &= (a_0 + a_1 \omega_n^2) M_n \end{aligned}$$

which shows that, when the damping matrix is of the form of eq. (20.13), the damping forces are also uncoupled with the transformation eq. (20.4). However, it can be shown that there are other matrices formed from the mass and stiffness matrices

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$$\{\xi\} = \frac{1}{2}[Q]\{a\} \quad (20.20)$$

where $[Q]$ is a square matrix having different powers of the natural frequencies. The solution of eq. (20.20) gives the constants $\{a\}$ as

$$\{a\} = 2[Q]^{-1}\{\xi\} \quad (20.21)$$

Finally the damping matrix is obtained after the substitution of eq. (20.21) into eq. (20.15).

It is interesting to observe from eq. (20.18) that in the special case when the damping matrix is proportional to the mass $\{C\} = a_0 [M]$ ($i = 0$), the damping ratios are inversely proportional to the natural frequencies; thus the higher modes of the structures will be given very little damping. Analogously, when the damping is proportional to the stiffness matrix ($[C] = a_1 [K]$), the damping ratios are directly proportional to the corresponding natural frequencies, as can be seen from eq. (20.18) evaluated for $i = 1$; and in this case the higher modes of the structure will be very heavily damped.

Illustrative Example 20.1.

Determine the absolute damping coefficients for the structure presented in Example 7.1. Assume 10% of the critical damping for each mode.

Solution:

From Example 7.1, we have the following information.

Natural frequencies:

$$\begin{aligned} \omega_1 &= 11.83 \text{ rad/sec} \\ \omega_2 &= 32.89 \text{ rad/sec} \end{aligned} \quad (a)$$

Modal matrix:

$$[\Phi] = \begin{bmatrix} 1.00 & 1.00 \\ 1.26 & -1.63 \end{bmatrix} \quad (b)$$

Mass matrix:

$$[M] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}$$

Stiffness matrix:

$$[K] = \begin{bmatrix} 75,000 & -44,300 \\ -44,300 & 44,300 \end{bmatrix}$$

Using eqs. (20.18) and (20.19) with $i = 0, 1$ to calculate the constants a_i needed in eq. (20.15), we obtain the following system of equations:

$$\begin{Bmatrix} 0.1 \\ 0.1 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 11.83 & (11.83)^3 \\ 32.89 & (32.89)^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

Solving this system of equations gives

$$\begin{aligned} a_1 &= 0.01851 \\ a_2 &= -0.00001146 \end{aligned}$$

We also calculate

$$[M]^{-1} = \begin{bmatrix} 0.007353 & 0 \\ 0 & 0.01515 \end{bmatrix}$$

and

$$[M]^{-1}[K] = \begin{bmatrix} 551475 & -325738 \\ -671145 & 671145 \end{bmatrix}$$

Then

$$\begin{aligned} \sum_{i=1}^2 a_i ([M]^{-1}[K])^i &= 0.01851 \begin{bmatrix} 551475 & -325738 \\ -671145 & 671145 \end{bmatrix} \\ &\quad - 0.00001146 \begin{bmatrix} 551.475 & -325.738 \\ -671.145 & 671.145 \end{bmatrix}^2 \\ &= \begin{bmatrix} 4.2172 & -1.4654 \\ -3.0193 & 4.7556 \end{bmatrix} \end{aligned}$$

Finally, substituting this matrix into eq. (20.15) yields the damping matrix as

$$[C] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix} \begin{bmatrix} 4.2172 & -1.4654 \\ -3.0193 & 4.7556 \end{bmatrix} = \begin{bmatrix} 573.5 & -199.3 \\ -199.3 & 313.9 \end{bmatrix}$$

There is yet a second method for evaluating the damping matrix corresponding to any set of specified modal damping ratio. The method may be explained starting with the relationship

$$[A] = [\Phi]^T [C] [\Phi] \quad (20.22)$$

where the matrix $[A]$ is defined as

$$[A] = \begin{bmatrix} 2\zeta_1\omega_1 M_1 & 0 & 0 & - \\ 0 & 2\zeta_2\omega_2 M_2 & 0 & - \\ 0 & 0 & 2\zeta_3\omega_3 M_3 & - \\ - & - & - & - \end{bmatrix} \quad (20.23)$$

in which the modal masses M_1, M_2, M_3, \dots are equal to one if the modal matrix $\{\Phi\}$ has been normalized. It is evident that the damping matrix $[C]$ may be evaluated by postmultiplying and premultiplying eq. (20.22) by the inverse of the modal matrix $[\Phi]^{-1}$ and its inverse transpose $[\Phi]^{-T}$, such that

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Solution:

From Example 7.2 we have the natural frequencies

$$\omega_1 = 11.83 \text{ rad/sec}$$

$$\omega_2 = 32.89 \text{ rad/sec}$$

and the mass matrix

$$[M] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}$$

To determine $[C]$, we could use either eq. (20.24) or eq. (20.25). From Example 7.2, the normalized modal matrix is

$$[\Phi] = \begin{bmatrix} 0.06437 & 0.567 \\ 0.0813 & -0.0924 \end{bmatrix}$$

and its inverse by eq. (20.27) is

$$[\Phi]^{-1} = \begin{bmatrix} 8.752 & 5.370 \\ 7.700 & -6.097 \end{bmatrix}$$

Substituting into eq. (20.23), we obtain

$$2\xi_1\omega_1M_1 = (2)(0.1)(11.83)(1) = 2.366$$

$$2\xi_2\omega_2M_2 = (2)(0.1)(32.89)(1) = 6.578$$

Then by eq. (20.24)

$$[C] = \begin{bmatrix} 8.752 & 7.700 \\ 5.370 & -6.097 \end{bmatrix} \begin{bmatrix} 2.366 & 0 \\ 0 & 6.578 \end{bmatrix} \begin{bmatrix} 8.752 & 7.700 \\ 5.370 & -6.097 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 572 & -198 \\ -198 & 313 \end{bmatrix}$$

which in this case of equal damping ratios in all the models checks with the damping matrix obtained in Example 20.1 for the same structure using eq. (20.15).

20.4 Program 11—Absolute Damping From Modal Damping Ratios

Program 11, "DAMPING," serves to calculate the absolute damping coefficients, that is, the elements of the damping matrix from known values of modal damping ratios.

Illustrative Example 20.3.

Use Program 11 to calculate the damping matrix for a structure with three degrees of freedom for which the squares of the natural frequencies (eigenvalues) are

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21 Generalized Coordinates and Rayleigh's Method

In the preceding chapters we concentrated our efforts in obtaining the response to dynamic loads of structures modeled by the simple oscillator, that is, structures that may be analyzed as a damped or undamped spring-mass system. Our plan in the present chapter is to discuss the conditions under which a structural system consisting of multiple interconnected rigid bodies or having distributed mass and elasticity can still be modeled as a one-degree of-freedom system. We begin by presenting an alternative method to the direct application of Newton's Law of Motion, the principle of virtual work.

21.1 Principle of Virtual Work

An alternative approach to the direct method employed thus far for the formulation of the equations of motion is the use of the principle of virtual work. This principle is particularly useful for relatively complex structural systems which contain many interconnected parts. The principle of virtual work was originally stated for a system in equilibrium. Nevertheless, the principle can readily be applied to dynamic systems by the simple recourse to D'Alembert's Principle, which establishes dynamic equilibrium by the inclusion of the inertial forces in the system.

The principle of virtual work may be stated as follows: For a system that is in

equilibrium, the work done by all the forces during an assumed displacement (virtual displacement) that is compatible with the system constraints is equal to zero. In general the equations of motion are obtained by introducing virtual displacements corresponding to each degree of freedom and equating the resulting work done to zero.

To illustrate the application of the principle of virtual work to obtain the equation of motion for a single-degree-of-freedom system, let us consider the damped oscillator shown in Fig. 21.1 (a) and its corresponding free body diagram in Fig. 21.1 (b). Since the inertial force has been included among the external forces, the system is in "equilibrium" (dynamic equilibrium). Consequently, the principle of virtual work is applicable. If a virtual displacement δu is assumed to have taken place, the total work done by the forces shown in Fig. 21.1 (b) is equal to zero, that is,

$$m\ddot{u}\delta u + c\dot{u}\delta u + ku\delta u - F(t)\delta u = 0$$

or

$$\{m\ddot{u} + c\dot{u} + ku - F(t)\}\delta u = 0 \quad (21.1)$$

Since δu is arbitrarily selected as not equal to zero, the other factor in eq. (21.1) must equal zero. Hence,

$$m\ddot{u} + c\dot{u} + ku - F(t) = 0 \quad (21.2)$$

Thus we obtained in eq. (21.2) the differential equation for the motion of the damped oscillator.

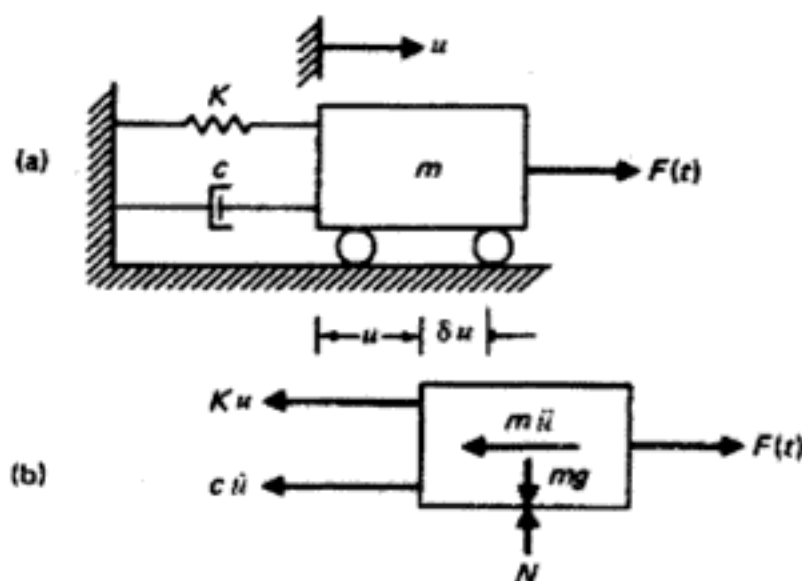


Fig. 21.1 Damped simple oscillator undergoing virtual displacement δu .

21.2 Generalized Single-degree-of-freedom System—Rigid Body

Most frequently the configuration of a dynamic system is specified by coordinates indicating the linear or angular positions of elements of the system. However, coordinates do not necessarily have to correspond directly to displacements; they may in general be any independent quantities that are sufficient in number to specify the position of all parts of the system. These coordinates are usually called generalized coordinates and their number is equal to the number of degrees of freedom of the system.

The example of the rigid-body system shown in Fig. 21.2 consists of a rigid bar with distributed mass supporting a circular plate at one end. The bar is supported by springs and dampers in addition to a single frictionless support. Dynamic excitation is provided by a transverse load $F(x, t)$ varying linearly on the portion AB of the bar. Our purpose is to obtain the differential equation of motion and to identify the corresponding expressions for the parameters of the simple oscillator representing this system.

Since the bar is rigid, the system in Fig. 21.2 has only one degree of freedom, and, therefore, its dynamic response can be expressed with one equation of motion. The generalized coordinate could be selected as the vertical displacement of any point such as A , B , or C along the bar, or may be taken as the angular position of the bar. This last coordinate designated $\theta(t)$ is selected as the generalized coordinate of the system. The corresponding free body diagram showing all the forces including the inertial forces and the inertial moments is shown in Fig. 21.3. In evaluating the displacements of the different forces, it is assumed that the displacements of the system are small and, therefore, vertical displacements are simply equal to the product of the distance to support D multiplied by the angular displacement $\theta = \theta(t)$.

The displacements resulting at the points of application of the forces in Fig. 21.3 due to a virtual displacement $\delta\theta$ are indicated in this figure. By the principle of virtual work, the total work done by the forces during this virtual displacement is equal to zero. Hence

$$\delta\theta[I_0\ddot{\theta} + I_1\ddot{\theta} + 4L^3\bar{m}\ddot{\theta} + mL^2\ddot{\theta} + cL^2\dot{\theta} + 4kL^2\theta - \frac{7}{6}L^2f(t)] = 0$$

or, since $\delta\theta$ is arbitrarily set not equal to zero, it follows that

$$(I_0 + I_1 + 4L^3\bar{m} + mL^2)\ddot{\theta} + cL^2\dot{\theta} + 4kL^2\theta - \frac{7}{6}L^2f(t) = 0, \quad (21.3)$$

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Illustrative Example 21.1.

For the system shown in Fig. 21.4, determine the generalized physical properties M^* , C^* , K^* and generalized loading $F^*(t)$. Let $U(t)$ at the point A_2 in Fig. 1.4 be the generalized coordinate of the system.

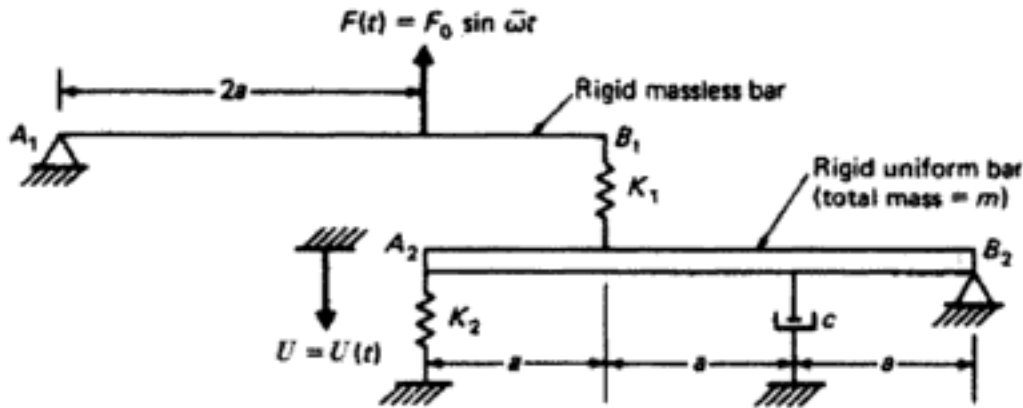


Fig. 21.4 System for Illustrative Example 21.1

Solution:

The free body diagram for the system is depicted in Fig. 21.5 which shows all the forces on the two bars of the system including the inertial force and the inertial moment. The generalized coordinate is $U(t)$ and the displacement of any point in the system should be expressed in terms of this coordinate; nevertheless, for convenience, we select also the auxiliary coordinate $U_1(t)$ as indicated in Fig. 21.5.

The summation of the moments about point A_1 of all the forces acting on bar $A_1 - B_1$, and the summation of moments about B_2 of the forces on bar $A_2 - B_2$. Give the following equations:

$$k_1 \left(\frac{2}{3} U - U_1 \right) 3a = 2aF_0 \sin \bar{\omega} t \quad (21.5)$$

$$\frac{I_0}{3a} \ddot{U} + \frac{3}{4} m a \ddot{U} + \frac{a}{3} c \dot{U} + k_1 \left(\frac{2}{3} U - U_1 \right) 2a + 3a k_2 U = 0 \quad (21.6)$$

Substituting U_1 from eq. (21.5) into eq. (21.6), we obtain the differential equation for the motion of the system in terms of the generalized coordinate $U(t)$, namely

$$M^* \ddot{U}(t) + C^* \dot{U}(t) + K^* U(t) = F^*(t)$$

where the generalized quantities are given by

$$M^* = \frac{I_0}{3a^2} + \frac{3m}{4}$$

$$C^* = \frac{c}{3}$$

$$K^* = 3k_2$$

and

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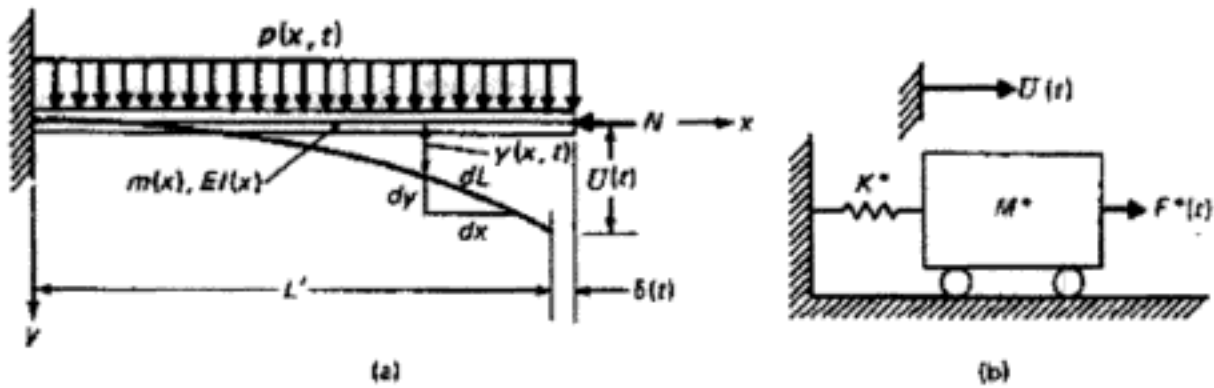


Fig. 21.6 Single-degree-of-freedom continuous system.

The equivalent one-degree-of-freedom system [Fig. 21.6(b)] may be defined simply as the system for which the kinetic energy, potential energy (strain energy), and work done by the external forces have at all times the same values in the two systems.

The kinetic energy T of the beam in Fig. 21.6 vibrating in the pattern indicated by eq. (21.7) is

$$T = \int_0^{L'} \frac{1}{2} m(x) \{\dot{\phi}(x) \dot{U}(t)\}^2 dx \quad (21.8)$$

Equating this expression for the kinetic energy of the continuous system to the kinetic energy of the equivalent single-degree-of-freedom system $\frac{1}{2} \dot{M}^* \dot{U}^2(t)$ and solving the resulting equation for the generalized mass, we obtain

$$M^* = \int_0^{L'} m(x) \phi^2(x) dx \quad (21.9)$$

The flexural strain energy V of a prismatic beam may be determined as the work done by the bending moment $M(x)$ undergoing an angular displacement $d\theta$. This angular displacement is obtained from the well-known formula for the flexural curvature of a beam, namely

$$\frac{d^2 u}{dx^2} = \frac{d\theta}{dx} = \frac{M(x)}{EI} \quad (21.10)$$

or

$$d\theta = \frac{M(x)}{EI} dx \quad (21.11)$$

since $du/dx = \theta$, where θ , being assumed small, is taken as the slope of the elastic curve. Consequently, the strain energy is given by

$$V = \int_0^{L'} \frac{1}{2} M(x) d\theta \quad (21.12)$$

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$$\frac{1}{2} K_G^* U(t)^2 = N \delta(t)$$

Substituting $\delta(t)$ from eq. (21.21) and the derivative du/dx from eq. (21.7), we have

$$\frac{1}{2} K_G^* U(t)^2 = \frac{1}{2} N \int_0^L \left\{ U(t) \frac{d\phi}{dx} \right\}^2 dx$$

or

$$K_G^* = N \int_0^L \left(\frac{d\phi}{dx} \right)^2 dx \quad (21.22)$$

Equations (21.9), (21.14), (21.17), (21.18), and (21.22) give, respectively, the generalized expression for the mass, stiffness, force, damping, and geometric stiffness for a beam with distributed properties and load, modeling it as a simple oscillator.

For the case of an axial compressive force, the potential energy in the beam decreases with a loss of stiffness in the beam. The opposite is true for a tensile axial force, which results in an increase of the flexural stiffness of the beam. Customarily, the geometric stiffness is determined for a compressive axial force. Consequently, the combined generalized stiffness K_c^* is then given by

$$K_c^* = K^* - K_G^* \quad (21.23)$$

Finally, the differential equation for the equivalent system may be written as

$$M^* \ddot{U}(t) + C^* \dot{U}(t) + K_c^* U(t) = F^*(t) \quad (21.24)$$

The critical buckling load N_{cr} is defined as the axial compressive load that reduces the combined stiffness to zero, that is,

$$K_c^* = K^* - K_G^* = 0$$

The substitution of K^* and K_G^* from eqs. (21.14) and (21.22) gives

$$\int_0^L EI \left(\frac{d^2 \phi}{dx^2} \right)^2 dx - N_{cr} \int_0^L \left(\frac{d\phi}{dx} \right)^2 dx = 0$$

and solving for the critical buckling load, we obtain

$$N_{cr} = \frac{\int_0^L EI (d^2 \phi / dx^2)^2 dx}{\int_0^L (d\phi / dx)^2 dx} \quad (21.25)$$

Once the generalized stiffness K^* and generalized mass M^* have been determined, the system can be analyzed by any of the methods presented in the preceding chapters for single-degree-of-freedom systems. In particular, the square of the natural frequency, ω^2 , is given from eqs. (21.9) and (21.14) by

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$$u(x, t) = \phi(x) U(t) \quad (21.42)$$

The generalized coordinate $f(t)$ in eq. (21.42) is selected as the lateral displacement at the top level of the building which requires that shape function be assigned a unit value at that level; that is, $\phi(H) = 1.0$ where H is the height of the building.

The equation of motion for the generalized single-degree-of-freedom system in Fig. 21.8(c) is obtained by equating to zero the sum of forces in the corresponding free-body diagram [Fig. 21.8(d)], that is

$$M^* \ddot{U} + C^* \dot{U} + K^* U = F^*(t) \quad (21.43)$$

where M^* , C^* , K^* , and $F^*(t)$ are, respectively, the generalized mass, generalized damping, generalized stiffness, and generalized force, and are given for a discrete system, modeled in Fig. 21.8(b), by

$$M^* = \sum_{i=1}^N m_i \Phi_i^2 \quad (21.44a)$$

$$C^* = \sum_{i=1}^N c_i \Delta \Phi_i^2 \quad (21.44b)$$

$$K^* = \sum_{i=1}^N k_i \Delta \Phi_i^2 \quad (21.44c)$$

$$F^*(t) = \sum_{i=1}^N F_i(t) \Phi_i \quad (21.44d)$$

where the upper index N in the summations is equal to the number of stories or levels in the building.

The various expressions for the equivalent parameters in eq. (21.44) are obtained by equating the kinetic energy, potential energy, and the virtual work done by the damping forces and by external forces in the actual structure with the corresponding expressions for the generalized single-degree-of-freedom system.

In eq. (21.44), the relative displacement $\Delta \phi_i$ between two consecutive levels of the building is given by

$$\Delta \phi_i = \phi_i - \phi_{i-1} \quad (21.45)$$

with $\phi_0 = 0$ at the ground level.

As shown in Fig. 21.8(b), m_i and $F_i(t)$ are, respectively, the mass and the external force at level i of the building, while k_i and c_i are the stiffness and damping coefficients corresponding to the i th story.

It is convenient to express the generalized damping coefficient C^* of eq. (21.44b) in terms of generalized damping ratio ξ^* ; thus by eq. (2.7)

$$C^* = \sum_{i=1}^N c_i \Delta \Phi_i^2 = \xi^* C_{cr}^* = 2 \xi^* M^* \omega \quad (21.46)$$

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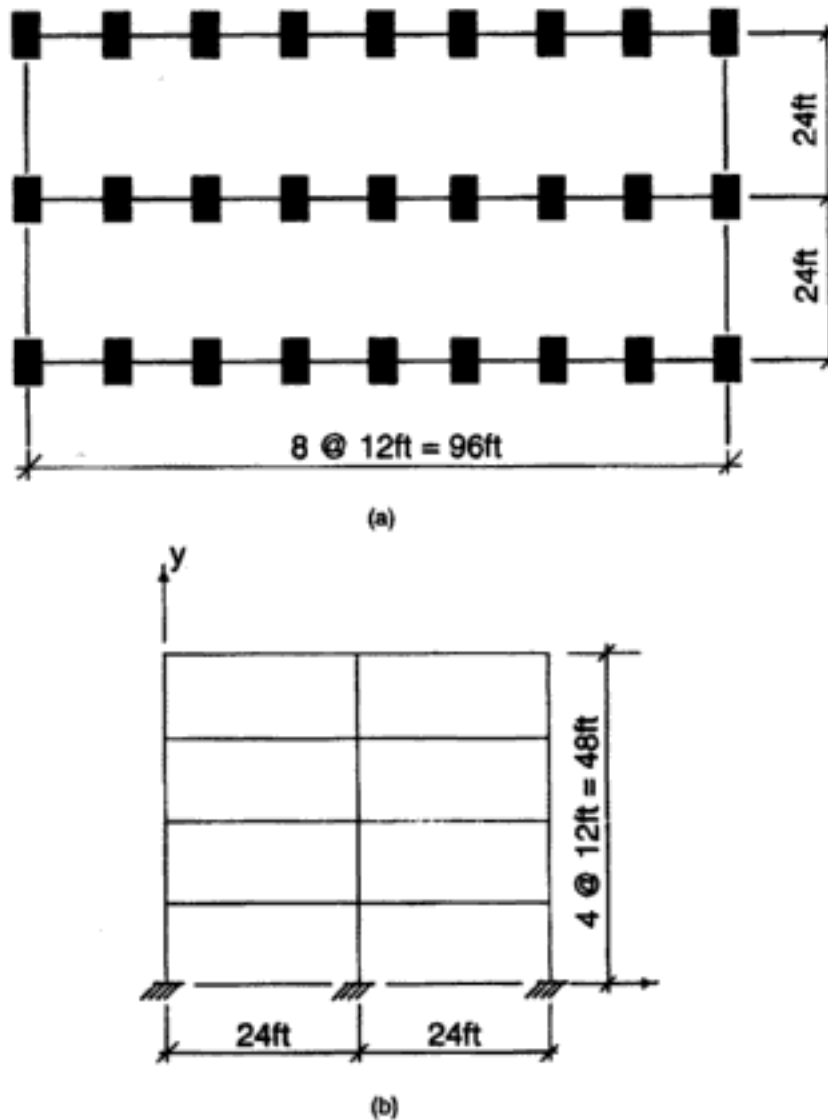


Fig. 21.10 Plan and elevation for a four-story building for Example 21.3; (a) Plan. (b) Elevation.

Solution:

1. Effective weight at various floors:

No live load needs to be considered on the roof. Hence, the effective weight at all floors, except at the roof, will be $140 + 0.25 \times 125 = 171.25$ psf, and the effective weight for the roof will be 140 psf. The plan area is $48 \text{ ft} \times 96 \text{ ft} = 4608 \text{ ft}^2$. Hence, the weights of various levels are:

$$W_1 = W_2 = W_3 = 4608 \times 0.17125 = 789.1 \text{ Kips}$$

$$W_4 = 4608 \times 0.140 = 645.1 \text{ Kips}$$

The total seismic design weight of the building is then

$$W = 789.1 \times 3 + 645.1 = 3012.4 \text{ Kips}$$

2. Story lateral stiffness:

It will be assumed that horizontal beam-and-floor diaphragms are rigid compared to

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The force in the spring, when displaced y units from the equilibrium position, is ku and the work done by this force on the mass for an additional displacement du is $-ku du$. This work is negative because the force ku acting on the mass is opposite to the incremental displacement du given in the positive direction of coordinate u . However, by definition, the potential energy is the value of this work but with opposite sign. It follows then that the total potential energy (PE) in the spring for a final displacement u will be

$$(PE) = \int_0^u kudu = \frac{1}{2}ku^2 \quad (21.57)$$

Adding eq. (21.56) and (21.57), and setting this sum equal to a constant, will give

$$\frac{1}{2}my\dot{u}^2 + \frac{1}{2}ku^2 = C_0 \quad (21.58)$$

Differentiation with respect to time yields

$$my\ddot{u} + ku\dot{u} = 0$$

Since \dot{y} cannot be zero for all values of t , it follows that

$$m\ddot{u} + ku = 0 \quad (21.59)$$

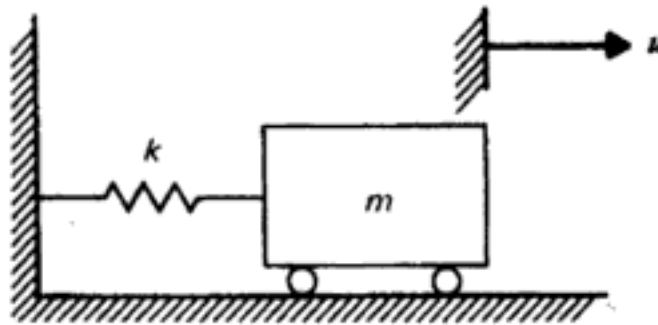


Fig.21.11 Spring-mass system in free vibration

This equation is identical with eq. (1.11) of Chapter 1 obtained by application of Newton's Law of Motion. Used in this manner, the energy method has no particular advantage over the equilibrium method. However, in many practical problems it is only the natural frequency that is desired. Consider again the simple oscillator of Fig. 21.11, and assume that the motion is harmonic. This assumption leads to the equation of motion of the form

$$u = C \sin (\omega t + \alpha) \quad (21.60)$$

and velocity

$$\dot{u} = \omega C \cos (\omega t + \alpha) \quad (21.61)$$

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has a length L and a total mass m_s . Use Rayleigh's Method to determine the fraction of the spring mass that should be added to the vibrating mass.

Solution:

The displacement of an arbitrary section of the spring at a distance s from the support will now be assumed to be $u_r = su/L$. Assuming that the motion of the mass m is harmonic and given by eq. (21.60), we obtain

$$u_r = \frac{s}{L}u = \frac{s}{L}C \sin(\omega t + \alpha) \quad (21.66)$$

The potential energy of the uniformly stretched spring is given by eq. (21.57) and its maximum value is

$$(PE)_{\max} = \frac{1}{2} k C^2 \quad (21.67)$$

A differential element of the spring of length ds has mass equal to $m_s ds/L$ and maximum velocity $\dot{u}_{r, \max} = \omega u_{r, \max} = \omega s C/L$. Consequently the total kinetic energy in the system at its maximum value is

$$T_{\max} = \int_0^L \frac{1}{2} \frac{m_s}{L} ds (\omega \frac{s}{L} C)^2 + \frac{1}{2} m \omega^2 C^2 \quad (21.68)$$

After integrating eq. (21.68) and equating it with eq. (21.67), we obtain

$$\frac{1}{2} k C^2 = \frac{1}{2} \omega^2 C^2 (m + \frac{m_s}{3}) \quad (21.69)$$

Solving for the natural frequency yields

$$\omega = \sqrt{\frac{k}{m + \frac{m_s}{3}}} \quad (21.70)$$

or in cycles per second (cps),

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m + \frac{m_s}{3}}} \quad (21.71)$$

The application of Rayleigh's Method shows that a better value for the natural frequency may be obtained by adding one-third of the mass of the spring to that of the main vibrating mass.

Rayleigh's Method may also be used to determine the natural frequency of a continuous system provided that the deformed shape of the structure is described as a generalized coordinate. The deformed shape of continuous structures and also of discrete structures of multiple degrees of freedom could in general be assumed

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value $(PE)_{\max}$ which is equal to the maximum potential energy is then

$$(PE)_{\max} = \frac{1}{2} FC = \frac{3EI}{2L^3} C^2 \quad (21.74)$$

since the force F is related to the maximum deflection by the formula from elementary strength of materials,

$$(PE)_{\max} = u = C = \frac{FL^3}{3EI} \quad (21.75)$$

The kinetic energy due to the distributed mass of the beam is given by

$$T = \int_0^L \frac{1}{2} \left(\frac{m_b}{L} \right) \dot{u}^2 dx \quad (21.76)$$

and using eq. (21.73) the maximum value for total kinetic energy will then be

$$T_{\max} = \frac{m_b}{2L} \int_0^L \left(\frac{3x^2L - x^3}{2L^3} \omega C \right)^2 dx + \frac{m}{2} \omega^2 C^2 \quad (21.77)$$

After integrating eq. (21.77) and equating it with eq. (21.74), we obtain

$$\frac{3EI}{2L^3} C^2 = \frac{1}{2} \omega^2 C^2 \left(m + \frac{33}{140} m_b \right) \quad (21.78)$$

and the natural frequency becomes

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{3EI}{L^3 \left(m + \frac{33}{140} m_b \right)}} \quad (21.79)$$

It is seen, then, that by concentrating a mass equal to $(33/140) m_b$ at the end of the beam, a more accurate value for the natural frequency of the cantilever beam is obtained compared to the result obtained by simply neglecting its distributed mass. In practice the fraction $33/140$ is rounded to $1/4$, thus approximating the natural frequency of a cantilever beam by

$$f = \frac{1}{2\pi} \sqrt{\frac{3EI}{L^3 \left(m + \frac{33}{140} m_b \right)}} \quad (21.80)$$

The approximation given by either eq. (21.79) or eq. (21.80) is a good one even for the case in which $m = 0$. For this case the error given by these formulas is about 1.5% compared to the exact solution which will be presented in Chapter 21.

Illustrative Example 21.6.

Consider in Fig. 21.14 the case of a simple beam carrying several concentrated

masses. Neglect the mass of the beam and determine an expression for the natural frequency by application of Rayleigh's Method.

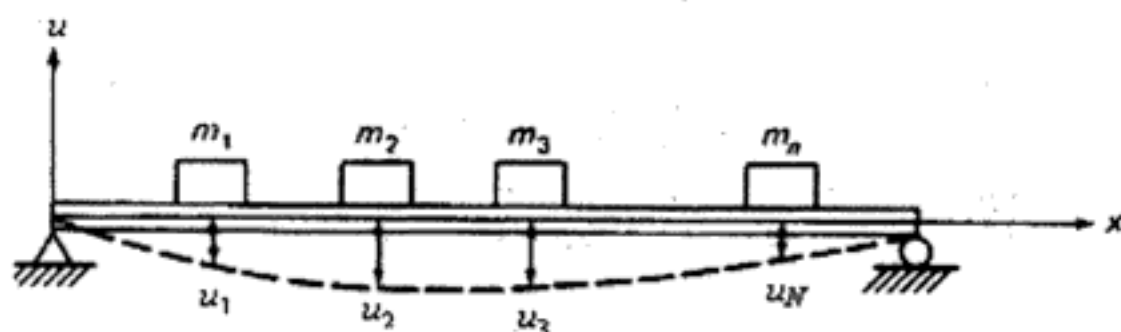


Fig. 21.14 Simple beam carrying concentrated masses.

Solution:

In the application of Rayleigh's Method, it is necessary to choose a suitable curve to represent the deformed shape that the beam will have during vibration. A choice of a shape that gives consistently good results is the curve produced by forces proportional to the magnitude of the masses acting on the structure. For the simple beam, these forces could be assumed to be the weights $W_1 = m_1g$, $W_2 = m_2g$, ..., $W_N = m_Ng$ due to gravitational action on the concentrated masses. The static deflections under these weights may then be designated by U_1 , U_2 , ..., U_N . The potential energy is then equal to work done during the loading of the beam, thus,

$$(PE)_{\max} = \frac{1}{2} W_1 u_1 + \frac{1}{2} W_2 u_2 + \dots + \frac{1}{2} W_N u_N \quad (21.81)$$

For harmonic motion in free vibration, the maximum velocities under the weights would be ωu_1 , ωu_2 , ..., ωu_N , and therefore the maximum kinetic energy would be

$$T_{\max} = \frac{1}{2} \frac{W_1}{g} (\omega u_1)^2 + \frac{1}{2} \frac{W_2}{g} (\omega u_2)^2 + \dots + \frac{W_N}{g} (\omega u_N)^2 \quad (21.82)$$

When the maximum potential energy, eq. (21.81), is equated with the maximum kinetic energy, eq. (21.82), the natural frequency is found to be

$$\omega = \sqrt{\frac{g(W_1 u_1 + W_2 u_2 + \dots + W_N u_N)}{W_1 u_1^2 + W_2 u_2^2 + \dots + W_N u_N^2}}$$

or

$$\omega = \sqrt{\frac{g \sum_{i=1}^N W_i u_i}{\sum_{i=1}^N W_i u_i^2}} \quad (21.83)$$

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21.8 Improved Rayleigh's Method

The concept of applying inertial forces as static loads in determining the deformed shape for Rayleigh's Method may be used in developing an improved scheme for the method. In the application of the improved Rayleigh's Method, one would start from an assumed deformation curve followed by the calculation of the maximum values for the kinetic energy and for the potential energy of the system. An approximate value for natural frequency is calculated by equating maximum kinetic energy with the maximum potential energy. Then an improved value for the natural frequency may be obtained by loading the structure with the inertial loads associated with the assumed deflection. This load results in a new deformed shape which is used in calculating the maximum potential energy. The method is better explained with the aid of numerical examples.

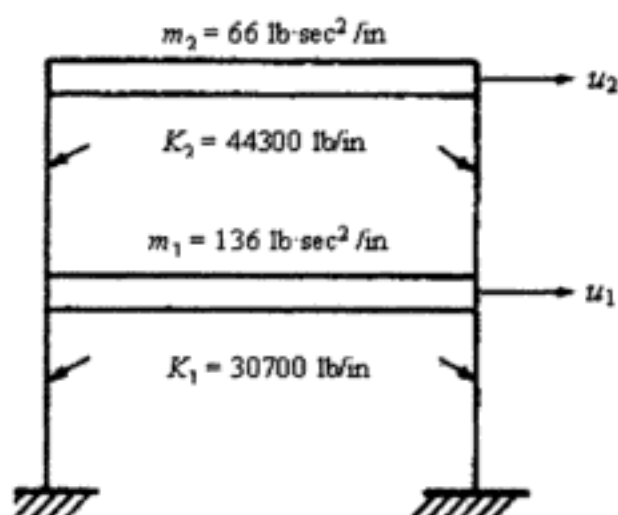


Fig. 21.16 Two-story frame for Example 21.7.

Illustrative Example 21.7.

By Rayleigh's Method, determine the natural frequency (lower or fundamental frequency) of the two-story frame shown in Fig. 21.16. Assume that the horizontal members are very rigid compared to the columns of the frame. This assumption reduces the system to only two degrees of freedom, indicated by coordinates u_1 and u_2 in the figure. The mass of the structure, which is lumped at the floor levels, has values $m_1 = 136 \text{ lb-sec}^2/\text{in}$ and $m_2 = 66 \text{ lb-sec}^2/\text{in}$. The total stiffness of the first story is $k_1 = 30,700 \text{ lb/in}$ and of the second story $k_2 = 44,300 \text{ lb/in}$, as indicated in Fig. 21.16.

Solution:

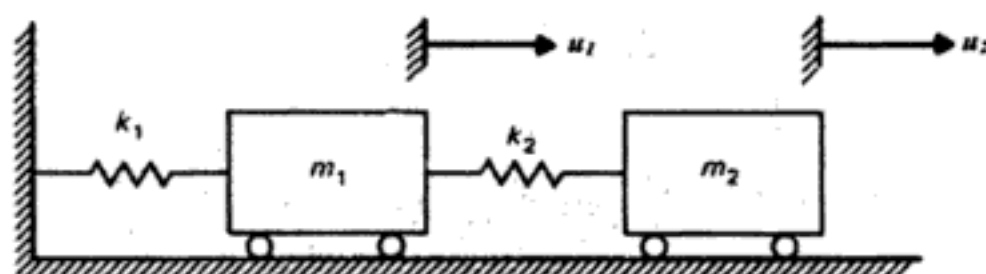
This structure may be modeled by the two mass systems shown in Fig. 21.17. In applying Rayleigh's Method, let us assume a deformed shape for which $u_1 = 1$ and $u_2 = 2$. The maximum potential energy is then

$$(PE)_{\max} = \frac{1}{2} K_1 u_1^2 + \frac{1}{2} K_2 (u_2 - u_1)^2$$

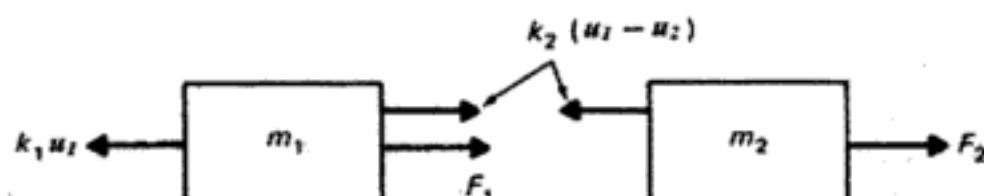
$$\begin{aligned}
 &= \frac{1}{2} (30,700)(1)^2 + \frac{1}{2} (44,300)(1)^2 \\
 &= 37,500 \text{ lb.in}
 \end{aligned}
 \tag{a}$$

and the maximum kinetic energy

$$\begin{aligned}
 T_{\max} &= \frac{1}{2} m_1 (\omega u_1)^2 + \frac{1}{2} m_2 (\omega u_2)^2 \\
 &= \frac{1}{2} (136) \omega^2 + \frac{1}{2} (66) (2\omega)^2 \\
 &= 200 \omega^2
 \end{aligned}
 \tag{b}$$



(a)



(b)

Fig. 21.17 Mathematical model for structure of Example 21.7.

Equating maximum potential energy with maximum kinetic energy and solving for the natural frequency gives

$$\omega = 13.69 \text{ rad/sec}$$

or

$$f = \frac{\omega}{2\pi} = 2.18 \text{ cps}$$

The natural frequency calculated as $f = 2.18$ cps is only an approximation to the exact value, since the deformed shape was assumed for the purpose of applying Rayleigh's Method. To improve this calculated value of the natural frequency, let us load the mathematical model in Fig. 21.17(a) with the inertial load calculated as

$$F_1 = m_1 \omega^2 u_1 = (136) (13.69)^2 (1) = 25,489$$

$$F_2 = m_2 \omega^2 u_2 = (66) (13.69)^2 (2) = 24,739$$

The equilibrium equations obtained by equating to zero the sum of the forces in the free body diagram shown in Fig. 21.17(b) gives

$$30,700 u_1 - 44,300 (u_2 - u_1) = 25,489$$

$$44,300 (u_2 - u_1) = 24,739$$

and solving

$$u_1 = 1.64$$

$$u_2 = 2.19$$

or in the ratio

$$u_1 = 1.00$$

$$u_2 = 1.34 \tag{c}$$

Introducing these improved values for the displacements u_1 and u_2 into eqs. (a) and (b) to recalculate the maximum potential energy and maximum kinetic energy results in

$$(PE)_{max} = 25 \tag{d}$$

$$T_{max} = 160.03 \omega^2 \tag{e}$$

and upon equating $(PE)_{max}$ and T_{max} , we obtain

$$\omega = 12.57 \text{ rad/sec}$$

or

$$f = 2.00 \text{ cps}$$

This last calculated value for the natural frequency $f = 2.00$ cps could be further improved by applying a new inertial load in the system based on the last value of the natural frequency and repeating anew cycle of calculations. Table 21.3 shows results obtained for four cycles.

The exact natural frequency and deformed shape, which are calculated for this system in Chapter 7, Example 7.1, as a two-degrees-of-freedom system, checks with the values obtained in the last cycle of the calculations shown in Table 21.3.

Table 21.3 Improved Rayleigh's Method Applied to Example 21.7

Cycle	Deformed Shape	Inertial Load		Natural Frequency
		F_1	F_2	
1	1:2.00			2.18 cps
2	1:1.34	25,489	24,739	2.00
3	1:1.32	21,489	18,725	1.88
4	1:1.27	19,091	12,230	1.88

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$$u_3 = \frac{92}{162} \frac{WL^3}{EI} \left[1 + 0.611 \left(\frac{D}{L} \right)^2 \right] \quad (f)$$

Table 21.4 Calculation of the Natural Frequency for the Shear Wall Modeled as Shown in Fig. 21.18

D/L *	u_1 ** (in)	u_2 ** (in)	u_3 ** (in)	ω *** (rad/sec)	f *** (cps)
0.00	0.09259	0.30247	0.56790	29.66	4.72
0.50	0.13600	0.37483	0.65465	27.67	4.40
1.00	0.26620	0.59193	0.91489	23.30	3.71
1.50	0.48322	0.95376	1.34862	19.05	3.03
2.00	0.78704	1.46032	1.95585	15.71	2.50
2.50	1.17765	2.11161	2.73658	13.21	2.10
3.00	1.65509	2.90764	3.69079	11.33	1.80

* $D/L = 0$ is equivalent, to neglect shear deformations.

** Factor of WL^3/EI .

*** Factor of $\sqrt{EI/WL^3}$.

The next illustrative example presents a table showing the relative importance that shear deformation has in calculating the natural frequency for a series of values of the ratio D/L .

Illustrative Example 21.10.

For the structure modeled as shown in Fig. 21.18, study the relative importance of shear deformation in calculating the natural frequency. Solution: In this study we will consider, for the wall, a range of values from 0 to 3.0 for the ratio D/L (depth-to-length ratio). The deflections u_1 , u_2 , u_3 at the floor levels are given by eqs. (1) of Example 21.9 and the natural frequency by eq. (21.83). The necessary calculations are conveniently shown in Table 21.4. It may be seen from the last column of Table 21.4, that for this example the natural frequency neglecting shear deformation ($D/L = 0$) is $f = 4.72 \sqrt{EI/WL^3}$ cps. For short walls ($D/L > 0.5$) the effect of shear deformation becomes increasingly important.

21.10 Summary

The concept of generalized coordinate presented in this chapter permits the analysis of multiple interconnected rigid or elastic bodies with distributed properties as single-degree-of-freedom systems. The analysis as one-degree-of freedom systems can be made provided that by the specification of a single coordinate (the generalized coordinate) the configuration of the whole system is determined. Such a system may

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Problem 21.2

Determine the generalized quantities M^* , C^* , K^* , and $F^*(t)$ for the structure shown in Fig. P21.2. Select $U(t)$ as the generalized coordinate.

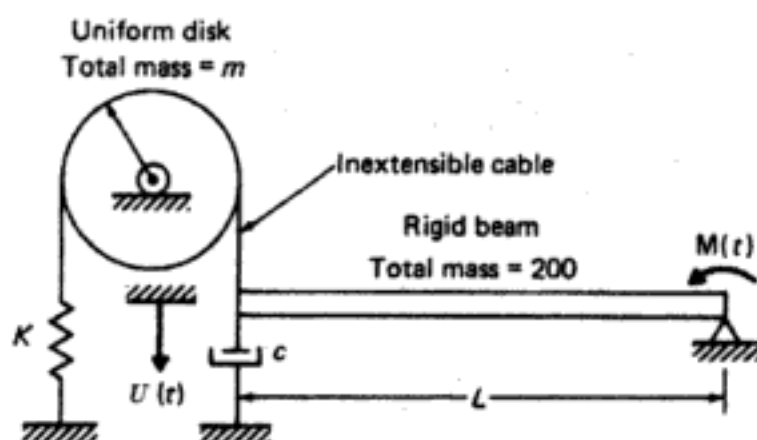


Fig. P21.2

Problem 21.3

Determine the generalized quantities M^* , C^* , K^* , and $F^*(t)$ for the structure shown in Fig. 216.3. Select $\theta(t)$ as the generalized coordinate.

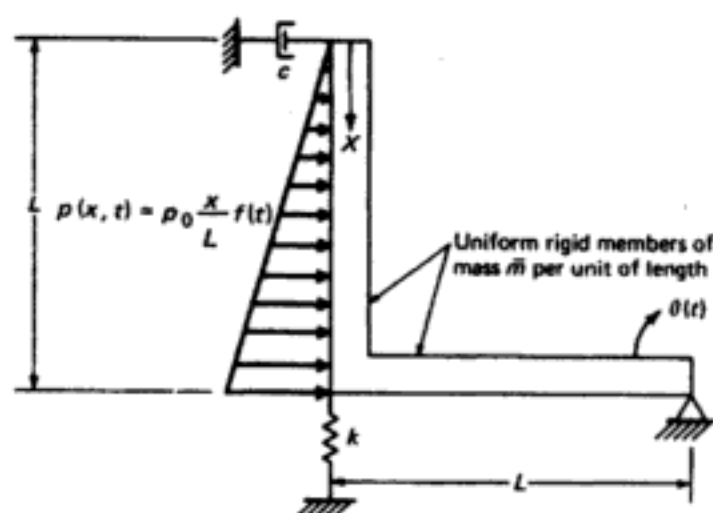


Fig. P21.3

Problem 21.4

For the elastic cantilever beam shown in Fig. P21.4, determine the generalized quantities M^* , K^* , and $F^*(t)$. Neglect damping. Assume that the deflected shape is given by $\phi(x) = 1 - \cos(\pi x/2L)$ and select $U(t)$ as the generalized coordinate as shown in Fig. P21.4. The beam is excited by a concentrated force $F(t) = F_0 f(t)$ at midspan.

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Problem 21.8

Determine the natural frequency of a simply supported beam with overhang which has a total uniformly distributed mass m_b , flexural rigidity EI , and dimensions shown in Fig. P21.8. Assume that during vibration the beam deflected curve is of the shape produced by a concentrated force applied at the free end of the beam.

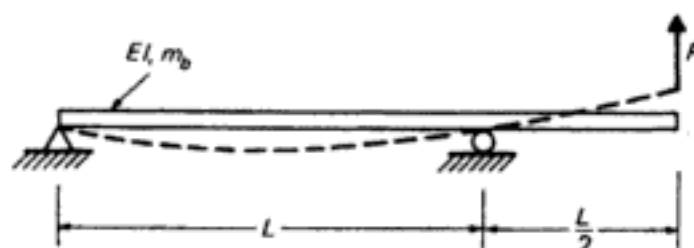


Fig. P21.8.

Problem 21.9

Determine the natural frequency of the simply supported beam shown in Fig. P21.9 using Rayleigh's Method. Assume the deflection curve given by $\phi(x) = U \sin \pi x / L$. The total mass of the beam is $m_b = 10 \text{ lb} \cdot \text{sec}^2/\text{in}$, flexural rigidity $EI = 10^8 \text{ lb} \cdot \text{in}^2$, and length $L = 100 \text{ in}$. The beam carries a concentrated mass at the center $m = 5 \text{ lb} \cdot \text{sec}^2/\text{in}$.

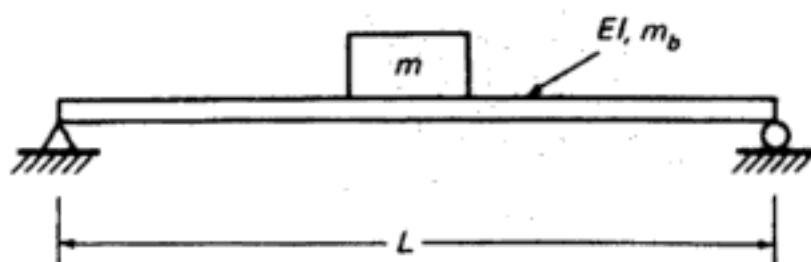


Fig. P21.9.

Problem 21.10

A two-story building is modeled as the frame shown in Fig. P21.10. Use Rayleigh's Method to determine the natural frequency of vibration for the case in which only flexural deformation needs to be considered. Neglect the mass of the columns and assume rigid floors. [Hint: Use eq. (21.83)].

Problem 21.11

Solve Problem 21.10 for the case in which the columns are short and only shear deformation needs to be considered. (The lateral force V for a fixed column of length L , cross-sectional area A , is approximately given by $V = AG\Delta/L$, where G is the shear modulus of elasticity and Δ the lateral deflection.)

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PART VI

Random Vibration

22 Random Vibration

The previous chapters of this book have dealt with the dynamic analysis of structures subjected to excitations which were known as a function of time. Such an analysis is said to be deterministic. When an excitation function applied to a structure has an irregular shape that is described indirectly by statistical means, we speak of a random vibration. Such a function is usually described as a continuous or discrete function of the exciting frequencies, in a manner similar to the description of a function by Fourier series. In structural dynamics, the random excitations most often encountered are either motion transmitted through the foundation or acoustic pressure. Both of these types of loading are usually generated by explosions occurring in the vicinity of the structure. Common sources of these explosions are construction work and mining. Other types of loading, such as earthquake excitation, may also be considered a random function of time. In these cases the structural response is obtained in probabilistic terms using random vibration theory.

A record of random vibration is a time function such as shown in Fig. 22.1. The main characteristic of such a random function is that its instantaneous value cannot be predicted in a deterministic sense. The description and analysis of random processes are established in a probabilistic sense for which it is necessary to use tools provided by the theory of statistics.

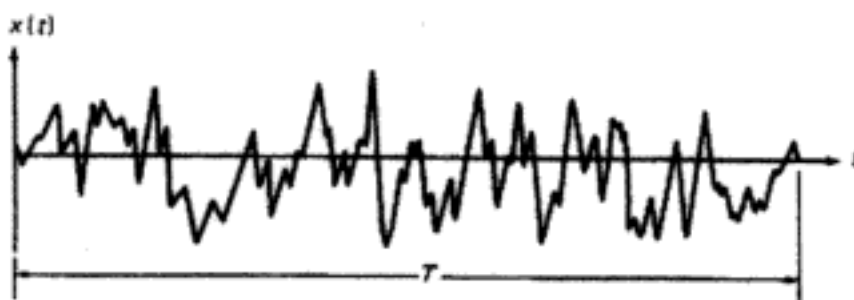


Fig. 22.1 Record of a random function of time

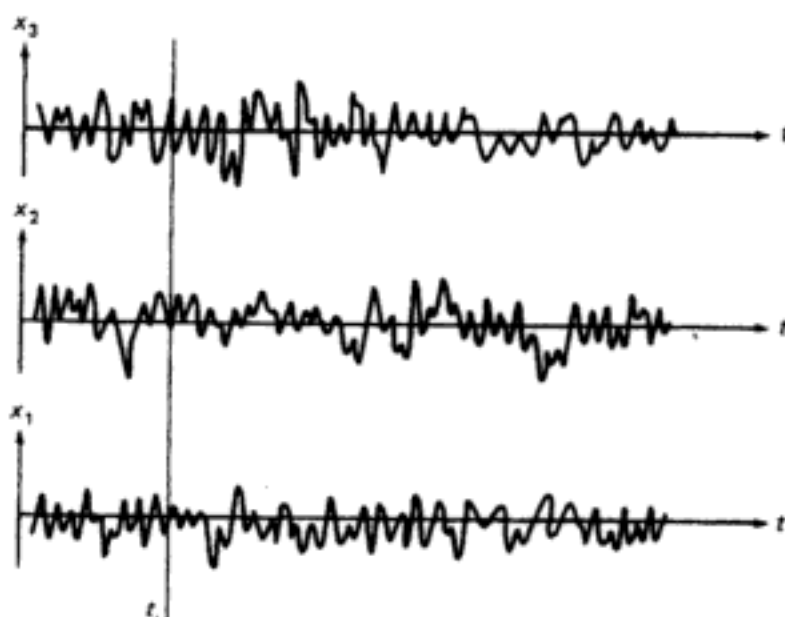


Fig. 22.2 An ensemble of random functions of time.

22.1 Statistical Description of Random Functions

In any statistical method a large number of responses is needed to describe a random function. For example, to establish the statistics of the foundation excitation due to explosions in the vicinity of a structure, many records of the type shown in Fig. 22.2 may be needed. Each record is called a sample, and the total collection of samples an ensemble. To describe an ensemble statistically, we can compute at any time t , the average value of the instantaneous displacements x . If such averages do not differ as we select different values of t , then the random process is said to be stationary. In addition, if the average obtained with respect to time for any member of the ensemble is equal to the average across the ensemble at an arbitrary time t , the random process is called ergodic. Thus in a stationary, ergodic process, a single record may be used to obtain the statistical description of a random function. We shall assume that all random process considered are stationary and ergodic. The random function of time shown in Fig. 22.1 has been recorded during an interval of time T . Several averages are useful in describing such a random function. The most common are the mean value \bar{x} which is defined as

$$\bar{x} = \frac{1}{T} \int_0^T x(t) dt \quad (22.1)$$

and the mean-square value $\overline{x^2}$ defined as

$$\overline{x^2} = \frac{1}{T} \int_0^T x^2(t) dt \quad (22.2)$$

Both the mean and the mean-square values provide measurements for the average value of the random function $x(t)$. The measure of how widely the function $x(t)$ differs from the average is given by its variance, σ_x^2 , defined as

$$\sigma_x^2 = \frac{1}{T} \int_0^T [x(t) - \bar{x}]^2 dt \quad (22.3)$$

When the expression under the integral is expanded and then integrated, we find that

$$\sigma_x^2 = \overline{x^2} - (\bar{x})^2 \quad (22.4)$$

which means that the variance can be calculated as the mean-square minus the square of the mean. Quite often the mean value is zero, in which case variance is equal to the mean square value. The root mean-square RMS_x of the random function $x(t)$ is defined as

$$RMS_x = \sqrt{\overline{x^2}} \quad (22.5)$$

The standard deviation σ_x of the random function $x(t)$ is the square root of the variance; hence from eq.(22.4)

$$\sigma_x = \sqrt{\overline{x^2} - (\bar{x})^2} \quad (22.6)$$

Illustrative Example 22.1.

Determine the mean value \bar{F} , the mean-square value $\overline{F^2}$, the variance σ_F^2 , and the root mean square values RMS_F of the forcing function $F(t)$ shown in Fig. 22.3.

Solution: Since the force $F(t)$ is periodic with period T , we can take the duration of the force equal to T ; hence by eq. (22.1) we have

$$\bar{F} = \frac{1}{T} \int_0^T F(t) dt = \frac{F_{\max}}{2}$$

and noting that

$$F(t) = \frac{2F_{\max}}{T} t \quad \text{for} \quad 0 < t < \frac{T}{2}$$

we obtain by eq. (22.2)

$$\overline{F^2} = \frac{2}{T} \int_0^{\frac{T}{2}} \left(\frac{2F_{\max}}{T} t \right)^2 t^2 dt = \frac{F_{\max}^2}{3}$$

The variance may now be calculated from eq. (22.4) as

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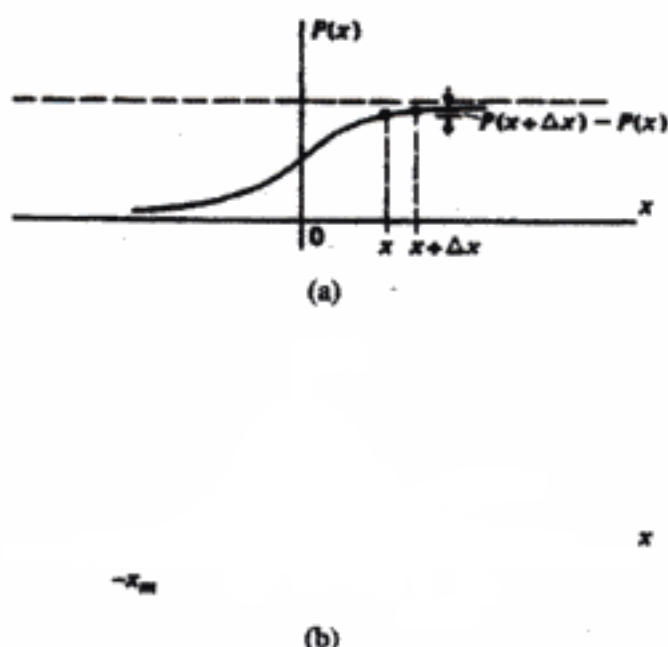


Fig. 22.5 (a) Cumulative probability function $P(x)$ and
(b) Probability density function $p(x)$ of the random variable $x = x(t)$

22.3 The Normal Distribution

The most commonly used probability density function is the normal distribution, also referred to as the Gaussian distribution, expressed by

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\bar{x})^2/\sigma^2} \quad (22.12)$$

Figure 22.6 shows the shape of this function. It may be observed that the normal distribution function is symmetric about the mean value \bar{x} . In Fig. 22.7 the standard normal distribution is plotted non-dimensionally in terms of $(x - \bar{x})/\sigma$. Values of $P(-\infty, x_2)$ [$x_1 = -\infty$ in eq.(22.10)] are tabulated in many sources including mathematical handbook.¹ The probability of x being between $x - \lambda\sigma$ and $x + \lambda\sigma$, where λ is any positive number, is given by the equation.

$$P[\bar{x} - \lambda\sigma < x < \bar{x} + \lambda\sigma] = \frac{1}{\sqrt{2\pi}\sigma} \int_{\bar{x}-\lambda\sigma}^{\bar{x}+\lambda\sigma} e^{-\frac{1}{2}(x-\bar{x})^2/\sigma^2} dx \quad (22.13)$$

Equation (22.13) represents the probability that x lies within λ standard deviations from \bar{x} . The probability of x lying more than λ standard deviations from \bar{x} is the probability of $|x - \bar{x}|$ exceeding $\lambda\sigma$, which is 1.0 minus the value given by eq.(22.13). The following table presents numerical values for the normal distribution associated with $\lambda = 1, 2$, and 3:

¹ *Standard mathematical Tables*, The Chemical Rubber Co. (CRC) 20th Ed. 1972, pp 566-575.

$P(x)$ \bar{x} x_1, x_2 x

Fig. 22.6 Normal probability density function.

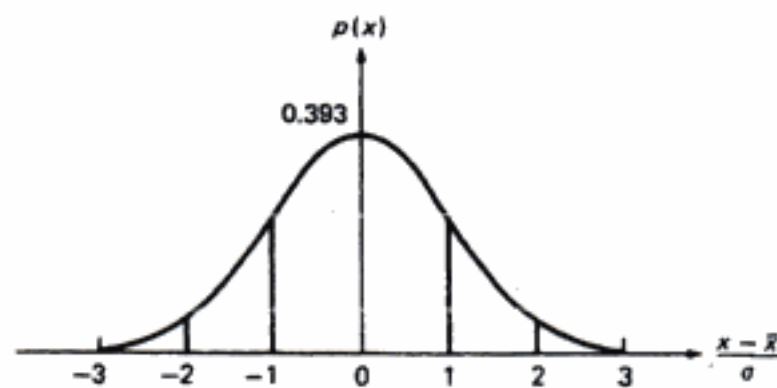


Fig. 22.7 Standard normal probability density function.

λ	$P[\bar{x} - \lambda \sigma < x < \bar{x} + \lambda \sigma]$	$P[x - \bar{x}] > \lambda \sigma$
1	68.3%	31.7%
2	95.4%	4.6%
3	99.7%	0.3%

22.4 The Rayleigh Distribution

Variables that are positive, such as the absolute value A of the peaks of vibration of a random function $x(t)$, often tend to follow the Rayleigh distribution, which is defined by the equation

$$p(A) = \frac{A}{\sigma^2} e^{-A^2/2\sigma^2}, \quad A > 0 \quad (22.14)$$

where σ is a parameter that may be interpreted as the standard deviation of a function $x(t)$.

The probability density $p(A)$ is zero for $A < 0$ and has the shape shown in Fig. 22.8 for positive values A .

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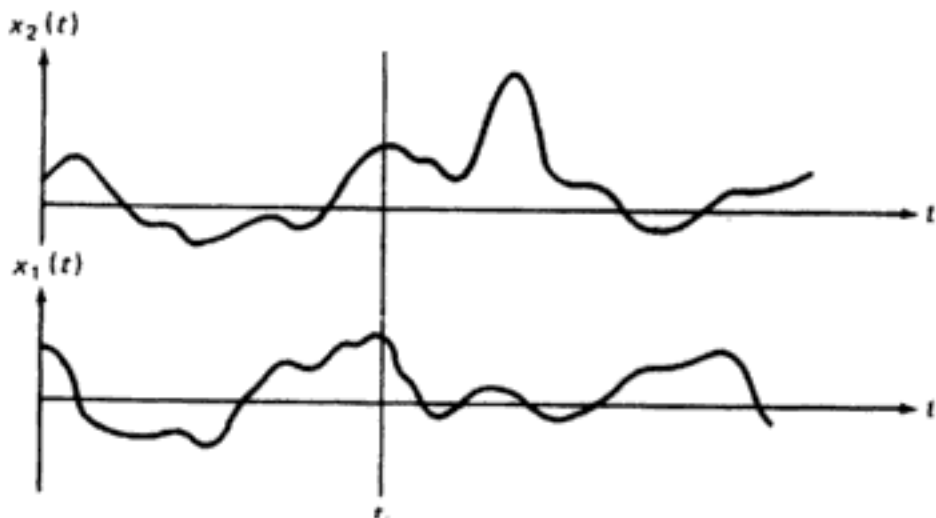


Fig. 22.9 Correlation between $x_1(t)$ and $x_2(t)$.

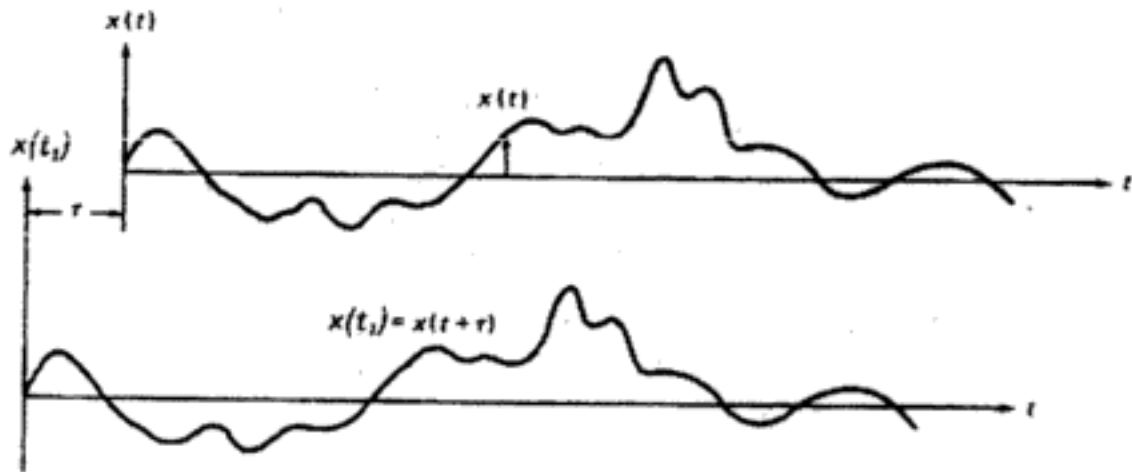


Fig. 22.10 Autocorrelation between $x(t)$ and $x(t + \tau)$.

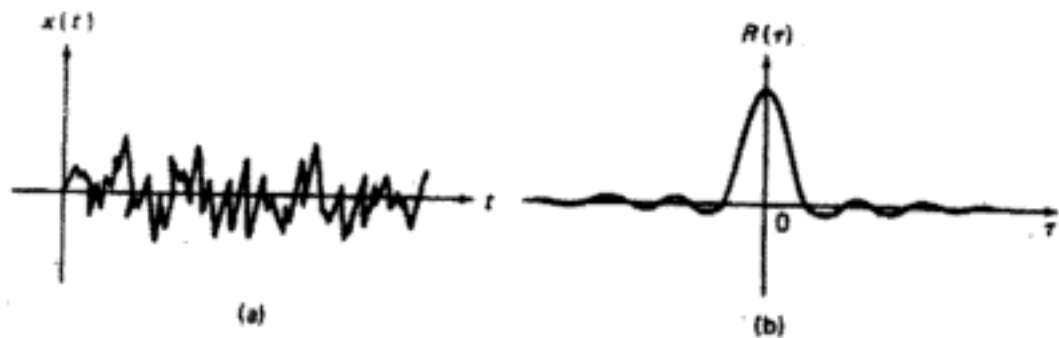


Fig. 22.11 Wide-band random process $x(t)$ and its autocorrelation function $R(\tau)$.

Since the second record of Fig. 22.10 can be considered to be delayed with respect to the first record, or the first record advanced with respect to the second record, it is evident that $R(\tau) = R(-\tau)$ is symmetric about the R axis and that $R(\tau)$ is always less than $R(0)$.

Highly random or wide-band functions such as the one shown in Fig. 22.11(a) lose their similarity within a short time shift. The autocorrelation of such a function, therefore, is a sharp spike at $\tau = 0$ that drops off rapidly as τ moves away from zero, as shown in Fig. 22.11(b). For the narrow-band record containing a dominant frequency as shown in Fig. 22.12(a), the autocorrelation has the characteristics indicated in Fig. 22.12(b) in that it is a symmetric function with a maximum at $\tau = 0$ and frequency ω_0 corresponding to the dominant frequency of $x(t)$.

Illustrative Example 22.2.

For the function depicted in Fig. 22.13, determine: (a) the mean, (b) the mean square value, and (c) the autocorrelation function. Also, plot the autocorrelation function.

Solution: because the function $x(t)$ is periodic, averages calculated over a long time approach those calculated over a single period. Considering the period between 0 and T , the function is described analytically by

$$\begin{aligned} x(t) &= x_0 & 0 < x < T/2 \\ &= 0 & T/2 < x < T \end{aligned} \quad (a)$$

(a) Mean value: by eq. (22.1)

$$\bar{x} = \frac{1}{T} \int_0^T x_0 dt = \frac{x_0}{2} \quad (b)$$

(b) Mean square value: by eq. (22.2)

$$\overline{x^2} = \frac{1}{T} \int_0^T x_0^2 dt = \frac{x_0^2}{2} \quad (c)$$

(c) Autocorrelation function:

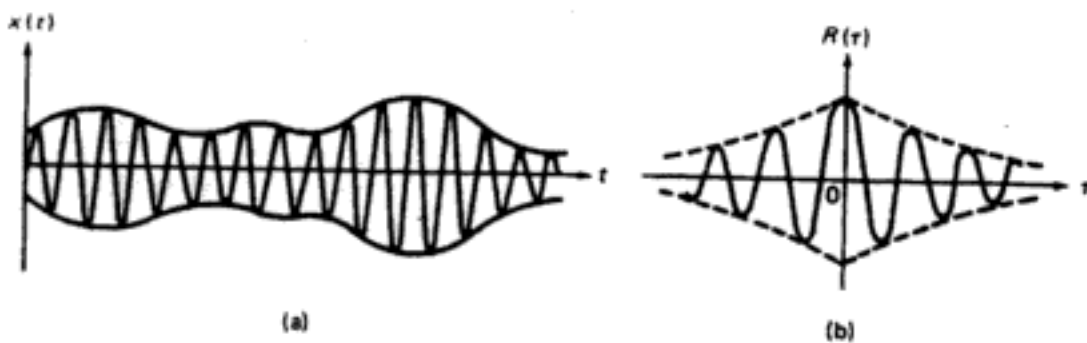


Fig. 22.12 Narrow-band random process $x(t)$ and its autocorrelation function $R(T)$.

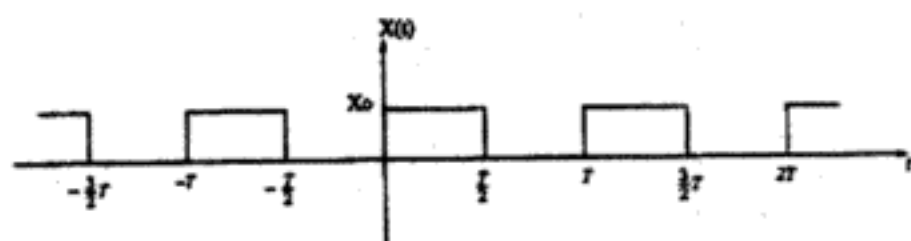


Fig. 22.13 Periodic function $x(t)$ of Illustrative Example 22.2.

To calculate the autocorrelation function, it is necessary to distinguish between the time shift $0 < \tau < T/2$ and $T/2 < \tau < T$ as shown, respectively, in Figs 22.14(a) and 22.14(b).

- (d) Using eq. (22.19) and considering Fig. 22.14(a), we obtain
for $0 < \tau < T/2$

$$\begin{aligned} R(\tau) &= \frac{1}{T} \int_0^T x(t) \cdot x(t + \tau) dt = \int_0^{T-\tau} x_0 \cdot x_0 dt \\ &= \frac{x_0^2}{T} \left(\frac{T}{2} - \tau \right) = x_0^2 \left(\frac{1}{2} - \frac{\tau}{T} \right) \quad \text{for } 0 < \tau < \frac{T}{2} \end{aligned} \quad (d)$$

in which the limits of integration are given by the overlapping portions of $x(t)$ and $x(t + T)$ as indicated by the shaded area in Fig. 22.14(a).

When $T/2 < \tau < T$, we obtain from Fig. 22.14(b)

$$\begin{aligned} R(\tau) &= \frac{1}{T} \int_{-\tau}^{\frac{T}{2}} x_0 \cdot x_0 dt = \frac{x_0^2}{T} \left[\frac{T}{2} - T + \tau \right] \\ &= x_0^2 \left[-\frac{1}{2} + \frac{\tau}{T} \right] \quad \text{for } \frac{T}{2} < \tau < T \end{aligned} \quad (e)$$

From examination of Fig. 22.14, it can be concluded that the autocorrelation function $R(\tau)$ for the function $x(t)$ must be periodic in t with period T . Therefore, from eq. (d), (e), and the fact that $R(\tau)$ is periodic, we can plot the autocorrelation function as shown in Fig. 22.15.

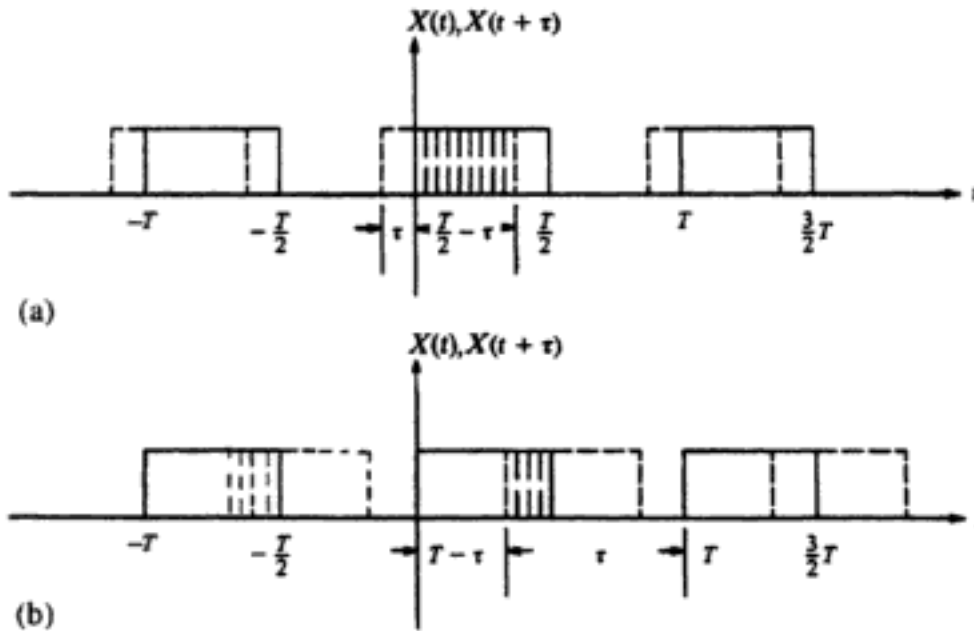


Fig. 22.14 Plot of $x(t)$ and $x(t+\tau)$; (a) for $0 < \tau < T/2$. (b) for $T/2 < \tau < T$.

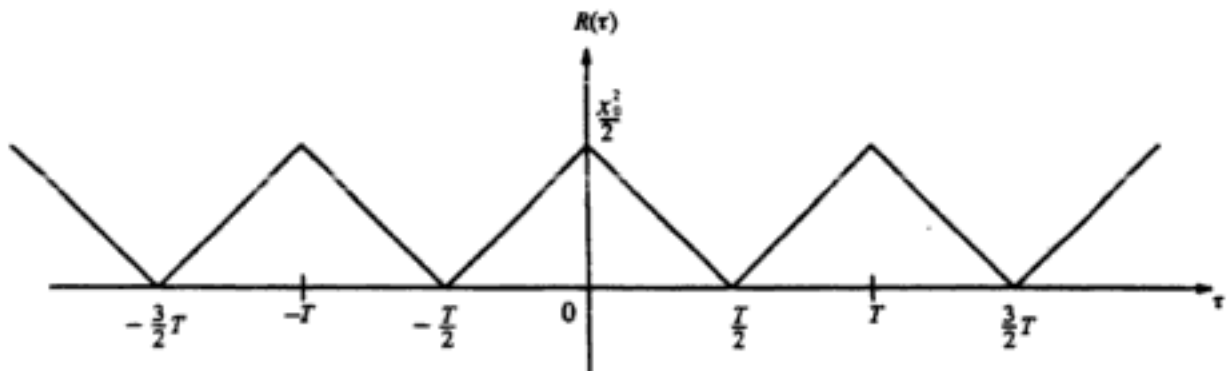


Fig. 22.15 Autocorrelation function $R(\tau)$ for the function of Illustrative Example 22.2.

22.6 The Fourier Transform

In Chapter 19, we used Fourier series to obtain the frequency components of periodic functions of time. In general, random vibrations are not periodic, and the frequency analysis requires the extension of Fourier series to the Fourier integral for non-periodic functions. Fourier transforms, which result from Fourier integrals, enable a more extensive treatment of the random vibration problem.

We begin by showing that the Fourier integral can be viewed as a limiting case of the Fourier series as the period goes to infinity. Toward this objective we consider the Fourier series in exponential form and substitute the coefficient C_n given by eq. (19.20) into eq. (19.19) to obtain:

$$F(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} F(\tau) \cdot e^{-in\omega\tau} \cdot e^{in\omega t} d\tau \quad (22.21)$$

In eq. (22.21) we have selected the integration period from $-T/2$ to $T/2$ and substituted the symbol τ for t as the dummy variable of integration. The frequency $\omega = n\varpi$ is specified here at discrete, equally spaced values separated by the increment

$$\Delta\omega = (n+1)\varpi - n\varpi = \varpi = \frac{2\pi}{T}$$

We substitute in eq. (22.21) ω for $n\varpi$ and $\Delta\omega/2\pi$ for $1/T$ and notice that as $T \rightarrow \infty$, $\Delta\omega \rightarrow d\omega$. Thus, in the limit, eq. (22.21) becomes

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} F(\tau) e^{-i\omega\tau} d\tau \right\} e^{i\omega t} d\omega \quad (22.22)$$

which is the Fourier integral of $F(t)$.

Since the function within the inner braces is a function of only ω , we can write this equation in two parts as

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt \quad (22.23)$$

and

$$F(t) = \int_{-\infty}^{\infty} C(\omega) e^{i\omega t} d\omega \quad (22.24)$$

The validity of these relations, according to classical Fourier transform theory, is subject to the condition that

$$\int_{-\infty}^{\infty} |F(t)| dt < \infty \quad (22.25)$$

The function $C(\omega)$ in eq. (22.23) is the Fourier transform of $F(t)$ and the function $F(t)$ in eq. (22.24) is the inverse Fourier transform of $C(\omega)$. The pair of functions $F(t)$ and $C(\omega)$ is referred to as a Fourier transform pair. Equation (22.23) resolves the function $F(t)$ into harmonic components $C(\omega)$, whereas eq. (22.24) synthesizes $F(t)$ from these harmonic components. In practice it is more convenient to use frequency f in cps rather than the angular frequency ω in rad/sec. Mathematically, since $\omega = 2\pi f$ and $d\omega = 2\pi df$, this also has the advantage of reducing the Fourier transform pair into a more symmetric form, namely,

$$F(t) = \int_{-\infty}^{\infty} C(f) e^{i2\pi ft} df \quad (22.26)$$

and

$$C(f) = \int_{-\infty}^{\infty} F(t) e^{-i2\pi ft} dt \quad (22.27)$$

22.7 Spectral Analysis

We have seen in Chapter 19 that the application of Fourier analysis to a periodic function yields the frequency components of the function given by either trigonometric terms [eq. (19.2)] or exponential terms [eq. (19.19)]. When the periodic function is known at N discrete, equally spaced times, the frequency components are then given by the terms in eq. (19.28). Our purpose in this section is to relate the Fourier analysis for a given function $x(t)$ to its mean square value $\overline{x^2}$.

The contributions of the frequency components of $x(t)$ to the value $\overline{x^2}$ are referred to as the spectral function of $x(t)$. Hence spectral analysis consists in expressing $\overline{x^2}$ in terms of either the coefficients of the Fourier series (i.e., a_n and b_n or equivalently C_n) when $x(t)$ is periodic, or in terms of the Fourier transform [i.e., $C(\omega)$] when $x(t)$ is not a periodic function.

We begin by performing the spectral analysis of a periodic function $x(t)$ expressed in Fourier series, eq. (19.2), which is

$$x(t) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\} \quad (22.28)$$

where the coefficients are given by eq. (19.3) as

$$\begin{aligned} a_0 &= \frac{1}{T} \int_0^T x(t) dt \\ a_n &= \frac{2}{T} \int_0^T x(t) \cos n\omega t dt \\ b_n &= \frac{2}{T} \int_0^T x(t) \sin n\omega t dt \end{aligned} \quad (22.29)$$

In eq. (22.29), T is the period of the function and $\omega = 2\pi/T$ its frequency. The substitution of $x(t)$ from eq. (22.28) for one of the factors in the definition of mean square value gives

$$\begin{aligned} \overline{x^2} &= \frac{1}{T} \int_0^T x^2(t) dt \\ &= \frac{1}{T} \int_0^T x(t) \left\{ a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \right\} dt \\ &= \frac{a_0}{T} \int_0^T x(t) dt + \sum_{n=1}^{\infty} \left[\frac{a_n}{T} \int_0^T x(t) \cos n\omega t dt + \frac{b_n}{T} \int_0^T x(t) \sin n\omega t dt \right] \end{aligned}$$

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$$\begin{aligned}
 \overline{F^2} &= \frac{1}{N} \sum_{j=0}^{N-1} F(t_j) \sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)} \\
 &= \sum_{n=0}^{N-1} C_n \left[\frac{1}{N} \sum_{j=0}^{N-1} F(t_j) e^{2\pi i(nj/N)} \right] \\
 &= \sum_{j=0}^{N-1} C_n C_n^*
 \end{aligned} \tag{22.34}$$

where C_n^* is the complex conjugate² of C_n . Hence

$$\overline{F^2} = \sum_{n=0}^{N-1} |C_n|^2 = |C_0|^2 + |C_1|^2 + |C_2|^2 + \dots + |C_{N-1}|^2 \tag{22.35}$$

The terms of the summation in eq. (22.35) are the required spectrum of the discrete function $F(t_j)$; that is, these terms are the frequency contributions to the mean square value $\overline{F^2}$. As we can see in eq. (22.35), the contribution of each frequency is equal to the square of the modulus of the corresponding complex coefficient C_n which is given by eq. (22.32).

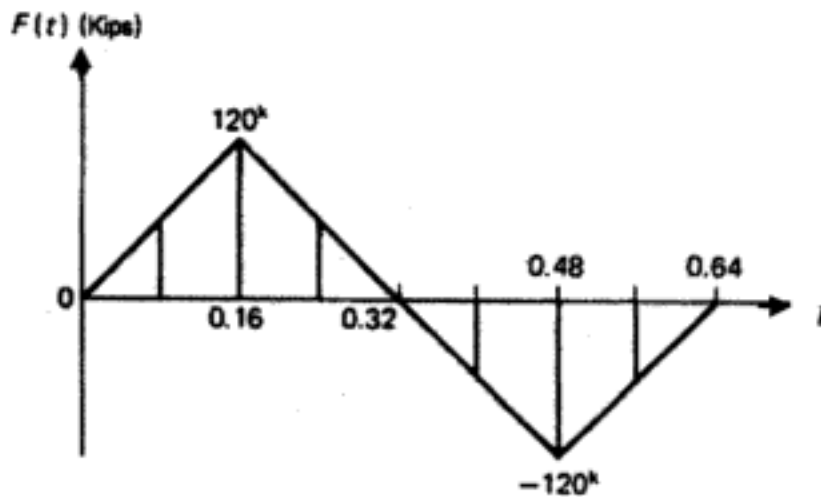


Fig. 22.16 Forcing function for Illustrative Example 22.3.

Illustrative Example 22.3.

Determine the spectrum of the forcing function $F(t)$ shown in Fig. 22.16. Assume that the function is defined at eight equally spaced time intervals. Use the spectrum to estimate the mean square value $\overline{F^2}$ and compare this result with the mean square calculated directly from the definition of $\overline{F^2}$.

² Since $F(t_j)$ is a real function, its conjugate $F(t_j)^* = F(t_j)$ and also $[e^{-2\pi i(nj/N)}]^* = e^{2\pi i(nj/N)}$, it follows

$$\text{that } \frac{1}{N} \sum_{j=0}^{N-1} F(t_j) e^{2\pi i(nj/N)} = C_n^*$$

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necessary to use more time intervals in the discrete Fourier Series. Table 22.2 shows results obtained using Program 4 with $M = 3, 4, 5, 6, 7$, corresponding to $N = 2^M = 8, 16, 32, 64, 128$ time intervals for the function $F(t)$ shown in Fig. 22.16. It may be observed that values displayed in the last column of the table are converging to the exact value $\overline{F^2} = 0.4800 \text{ E10}$ as the number of time intervals, N , is increased.

22.8 Spectral Density Function

If a random process $x(t)$ is normalized (or adjusted) so that the mean value of the process is zero, then, provided that $x(t)$ has no periodic components, the autocorrelation function $R_x(\tau)$ approaches zero as τ increases, that is,

$$\lim_{\tau \rightarrow \infty} R_x(\tau) = 0$$

We therefore expect that $R_x(\tau)$ should satisfy the condition in eq. (22.25). We can then use eqs. (22.23) and (22.24) to obtain the Fourier transform $S_x(\omega)$ of the autocorrelation function $R_x(\tau)$ and its inverse as

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} d\tau \quad (22.36a)$$

and

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega) e^{i\omega\tau} d\omega \quad (22.36b)$$

In eq.(22.36a), $S_x(\omega)$ is called the spectral density function of $x(t)$. The most important property of $S_x(\omega)$ becomes apparent by letting $\tau = 0$ in eq. (22.36b). In this case

$$R_x(0) = \int_{-\infty}^{\infty} S_x(\omega) d\omega \quad (22.37)$$

which by eq. (22.20) is equal to mean square value of the function $x(t)$, that is,

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(\omega) d\omega \quad (22.38)$$

The mean square value of a random process is therefore given by the area under the graph of the spectral density function as shown in Fig. 22.17. Consequently, the contribution of an incremental frequency $\Delta\omega$ to the mean square value is

$$\Delta \overline{x^2} = S_x(\omega) \Delta\omega \quad (22.39)$$

The Fourier transform pair for the discrete autocorrelation function $R_j = R(\tau)$ and for the discrete spectral density function $S_n = S(\omega_n)$ of a time function $x = x(t)$, is defined from eqs. (19.27) and (19.28) as

$$R_j = \sum_{n=0}^{N-1} S_n e^{j\omega_j n / N} \quad (22.40a)$$

and

$$S_n = \frac{1}{N} \sum_{j=0}^{N-1} R_j e^{-2\pi j(n/N)} \quad (22.40b)$$

It may be seen then, that the terms in eq.(22.40b) provide the contributions, $\Delta \overline{x^2}$, to mean square value $\overline{x^2}$, at the frequency ω_n as

$$\Delta \overline{x^2} = S_n \quad (22.41)$$

The contribution to the mean square value may also be expressed, in terms of the Fourier transform coefficient C_n and its conjugate C_n^* , using eq. (22.34) as

$$\Delta \overline{x^2} = C_n C_n^* \quad (22.42)$$

Therefore, it follows from eqs. (22.41) and (22.42) that.

$$S_n = C_n C_n^* \quad (22.43)$$

The spectral density of a given record can be obtained electronically by an instrument called a frequency analyzer or spectral density analyzer. The output of an accelerometer or other vibration transducer is fed into the instrument, which is essentially a variable frequency narrow-band filter with a spectral meter to display the filtered output. With this instrument the experimenter searches for the predominant frequencies present in a vibration signal. The output of the spectral density analyzer is the contribution to the mean square value $\overline{x^2}$ of the input signal $x(t)$ for a small range $\Delta\omega$ around the set frequency.

When dealing with theory, the natural unit for the frequency is rad/sec. However, in most practical problems the frequency is expressed in cycles per second or Hertz (abbreviated Hz). In the latter case, we rewrite eq. (22.39) as

$$\Delta \overline{x^2} = S_x(f) \Delta f \quad (22.44)$$

where f is the frequency in Hertz. Since $\Delta\omega = 2\pi\Delta f$, it follows from eqs. (22.39) and (22.44) that

$$S_x(f) = 2\pi S_x(\omega) \quad (22.45)$$

When the spectral density function for the excitation is known, its mean-square value may be determined from eq. (22.44) as

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$$\begin{aligned}
 R_x(\tau) &= 2 \int_0^{\infty} S_x(\omega) \cos \omega \tau d\omega \\
 &= 2 \int_{\omega_1}^{\omega_1 + \Delta\omega} S_0 \cos \omega \tau d\omega \\
 &= \frac{2S_0}{\tau} \left[\sin \omega \tau \right]_{\omega_1}^{\omega_1 + \Delta\omega} \\
 &= \frac{2S_0}{\tau} [\sin(\omega_1 + \Delta\omega)\tau - \sin \omega_1 \tau] \\
 R_x(\tau) &= \frac{4S_0}{\tau} \cos\left(\omega_1 + \frac{\Delta\omega}{2}\right)\tau \cdot \sin \frac{\Delta\omega}{2} \tau
 \end{aligned}
 \tag{a}$$

$$\tag{b}$$

The autocorrelation function for a narrow-band random process given by eq. (b) has the form shown in Fig. 22.20, where the predominant frequency of $R_x(\tau)$ is the average value $(\omega_1 + \Delta\omega/2)$. The autocorrelation for such a process has a maximum of $2S_0 \Delta\omega$ when $\tau = 0$ and decreases like a cosine graph as τ moves away from $\tau = 0$.

The autocorrelation function $R_x(\tau)$ for a wide-band random process whose spectral density function extends in the range ω_1 to ω_2 as shown in Fig. 22.19(b) can be obtained from the result of Illustrative Example 22.4. In this case, letting the lower frequency $\omega_1 = 0$ and the upper frequency $\Delta\omega = \omega_2$ we obtain from eq.(b) of Illustrative Example 22.4

$$R_x(\tau) = \frac{4S_0}{\tau} \cos\left(\frac{\omega_2 \tau}{2}\right) \sin\left(\frac{\omega_2 \tau}{2}\right) = \frac{2S_0}{\tau} \sin \omega_2 \tau \tag{22.51}$$

which has the form shown in Fig. 22.22(a).

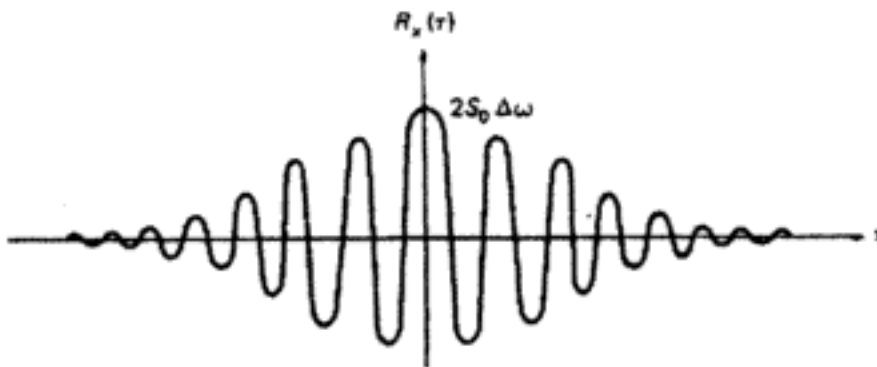


Fig. 22.20 Autocorrelation for a narrow-band random process

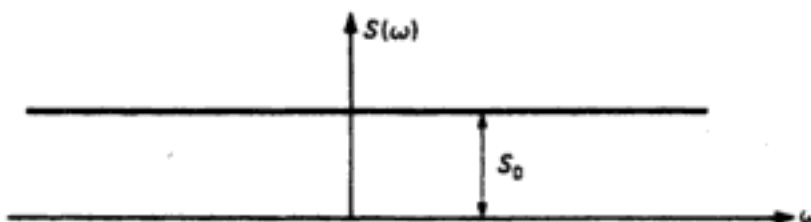


Fig. 22.21 Spectral density function for Illustrative Example 22.5.

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which by eq. (22.39), may be expressed as

$$\overline{\Delta F^2} = S_F(\omega_n) \Delta \omega \quad (22.64)$$

Now, using eqs. (22.63) and (22.64), we may write eq. (22.62) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 S_F(\omega_n) \Delta \omega \quad (22.65)$$

or as

$$\overline{u^2} = \sum_{n=0}^{N-1} S_u(\omega_n) \Delta \omega \quad (22.66)$$

where

$$S_u(\omega_n) = |H_n|^2 S_F(\omega_n) \quad (22.67)$$

When the frequency is expressed in cps, we may write eq. (22.66) as

$$\overline{u^2} = \sum_{n=0}^{N-1} S_u(f_n) \Delta f \quad (22.68)$$

where

$$S_u(f_n) = |H_n|^2 S_F(f_n) \quad (22.69)$$

Equation (22.69) states the important result that, when the frequency response function, H_n is known, the spectral density $S_u(f_n)$ for the response can be calculated from the spectral density, $S_F(f_n)$, of the excitation.

Illustrative Example 22.6.

Determine the mean square value of the response for the single-degree-of-freedom system in which $k = 100,000$ lb/in, $m = 100$ lb · sec²/in, $c = 632$ lb · sec/in subjected to the $F(t)$ shown in Fig. 22.16. Choose $N = 8$ for the number of intervals.

Solution: The mean square value $\overline{u^2}$ of the response is given by eq. (22.62) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 |C_n|^2$$

From eq. (22.61)

$$|H_n|^2 = \frac{1}{k^2} \cdot \frac{1}{(1-r_n^2)^2 + (2r_n\xi_n)^2}$$

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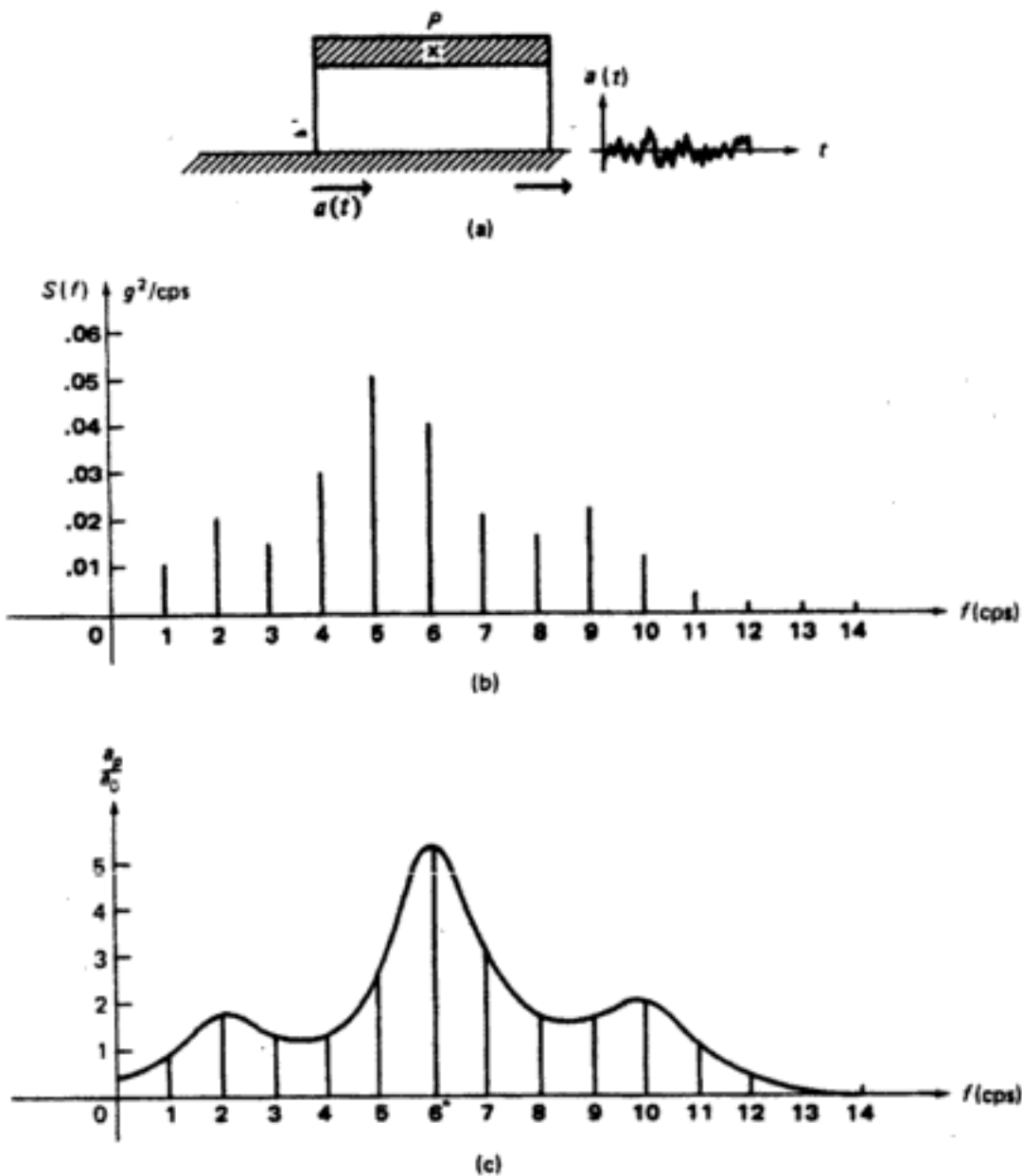


Fig. 22.24 (a) Structure subjected to random acceleration at the base. (b) Spectral density function of the excitation. (c) Relative frequency response at point P .

Similarly, the probability that the peak acceleration A_p at point P will exceed a specified value (1σ or 3σ) is found using the Rayleigh distribution (see Table in Sec. 22.4) as

$$P[A_p > 1.345g] = 60.65\%$$
$$P[A_p > 4.041g] = 1.11\%$$

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$$F(t) = \int_{-\infty}^{\infty} C(\omega) e^{i\omega t} d\omega \quad (22.24)$$

repeated

where the coefficient $C(\omega)$ is given by Fourier transform, eq. (22.23) as

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} d\omega \quad (22.23)$$

repeated

The differential equation of motion for the system shown in Fig. 22.25 is

$$m\ddot{u} + c\dot{u} + ku = F(t) \quad (22.71)$$

which for an excitation of a unit exponential function, $F(t) = e^{i\omega t}$, as presented in Chapter 3, has a steady-state solution, $u_1(t)$ of the form

$$u_1(t) = H(\omega) e^{i\omega t} \quad (22.72)$$

in which

$$H(\omega) = \frac{1}{k - m\omega^2 + ic\omega}$$

Consequently, the response $u(t)$ due to the force expressed by the inverse of Fourier transform in eq. (22.24) is given by

$$u(t) = \int_{-\infty}^{\infty} C(\omega) H(\omega) e^{i\omega t} d\omega \quad (22.73)$$

Consider now the special case where the total excitation consists of a single impulse of magnitude 1 applied at time $t = 0$.

In this case, $C(\omega)$ in eq. (22.23) becomes

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt = \frac{1}{2\pi} \quad (22.74)$$

because $e^{-i\omega t} = 1$ at $t = 0$ and

$$\int_{-\infty}^{\infty} F(t) dt = 1 \quad (22.75)$$

for a unit impulse.

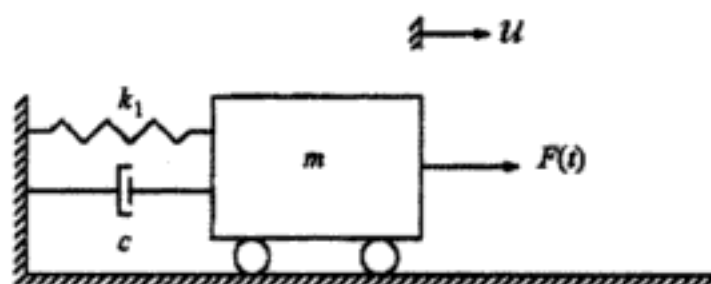


Fig. 22.25 Model of a damped single-degree-of-freedom system.

The expression given by eq. (22.75) is known as the Dirac's δ function. Thus, from eq. (22.73) and (22.74), the response, $h(t)$, to unit impulse is given by

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega \quad (22.76)$$

which is recognized as the inverse Fourier transform of

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt \quad (22.77)$$

We observe that the unit impulse response $h(t)$ of eq. (22.76) and the complex frequency response $H(\omega)$ of eq. (22.77) form a Fourier transform pair.

22.11.2 Response to Random Excitation: Two-Degree-of-Freedom System

The concepts and methodology for determining the response of multiple-degree-of-freedom systems can be adequately presented through the treatment of a two-degree-of-freedom system. We begin by presenting a two-degree-of-freedom system subjected to a single random excitation. This case is then extended by subjecting this two-degree-of-freedom system to a second excitation, in addition to the first random excitation. Finally, we present the general case of a multiple-degree-of-freedom system subjected to multiple random excitations.

We consider in Fig. 22.26(a) the structural model of a two-degree-of-freedom system subjected to a random stationary force $F_2(t)$ applied at coordinate u_2 . The equations obtained by establishing the dynamic equilibrium in the free-body-diagram in Fig. 22.26(b) are:

$$\begin{aligned} m_1 \ddot{u}_1 + (k_1 + k_2)u_1 - k_2 u_2 &= 0 \\ m_2 \ddot{u}_2 - k_1 u_1 + k_2 u_2 &= F_2(t) \end{aligned} \quad (22.78)$$

The natural frequencies and modal shapes are then found, as presented in Chapter 11, by setting $F_2(t) = 0$ in eq. (22.78) and substituting a trial solution $u_1 = a_1 \sin \omega t$ and $u_2 = a_2 \sin \omega t$. Hence, we obtain

$$\begin{aligned} (k_1 + k_2 - m_1 \omega^2) a_1 - k_2 a_2 &= 0 \\ -k_2 a_1 + (k_2 - m_2 \omega^2) a_2 &= 0 \end{aligned} \quad (22.79)$$

For a nontrivial solution of eq. (22.79), we set equal to zero the determinant of the coefficients:

$$\begin{vmatrix} k_1 + k_2 - m_1 \omega^2 & -k_2 \\ -k_2 & k_2 - m_2 \omega^2 \end{vmatrix} \quad (22.80)$$

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The expansion of the determinant in eq. (22.80) results in the following quadratic equation in ω^2 :

$$m_1 m_2 \omega^4 - [(k_1 + k_2)m_2 + k_2 m_1] \omega^2 + k_1 k_2 = 0 \quad (22.81)$$

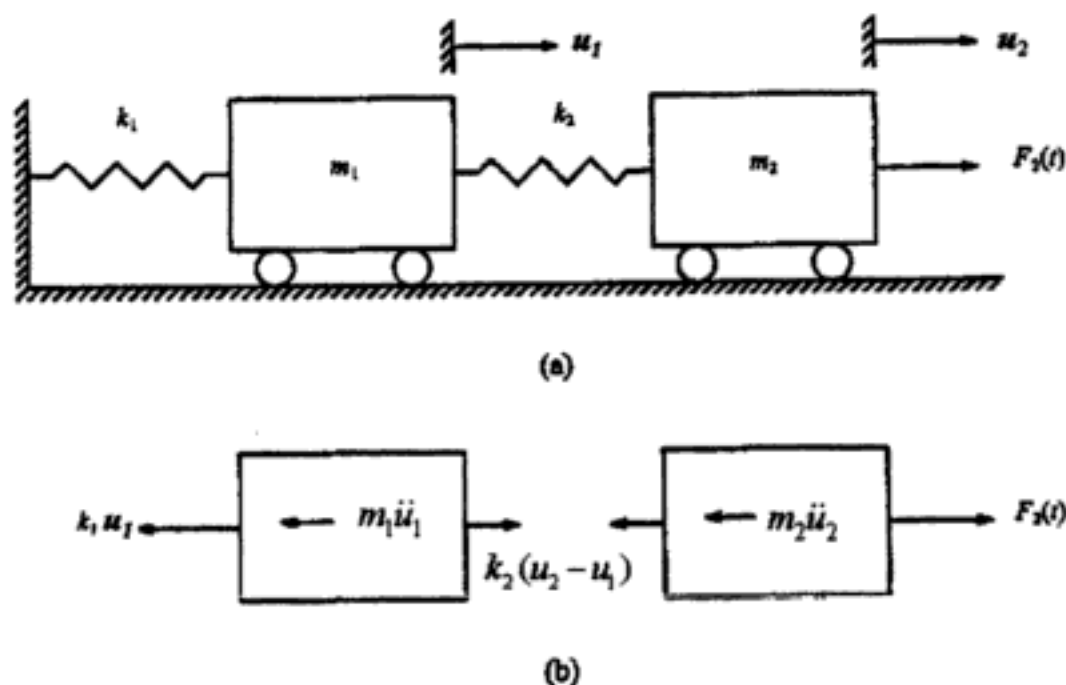


Fig. 22.26 Two-degree-of-freedom system. (a) Mathematical model.
(b) Free-body diagrams.

Equation (22.81) provides the roots ω_1^2 and ω_2^2 which are the square values of the natural frequencies for this two-degree-of-freedom system. The substitution of ω_1^2 into one of the equations in eq. (22.79) results in the first mode a_{11} , a_{21} , and the subsequent substitution of ω_2^2 , in the second mode a_{12} , a_{22} . These modal shapes are conveniently normalized by dividing the components a_{11} and a_{21} of the first mode by $\sqrt{m_1 a_{11}^2 + m_2 a_{21}^2}$ and the components, a_{12} and a_{22} , of the second mode by $\sqrt{m_1 a_{12}^2 + m_2 a_{22}^2}$ to obtain the modal matrix given by eq. (22.82) in which the normalized modes are arranged in the columns of this matrix.

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \quad (22.82)$$

The uncoupled modal equation, as presented in Chapter 6, is obtained by introducing in eq. (22.78) the linear transformation

$$\begin{aligned} u_1(t) &= \phi_{11} z_1(t) + \phi_{12} z_2(t) \\ u_2(t) &= \phi_{21} z_1(t) + \phi_{22} z_2(t) \end{aligned} \quad (22.83)$$

resulting in

$$\begin{aligned}\ddot{z}_1 + \omega_1^2 z_1 &= \phi_{21} F_2(t) \\ \ddot{z}_2 + \omega_2^2 z_2 &= \phi_{22} F_2(t)\end{aligned}\quad (22.84)$$

in which z_1 and z_2 are the modal displacement functions.

Damping in the system may readily be included by adding in eqs. (22.84) damping terms expressed in function of the modal damping ratios ξ_1 and ξ_2 . Hence,

$$\begin{aligned}\ddot{z}_1 + 2\xi_1\omega_1\dot{z}_1 + \omega_1^2 z_1 &= \phi_{21} F_2(t) \\ \ddot{z}_2 + 2\xi_2\omega_2\dot{z}_2 + \omega_2^2 z_2 &= \phi_{22} F_2(t)\end{aligned}\quad (22.85)$$

The solution of eqs. (22.85) may be expressed by Duhamel's integral which for a damped system is given from eq. (4.24) as

$$z_1(t) = \int_{-\infty}^{\infty} \phi_{21} F_2(t - \lambda_1) h_1(\lambda_1) d\lambda_1 \quad (22.86)$$

and

$$z_2(t) = \int_{-\infty}^{\infty} \phi_{22} F_2(t - \lambda_2) h_2(\lambda_2) d\lambda_2$$

in which the unit impulse functions $h_1(t)$ and $h_2(t)$ are given for $t \geq 0$ by

$$h_1(t) = \frac{e^{-\xi_1\omega_1 t}}{\omega_1 \sqrt{1 - \xi_1^2}} \sin[\omega_1 \sqrt{1 - \xi_1^2} t] \quad (22.87)$$

and

$$h_2(t) = \frac{e^{-\xi_2\omega_2 t}}{\omega_2 \sqrt{1 - \xi_2^2}} \sin[\omega_2 \sqrt{1 - \xi_2^2} t] \quad (22.88)$$

and for $t < 0$ by

$$h_1(t) = h_2(t) = 0$$

The response is then obtained in terms of the displacements $u_1(t)$ and $u_2(t)$ by substituting the solution of the modal equations, $z_1(t)$ and $z_2(t)$, into eq. (22.83).

We proceed now to the determination of the mean square value of the response, $u_1(t)$ in order to estimate its standard deviation and its confidence interval, due to random excitation. The mean square value of the response may be calculated from the autocorrelation function R_{u_1} using eqs. (22.19) and (22.20) as

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The spectral density function $S_{u1}(\omega)$ for the response is then given by eq. (22.36a) as

$$S_{u1}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{u1}(\omega) e^{-i\omega\tau} d\tau \quad (22.92)$$

The multiplication of eq. (22.91) by

$$\frac{1}{2\pi} e^{-i\omega\tau} d\tau = \frac{1}{2\pi} e^{i\omega(\tau-\lambda_2+\lambda_1)} e^{i\omega(\lambda_2-\lambda_1)} d\tau$$

followed by integration yields

$$\begin{aligned} S_{u1}(\omega) = & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} R_2(\tau - \lambda_2 + \lambda_1) e^{i\omega(\tau - \lambda_2 + \lambda_1)} [\phi_{11}^2 \phi_{21}^2 h_1(\lambda_1) h_2(\lambda_1) \\ & + \phi_{11} \phi_{12} \phi_{21} \phi_{22} h_1(\lambda_1) h_2(\lambda_2)] + \phi_{12} \phi_{11} \phi_{22} \phi_{21} h_1(\lambda_2) h_2(\lambda_1) \\ & + \phi_{12}^2 \phi_{22}^2 h_2(\lambda_1) h_2(\lambda_2)] e^{i\omega(\lambda_2 - \lambda_1)} d\tau d\lambda_1 d\lambda_2 \end{aligned}$$

After using eq. (22.77), $S_{u1}(\omega)$ may be expressed as

$$S_{u1}(\omega) = \sum_{j=1}^2 \sum_{k=1}^2 \phi_{1j} \phi_{1k} \phi_{2j} \phi_{2k} S_2(\omega) H_j(\omega) H_k^*(\omega) \quad (22.93)$$

in which $H_k^*(\omega)$ is the complex conjugate of $H_k(\omega)$ and

$$S_2(\omega) = \frac{1}{2\pi} \int R_2(\tau - \lambda_2 + \lambda_1) e^{i\omega(\tau - \lambda_2 + \lambda_1)} d\tau$$

Equation (22.93) provides the spectral density function, $S_{u1}(\omega)$, for the response at coordinate 1 of a two-degree-of-freedom system excited by a random force at coordinate 2 expressed by the spectral density function $S_{u2}(\omega)$. Now we use eq. (22.43) to obtain the discrete spectral function, $S_{u1}(\omega_n)$, evaluated at frequency ω_n as

$$S_{u1}(\omega) = \sum_{j=1}^2 \sum_{k=1}^2 \phi_{1j} \phi_{1k} \phi_{2j} \phi_{2k} C_2(\omega_n) C_2^*(\omega_n) H_j(\omega_n) H_k^*(\omega_n) \quad (22.94)$$

where $C_2(\omega_n)$ is the discrete Fourier transform coefficient of the random force at coordinate 2 and $C_2^*(\omega_n)$ its complex conjugate.

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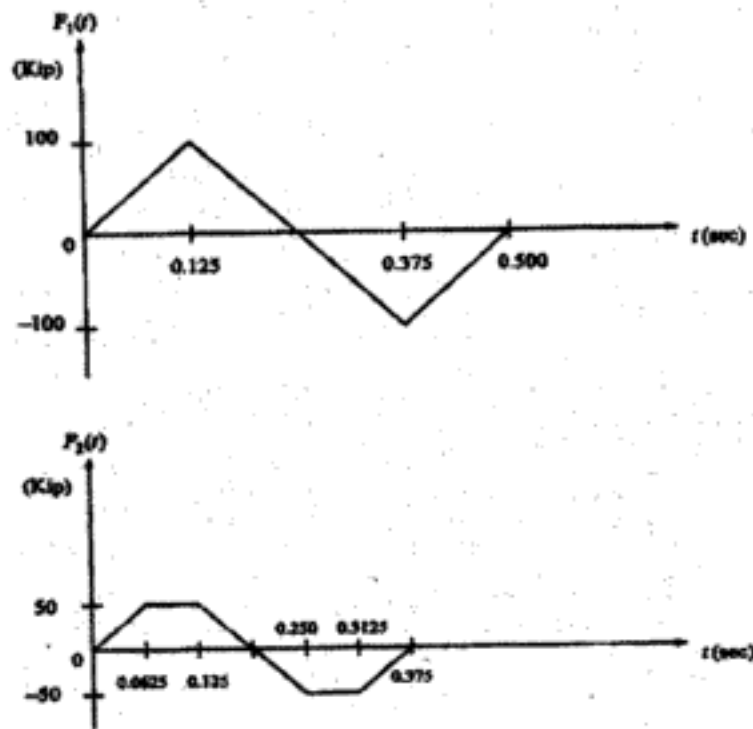


Fig. 22.27 Excitation Functions for Illustrative Example 22.8.

Solution:

INPUT DATA AND OUTPUT RESULTS FOR ILLUSTRATIVE EXAMPLE 22.8

PROGRAM 22 RANDOM

FILE : C: D22

INPUT DATA:

NUMBER OF DEGREES OF FREEDOM	ND= 2
NUM OF EXT. FORCES	NF= 2
TIME DURATION FOR THE FORCES	TD= 1
EXPONENT OF $N=2**M$	M= 3
GRAVITATIONAL INDEX	G= 0

FORCE #,	COORD.#	FORCE APPLIED AT,	POINTS DEFINING THE FORCE
1	2	2	6

EXCITATION FUNCTION FOR FORCE # 1

TIME	EXCITATION
0.0000	0.0000
0.1250	100.0000
0.3750	-100.0000
0.5000	0.0000
1.000	0.0000

EXCITATION FUNCTION FOR FORCE # 2

TIME	EXCITATION
0.0000	0.0000
0.0625	50.0000
0.1250	50.0000
0.2500	-50.0000
0.3125	-50.0000
0.3750	0.0000
1.0000	0.0000

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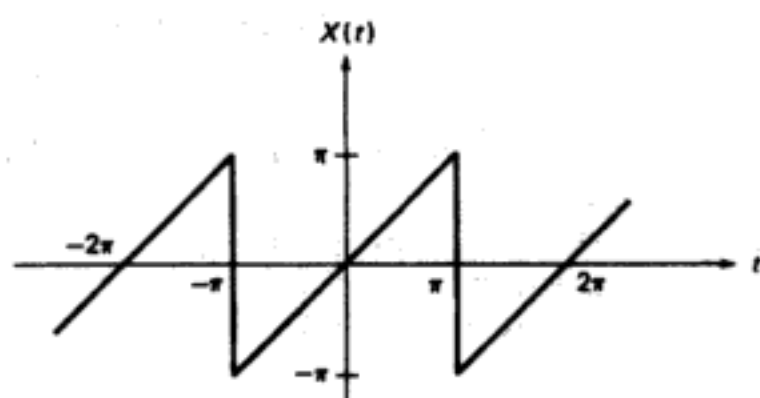


Fig. P22.2

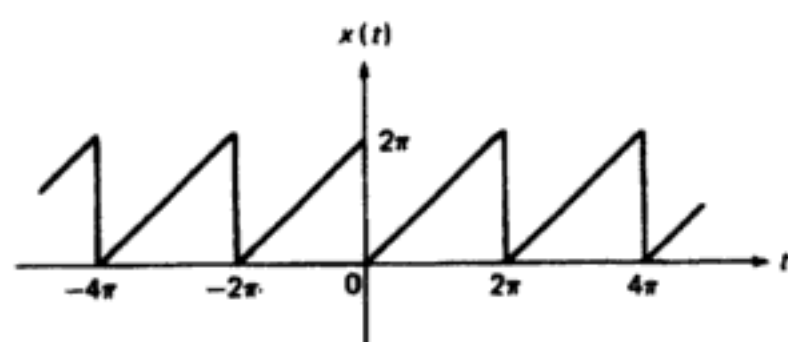


Fig. P22.3

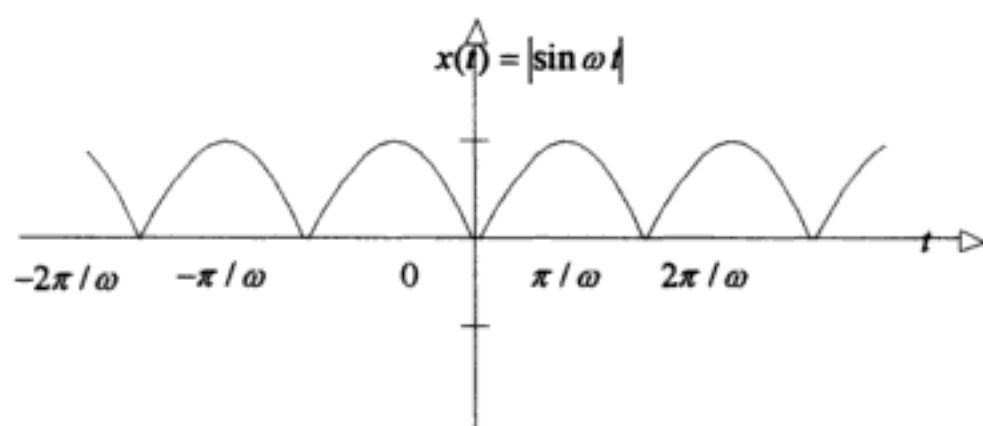


Fig. P22.4

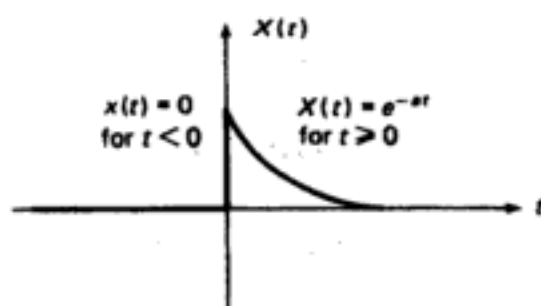


Fig. P22.5

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PART VII

Earthquake Engineering

23 Uniform Building Code 1997: Equivalent Lateral Force Method

Several seismic building codes are currently in use in different regions of the United States. The International Building Code-2000 (IBC-2000) and The Uniform Building Code-1997 (UBC-1997), published by the International Conference of Building Officials, are the building codes most extensively used, particularly in the western part of the country. In addition to these building codes, two other major building codes are used: 1) The BOCA, or National Building Code [Building Officials and Code Administrators International 1999]; and 2) The Standard Building Code, of the Southern Building Code Congress International (1999).

In addition to the preceding codes, there is the ASCE Standard: Minimum Design Loads in Buildings and Other Structures, ASCE-98 (1998), (revision of ANSI (ASCE7-93), which contains sections with recommendations for Earthquake Engineering design of buildings. Several organizations involved in earthquake-resistant design have published recommendations that form the basis for requirements in the official codes. These organizations include 1) the Structural Engineers Association of California [SEAOC (1990)], 2) the Applied Technology Council [ATC3-06 (1978)], 3) the Building Seismic Safety Council [BSSC (1994)], and 4) The Federal Emergency Agency [FEMA (1997)] which leads the National Earthquake Hazard Reduction Program (NEHRP) with publications issued by the Building Seismic Safety Council. These organizations periodically issue recommendations and requirements for earthquake-resistant design of structures that are based on a combination of theory, experimentation and practical observations.

Building codes are intended to provide guidelines and formulas that constitute minimum legal requirements for design and construction within a particular region. These requirements are intended to achieve satisfactory performance of the structure when subjected to seismic excitation, although they are not optimal; the safety of the structure is not assured in the event of a major earthquake. The objective of seismic codes is that a minor or moderate earthquake will not damage the structure and that a

major earthquake will not produce collapse of the structure. To understand how structures are affected by earthquake, it is necessary to understand the ground motion produced by an earthquake.

23.1 Earthquake Ground Motion

Most earthquakes are caused by sudden energy release at a dislocation or rupture in crustal plates generated at a point in the interior of the earth known as the focus or hypocenter. The point on the earth's surface directly above the focus is the epicenter. The magnitude of an earthquake is commonly measured by the Richter Magnitude, which is based on the reading registered by an instrument known as Wood-Anderson seismograph at a specified distance of 100 km from the epicenter of the earthquake. Specifically, The Richter Magnitude¹ (M) for an earthquake is defined as

$$M = \log_{10} \frac{A}{A_0} \quad (23.1)$$

where A is the maximum amplitude registered by a Wood-Anderson instrument located at 100 km from the epicenter and A_0 is the reference amplitude of one thousandth of a millimeter (for A expressed in millimeters, $A_0 = 0.001$). When there is no instrument located at this specified distance, an estimation of the Richter Magnitude as calculated based on readings registered at seismographs in the region affected by the earthquake. Earthquakes of magnitude 5.0 or greater generate motion sufficiently severe to be potentially damaging to structures. The energy E in Joules released by an earthquake can be estimated by the formula

$$E = 10^{4.8+1.5M} \text{ (Joules)} \quad (23.2)$$

where M is the Richter Magnitude of the earthquake. Equation (23.2) reveals that the energy increases by a factor of about 32 for an increase of 1 unit in the magnitude M and by a factor of 1000 for an increase of 2 units in the magnitude of M . Although the Richter magnitude provides a measure of the total energy released by an earthquake, it does not describe the damaging effects caused by an earthquake at a particular location. Such a description is provided by the modified Mercally intensity scale (MMI), with a total of 12 intensity values usually expressed in Roman numerals. Table 23.1 shows the MMI scale. The MMI value assigned to an earthquake at a particular site based on the observation of resultant damage is useful in lieu of instrument records of ground motion. Intensities up to VI usually do not produce damage while intensities from VI to XII result in progressively greater damage to buildings and other structures.

¹ Several definitions for the magnitude of earthquakes are in use in different countries, all emanating from the original formulation by C. I. Richter. A brief description of seismic magnitude scales is given in the *International Handbook of Earthquake Engineering Codes, Programs and Examples*, edited by M. Paz, 1994.

Graphical records or time histories acceleration (accelerograms) of earthquakes are obtained with instruments called strong-motion accelerographs. These instruments are installed on the ground in basements or in other locations of buildings or other structures. They are commonly designed to register the three orthogonal components of the ground acceleration. Figure 23.1 shows the N-S component of the accelerogram of the El Centro earthquake of 1940, the velocity and displacement obtained by integration of the Accelerogram as depicted in this figure. The earthquake accelerograms can also be analyzed to obtain direct estimates of peak ground motion, duration of the strong earthquake shaking, and the frequency content of an earthquake.

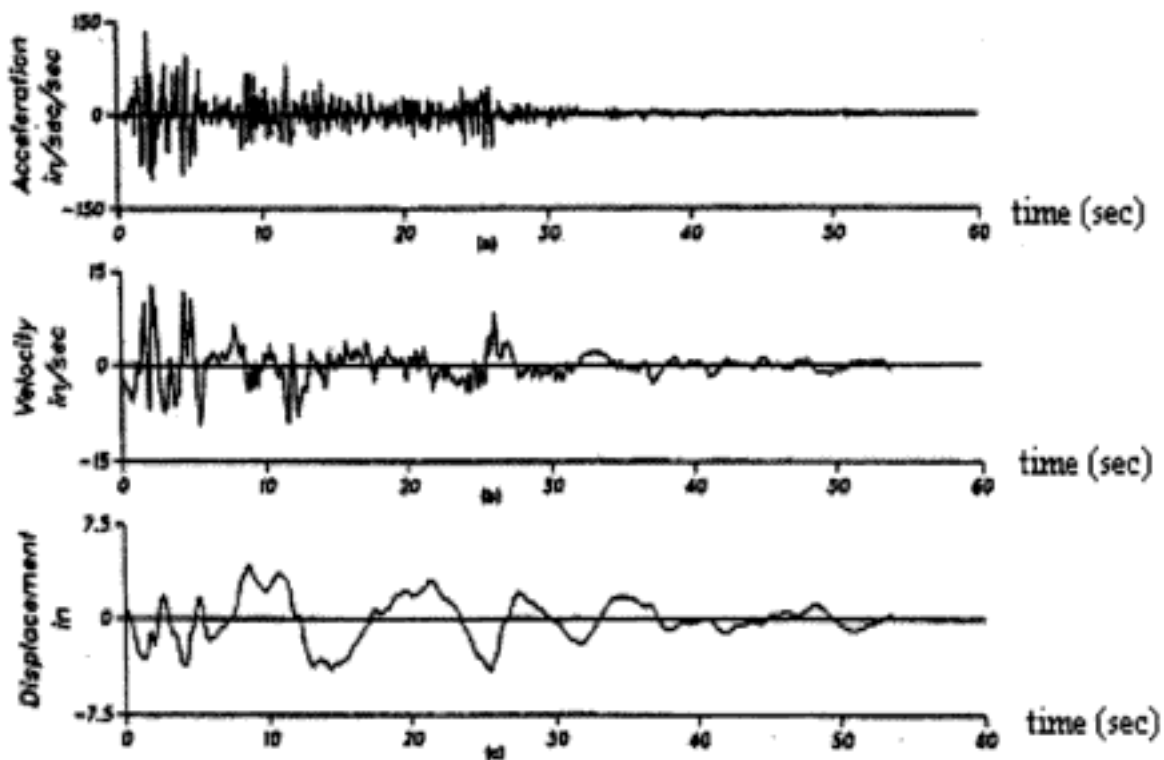


Fig. 23.1 North-South component of the El Centro earthquake, California, 1940.
(a) Acceleration, (b) Velocity, (c) Displacement.

Table 23.1 Modified Mercalli Intensity Scale (1956 Version)*

Intensity Value	Description
I	Not felt. Marginal and long-period effects of large earthquakes.
II	Felt by person at rest, on upper floors, or favorably placed.
III	Felt indoors. Hanging objects swing. Vibration like passing of light trucks. May not be recognized as an earthquake.
IV	Hanging objects swing. Vibration like passing of heavy trucks; or sensation of a jolt like a heavy ball striking the walls. Standing cars rock. Windows, dishes, doors rattle. Glasses clink. Crockery clashes. In the upper range of IV, wooden walls and frames creak.
V	Felt outdoors; direction estimated. Sleepers awakened. Liquids disturbed, some spilled. Small unstable objects displaced or upset. Doors swing, close, open. Shutters, pictures move. Pendulum clocks stop, start, change rate.
VI	Felt by all. Many frightened and run outdoors. Persons walk unsteadily. Windows, dishes, glassware broken. Knickknacks, books, etc. off shelves. Pictures off walls. Furniture moved or overturned. Weak plaster and masonry D** cracked. Small bells ring (church, school). Trees, bushes shaken visibly, or heard to rustle.
VII	Difficult to stand. Noticed by drivers. Hanging objects quiver. Furniture broken. Damage to masonry D**, including cracks. Weak chimneys broken at roofline. Fall of plaster, loose bricks, stones, tiles, cornices, also unbraced parapets and architectural ornaments. Some cracks in masonry C**. Waves on ponds, water turbid with mud. Small slides and caving in along sand or gravel banks. Large bells ring. Concrete irrigation ditches damaged.
VIII	Steering of cars affected. Damage to masonry C; partial collapse. Some damage to masonry B; none to masonry A. Fall of stucco and some masonry walls. Twisting, fall of chimneys, factory stacks, monuments, towers, elevated tanks. Frame houses moved on foundations if not bolted down; loose panel walls thrown out. Decayed piling broken off. Branches broken from trees. Changes in flow or temperature of springs and wells. Cracks in wet ground and on steep slopes.
IX	General panic. Masonry D** destroyed, masonry C** heavily damaged, sometimes with complete collapse, masonry B seriously damaged. General damage to foundations. Frame structures, if not bolted, shifter off foundations. Frames racked. Serious damage to reservoir. Underground pipes broken. Conspicuous cracks in ground. In alleviated area and mud ejected, earthquake fountains, sand craters.
X	Most masonry and frame structures destroyed with their foundations. Some well-built wooden structures and bridges destroyed. Serious damage to dams, dikes, embankments. Large landslides. Water thrown on banks of canals, rivers, lakes, etc. Sand and mud shifted horizontally on beaches and flat land. Rails bent slightly.
XI	Rails bent greatly. Underground pipelines completely out of service.
XII	Damage nearly total. Large rock masses displaced. Lines of sight and level distorted. Objects thrown in the air.

* Original 1931 version in Wood, H. O., and Neumann, F., 1931, Modified Mercalli intensity scale of 1931: *Seismological Society of America Bulletin*, v. 53, no. 5, pp. 979-987. 1956 version prepared by Charles F. Richter, in *Elementary Seismology*, 1958, pp. 137-138, W. H., Freeman and Company.

** Masonry A, B, C, D. To avoid ambiguity of language, the quality of masonry, brick or otherwise is specified by the following lettering:

Masonry A. Good workmanship, mortar and design; reinforced, especially laterally, and bound together by using steel, concrete, etc., designed to resist lateral forces.

Masonry B. Good workmanship and mortar; reinforced, but not designed in detail to resist lateral forces.

Masonry C. Ordinary workmanship and mortar, no extreme weaknesses like failing to the in at corners, but neither reinforced nor designed against horizontal forces.

Masonry D. Weak materials; such as adobe; poor mortar; low standards of workmanship; weak horizontally.

23.2 Equivalent Lateral Force Method

The earthquake resistant design regulations of the 1997 Uniform Building Code [UBC-97 (International Conference of Building Officials 1997)] are based mainly on the publication by the Structural Engineers Association of California (SEAOC-1990) entitled *Recommended Lateral Force Requirements and Tentative Commentary*, which is based to some extent on the provisions of the Applied Technology Council (1978) recommendations (ATC-06-1978) and of the Building Seismic Safety Council (1994) guidelines (BSSC-1994). The key provisions of the UBC-1997 for earthquake resistant design using the Equivalent Lateral Force Method are presented in this chapter.

23.3 Earthquake-Resistant Design Methods

Two methods for earthquake-resistant design are defined: 1) the Lateral Force Method, and 2) the Dynamic Method. The first method is applicable to all structures in Zone 1 and those in Zone 2 (defined in the seismic zone map of Fig. 23.2) designed for Occupancy Importance Category 4 and 5 as described in Table 23.2. The Equivalent Lateral Force method is also applicable, in any seismic zone, to regular structures under 240 ft in height and to irregular structures¹ of not more than five stories nor over 65 ft in height. The dynamic method *may* be used for any structure but *must* be used for structures over 240 ft in height, irregular structures over 5 stories or 65 ft in height, and structures located in seismic Zones 3 and 4 with dissimilar structural systems. The dynamic method must also be used for structures which have a fundamental period greater than 0.7 sec and are located on Soil Profile Type S_F as specified in Table 23.7.

23.4 Seismic Zone Factor

Six seismic zones (labeled 0, 1, 2A, 2B, 3 and 4) are assigned to the different regions of the United States as indicated in the seismic zone map reproduced in Fig. 23.2. The seismic Zone Factor, Z , is then assigned to the seismic Zones as follows:

Zone:	0	1	2A	2B	3	4
Z :	0	0.075	0.15	0.20	0.30	0.40

¹ Irregular structures are those that have significant physical discontinuities in configuration or in their lateral force-resisting systems. Specific features for vertical structural irregularities are described in Table 23.9 and for plane-structural irregularities in Table 23.10.

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but it cannot be less than

$$V = 0.11 C_a I W \quad (23.5)$$

In addition, in the zone of highest seismicity region (Zone 4), the Design Base Shear Force cannot be less than

$$V = \frac{0.8 Z N_v I}{R} W \quad (23.6)$$

The various factors and coefficients in eqs. (23.3) through (23.6) are defined as follows:

I is the Occupancy Importance Factor having a numerical value related to the use of the structure as classified in Table 23.2. The Occupancy Importance Factor is equal to 1.0 for general structures but equal to 1.25 for essential and hazardous structures.

C_a and C_v are the Seismic Coefficients related to the expected ground acceleration at the site. These coefficients are functions of the Seismic Zone Factor Z and of the Soil Profile Type S as classified for C_a in Table 23.3 and for C_v in Table 23.4. In addition, these coefficients also depend on the Near Source Factors N_a and N_v which are given in Tables 23.5A and 23.5B, respectively. The Near Source Factors N_a and N_v are functions of the closest distance to a known seismic source and the Seismic Source Type (A, B or C) as classified by the Code (Table 23.6)

S is the Soil Profile Type reflecting the soil characteristics at the site described in Table 23.7. The Soil Profile Types are designated as S_a through S_f .

R is the Structural System Factor ranging from 2.8 to 8.5 as given in Table 23.8. The factor R is a measure of the capacity of the structural system to absorb energy in the inelastic range.

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Table 23.5A Near Source Factor N_a [UBC-97: Table 16-S]

Seismic Source Type	Closest distance to known seismic source		
	≤ 2 km	5 km	≥ 10 km
A	1.5	1.2	1.0
B	1.3	1.0	1.0
C	1.0	1.0	1.0

Table 23.5B Near Source Factor N_v [UBC-97: Table 16-T]

Seismic Source Type	Closest distance to known seismic source			
	≤ 2 km	5 km	10 km	≥ 15 km
A	2.0	1.6	1.2	1.0
B	1.6	1.2	1.0	1.0
C	1.0	1.0	1.0	1.0

Table 23.6 Near Source Factor Type [UBC-97: Table 16-u]

Seismic Source Type	Seismic Source Description	Seismic Source Definition	
		Max Moment magnitude	Slip Rate SR (mm/year)
A	Fault capable of producing large Magnitude events and have high Seismicity activity	$M \geq 7.0$	$SR \geq 5$
B	All faults other than types A and C	$M \geq 7.0$ $M < 7.0$ $M \geq 6.5$	$SR < 5$ $SR > 2$ $SR < 2$
C	Fault capable of producing large Magnitude events and have high Seismicity activity	$M < 6.5$	$SR \leq 2$

Table 23.7 Soil Profile Types [UBC-97: Table 16-J]

Average Soil Properties for Top 100 Feet (30 480 mm) of Soil Profile				
Soil Profile Type	Soil Profile Name/Generic Description	Shear Wave Velocity, feet/second (m/s)	Standard Penetration Test, (blows/foot)	Undrained Shear Strength, psf (kPa)
S_A	Hard Rock	$> 5,000$ (1500)	—	—
S_B	Rock	2,500 to 5,000 (760 to 1,500)	—	—
S_C	Very Dense Soil and Soft Rock	1,200 to 2,500 (360 to 700)	> 50	$> 2,000$ (100)
S_D	Stiff Soil Profile	600 to 1,200 (180 to 360)	15 to 50	1,000 to 2,000 (50 to 100)
S_E	Soft Soil Profile	< 600 (180)	< 15	$< 1,000$ (50)
S_F	Soil Requiring Site-specific Evaluation			

Table 23.8 Structural System Factor, *R* [UBC-97: Table 16-N]

Basic Structural System	Lateral-Force-Resisting system Description	R
1. Bearing wall system	1. Light-framed walls with shear panels	
	a. Wood structural panel walls for structures three stories or less	5.5
	b. All other light-framed walls	4.5
	2. Shear walls	
	a. Concrete	4.5
	b. Masonry	4.5
	3. Light steel-framed bearing walls with tension only bracing	2.8
	4. Braced frames where bracing carries gravity load	
	a. Steel	4.4
2. Building frame systems	b. Concrete	2.8
	c. Heavy timber	2.8
	1. Steel eccentrically braced frame (EBF)	7.0
	2. Light-framed walls with shear panels	
	a. Wood structural panel walls for structures three stories or less	6.5
	b. All other light-framed walls	5.0
	3. Shear Walls	
	a. Concrete	5.5
	b. Masonry	5.5
	4. Ordinary braced frames	
	a. Steel	5.6
	b. Concrete	5.6
	c. Heavy timber	5.6
	5. Special truss moment frames of steel (STMF)	
	a. Steel	6.4
resisting frame system	1. Special moment-resisting frame (SMRF)	
	a. Steel	8.5
	b. Concrete	8.5
	2. Masonry moment-resisting wall frame (MMRWF)	6.5
	3. Concrete intermediate moment-resisting frame (MRF)	5.5
	4. Ordinary moment-resisting frame (OMRF)	
	a. Steel	4.5
	b. Concrete	3.5
	5. Special truss moment frames of steel (STMF)	6.5
4. Dual Systems	1. Shear Walls	
	a. Concrete with SMRF	8.5
	b. Concrete with steel OMRF	4.2
	c. Concrete with concrete IMRF	6.5
	d. Masonry with steel OMRF	5.5
	e. Masonry with steel IMRF	4.2
	f. Masonry with concrete IMRF	4.2
	g. Masonry with masonry MMRWF	6.0
	2. Steel EBF	
	a. With steel SMRF	8.5
	b. With steel OMRF	4.2
	3. Ordinary braced frames	
	a. Steel with steel SMRF	6.5
	b. Steel with steel OMRF	4.2
	c. Concrete with concrete SMRF	6.5
	d. Concrete with concrete IMRF	4.2
	4. Special concentrically braced frames	
	a. Steel with steel SMRF	7.5
	b. Steel with steel OMRF	4.2
5. Cantilevered column building systems		
	1. Cantilevered column elements	2.2
frame interaction systems	1. Concrete	5.5

23.6 Distribution of Lateral Seismic Forces

The total base shear force V calculated in Section 23.6 is distributed at each level over the height of the structure, as Lateral Seismic Forces, F_x plus an additional force F_t applied at the top of the building. The Lateral Seismic Forces, F_x and F_t are calculated by

$$F_x = \frac{(V - F_t)w_x h_x}{\sum_{i=1}^N w_i h_i} \quad (23.9)$$

and

$$\begin{aligned} F_t &= 0.07 T V < 0.25 V & \text{if } T > 0.7 \text{ sec} \\ F_t &= 0 & \text{if } T \leq 0.7 \text{ sec} \end{aligned} \quad (23.10)$$

Consequently

$$V - F_t = \sum_{i=1}^N F_x$$

or

$$V = F_t + \sum_{i=1}^N F_x \quad (23.11)$$

where

N = total number of stories above the base of the building

F_x, F_t, F_N = lateral force applied at level x, i or N

F_t = portion of the base force V at the top of the structure in addition to F_N

h_x, h_i = height at level x or i above the base

w_x, w_i = seismic weight of the x th or i th level.

The distribution of the lateral forces F_x and F_t is shown in Fig. 23.3 for a regular multistory building. The code stipulates that the force F_x at level x , be applied over the area of the building according to the mass distribution at that level.

23.7 Story Shear Force

The story shear force V_x at any story x is given by the sum of the lateral seismic forces at that story and above, that is

$$V_x = F_t + \sum_{i=x}^N F_i \quad (23.12)$$

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where

δ_{\max} = the maximum displacement at level x

δ_{avg} = the average displacement at the extreme points of the structure at level x

23.9 Overturning Moment

The code requires that overturning moments be determined at each level of the building. The overturning moment is calculated by static using the lateral design forces F_x and F_t [eqs. (23.9) and (23.10)] which act on levels of the building above the level under consideration. Hence, the overturning moment M_x at level x of the building is given by

$$M_x = F_t(h_N - h_x) + \sum_{i=x+1}^N F_i(h_i - h_x) \quad (23.14)$$

where

h_N, h_x, h_i = height of level N, x, i

F_t, F_i = Static Lateral Forces [eqs. (23.9) and (23.10)]

23.10 P -Delta Effects ($P - \Delta$)

The so-called $P - \Delta$ effect refers to the additional moment produced by the vertical loads and the lateral deflection of columns, or other elements of the building resisting lateral forces. Figure 23.4 shows a column supporting an axial compressive force P , a shear force V , and bending moments M_a and M_b at the two ends. Due to this load, the column undergoes a relative lateral displacement or drift Δ . In this case, the $P - \Delta$ effect results in a secondary moment $M_s = P\Delta$, which is resisted by an additional shear force $P\Delta / L$ in the column.

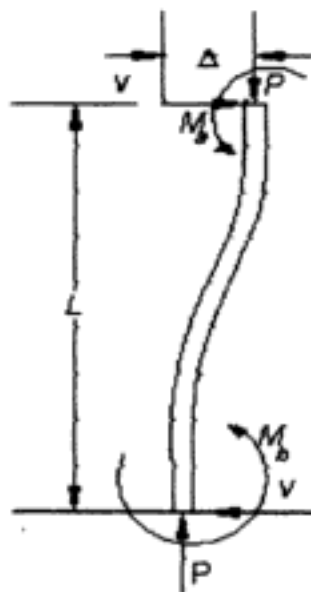


Fig. 23.4 Deflected column showing the $P - \Delta$ effect.

It should be realized that this simple calculation of the $P-\Delta$ effect involves an approximation, since the secondary moment $M_x = P\Delta$ will further increase the drift in the column and consequently will produce an increment of the secondary moment and shear force in the column.

An acceptable method to estimate the final drift is to add to the drift Δ_x for each story due to the primary overturning moment M_x , the additional drift resulting from the incremental overturning moments $M_x \theta_x$, $(M_x \theta_x) \theta_x$, $[(M_x \theta_x) \theta_x] \theta_x$, etc., in which θ_x is the ratio between the primary moment M_x and the secondary moment M_x as defined by eq. (23.16).

Hence, the total story drift Δ_{xTOTAL} due to the $P-\Delta$ effect is given by the geometric series

$$\Delta_{xTOTAL} = \Delta_x + \Delta_x \theta_x + \Delta_x \theta_x^2 + \dots$$

which is equal to

$$\Delta_{xTOTAL} = \Delta_x \left(\frac{1}{1 - \theta_x} \right) \quad (23.15)$$

Consequently, the $P-\Delta$ effect may be considered by multiplying, for each story, the calculated story drift and the calculated story shear force by the amplification factor $1/(1 - \theta_x)$ and then proceeding to recalculate the overturning moments and other seismic effects for these amplified story shear forces.

The code specifies that the resulting member forces and moments, as well as the story drift induced by the $P-\Delta$ effect, shall be considered in the evaluation of overall structural frame stability. However, the $P-\Delta$ effect need not be considered when the ratio of secondary moment (resulting from the story drift) to the primary moment (due to the static lateral forces), for any story, does not exceed 0.10. This ratio may be evaluated at each story as the product of the total dead and live loads P_x above the story, times the seismic story drift Δ_x divided by the product of the seismic shear force times the height of the story. That is, the ratio θ_x , at level x , between the secondary moment M'_x resulting from the $P-\Delta$ effect and the primary moment M_x due to the seismic lateral forces, may be calculated from the following formula:

$$\theta_x = \frac{M'_x}{M_x} = \frac{P_x \Delta_x}{V_x h_{sx}} \quad (23.16)$$

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where

R = Structural Response System Factor from Table 23.8.

Δ_s = Elastic story drift due to the seismic lateral forces F_x , F_h , calculated by eqs.(23.9) and (23.10).

However, the maximum calculated story drift is limited to $\Delta_M \leq 0.025h_s$ for structures with $T \leq 0.7$ sec and too $\Delta_M \leq 0.020h_s$ for structures with $T > 0.7$ sec, where h_s is the story height.

23.13 Diaphragm Design Forces

The code requires that horizontal diaphragms (floors and roof) be designed to resist the force F_{Px} at level x given by

$$F_{Px} = \frac{F_t + \sum_{i=x}^N F_i}{\sum_{i=x}^N w_i} w_{Px} \quad (23.18)$$

in which

F_t , F_i are the equivalent seismic forces [eqs. (23.9) and (23.10)]

w_i is the portion of the total seismic weight W located or assigned at level i

w_{Px} is the weight of the diaphragm and the elements tributary at level x including 25% of floor live load in storage and warehouse occupancies

The force F_{Px} determined by eq. (23.18) need not exceed

$$F_{Px} \leq C_a I w_{Px} \quad (23.19)$$

but shall not be less than

$$F_{Px} \geq 0.5 C_a I w_{Px} \quad (23.20)$$

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Table 23.9 Type and definition of vertical-structural irregularities
[UBC-97: Table 16-L]

1.	Stiffness irregularity-soft story A soft story is one in which the lateral stiffness is less than 70% of that in the story above or less than 80% of the average stiffness of the three stories above.
2.	Weight (mass) irregularity Mass irregularity shall be considered to exist where the effective mass of any story is more than 150% of the effective mass of an adjacent story. A roof that is lighter than the floor below need not be considered.
3.	Vertical geometric irregularity Vertical geometric irregularity shall be considered to exist where the horizontal dimension of the lateral-force-resisting system in any story is more than 130% of that in an adjacent story. One-story penthouses need not be considered.
4.	In-plane discontinuity in vertical-lateral-force-resisting element An in-plane offset of the lateral-load-resisting elements greater than the length of those elements.
5.	Discontinuity in capacity-weak story A weak story is one in which the story strength is less than 80% of that in the story above. The story strength is the total strength of all seismic-resisting elements sharing the story shear for the direction under consideration.

Table 23.10 Type and definition of plan-structural irregularities
[UBC-97: Table 16-M]

1.	Torsional Irregularity; to be considered when diaphragms are flexible Torsional irregularity shall be considered to exist when the maximum story drift, computed including accidental torsion, at one end of the structure transverse to an axis is more than 1.2 times the average of the story drifts of the two ends of the structure.
2.	Re-entrant corners Plan configurations of a structure and its lateral-force-resisting system contain re-entrant corners, where both projections of the structure beyond a re-entrant corner are greater than 15% of the plan dimension of the structure in the given direction.
3.	Diaphragm discontinuity Diaphragms with abrupt discontinuities or variations in stiffness, including those having cutout or open areas greater than 50% from one story to the next.
4.	Out-of-plane offsets Discontinuation in a lateral force path, such as out-of-plane offsets of the vertical elements.
5.	Nonparallel systems The vertical lateral-load-resisting elements are not parallel to or symmetric about the major orthogonal axes of the lateral-force-resisting system.

Illustrative Example 23.1

A four-story reinforced concrete (modulus of elasticity, $E = 3,000$ ksi) framed building has the dimensions shown in Fig. 23.5. The sizes of the exterior columns (nine each on lines A and C) are 12 in x 20 in, and the interior columns (nine on line B) are 12 in x 24 in for the bottom two stories, and respectively, 12 in x 16 in and 12 in x 20 in for the two highest stories. The height between floors is 12 ft. The dead load per unit area of floor is estimated to be 140 psf; it consists of floor slab, beam, half the weight of the columns above and below the floor, partition walls, etc. The normal live load is assumed as 125 psf. The soil below the foundation is assumed to be hard rock. The building site is located in Seattle, Washington (Seismic Zone 3). The building is intended to be used as a warehouse.

Perform the seismic analysis for this structure (in the direction normal to lines A, B, C) in accordance with the Uniform Building Code of 1997.

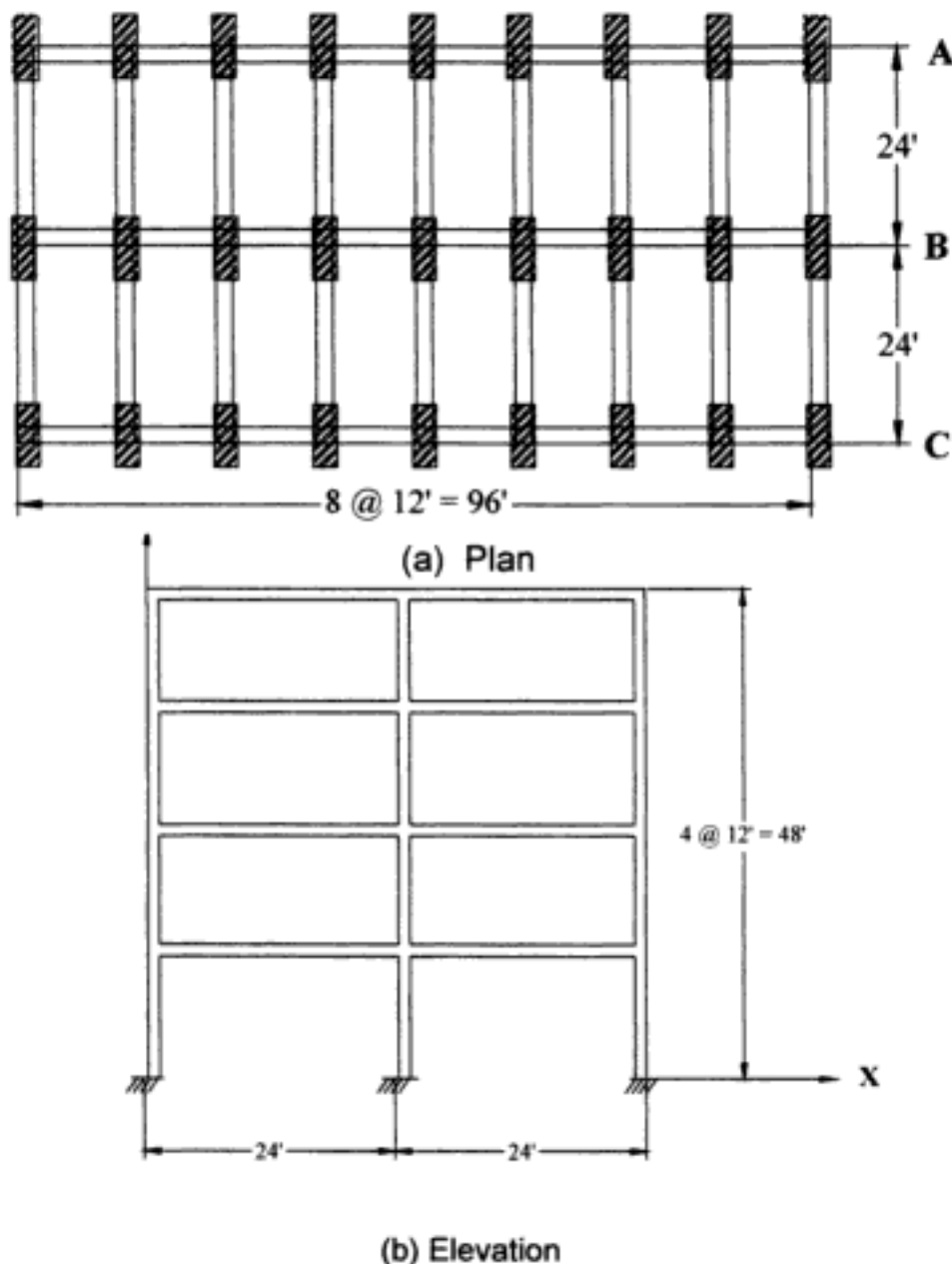


Fig. 23.5 Plan and elevation for a four-story building of Illustrative Example 23.1.

Solution:

1. Seismic Weight at Various Floors:

For a warehouse, the design load should include a minimum of 25% of the live load. No live load needs to be considered in the roof. Hence, the effective weight at all floors, except at the roof, will be $140 + 0.25 \times 125 = 171.25$ psf, and the effective weight for the roof will be 140 psf. The plan area is $48 \text{ ft} \times 96 \text{ ft} = 4608 \text{ ft}^2$. Hence, the seismic weights of various levels are:

$$w_1 = w_2 = w_3 = 4608 \times 0.17125 = 789.1 \text{ Kip}$$

$$w_4 = 4608 \times 0.140 = 645.1 \text{ Kip}$$

The total seismic weight of the building is then

$$W = 789.1 \times 3 + 645.1 = 3012.4 \text{ Kip}$$

2. Fundamental Period:

$$T = C_t h_N^{3/4} \quad \text{eq. (23.7) repeated}$$

where

$C_t = 0.030$ (for reinforced concrete moment-resisting frame)

$h_N = 48 \text{ ft}$ (total height of the building)

$$T = 0.030 \times 48^{3/4} = 0.55 \text{ sec}$$

3. Base shear Force:

$$V = \frac{C_v I}{R T} W \quad \text{eq. (23.3) repeated}$$

$Z = 0.3$ [Site in Zone 3 (Fig. 23.2)]

$R = 8.5$ [Special moment-resisting frame (Table 23.8)]

Soil Profile = S_B [Rock (Table 23.7)]

$C_a = 0.3$ [Table 23.3 with Soil Profile B and $Z = 0.3$]

$C_v = 0.3$ [Table 23.4 with Soil Profile B and $Z = 0.3$]

$I = 1.0$ [Warehouse (Table 23.2)]

Then:

$$V = \frac{0.3 \times 1.0}{8.5 \times 0.55} \times 3012.4 = 193.3 \text{ Kip}$$

Check: Not to exceed:

$$V = \frac{2.5 C_a I}{R} W \quad \text{eq. (23.4) repeated}$$

$$V = \frac{2.5 \times 0.3 \times 1.0}{8.5} \times 2012.4 = 265.8 > 193.3 \quad \text{OK}$$

Check: Cannot be less than:

$$V = 0.11 C_a I W \quad \text{eq. (23.5) repeated}$$

$$= 0.11 \times 0.3 \times 1.0 \times 3012.4 = 99.4 < 193.3 \quad \text{OK}$$

4. Seismic Equivalent Forces:

$$F_x = \frac{(F - F_t) w_x h_x}{\sum_{i=1}^N w_i h_i} \quad \text{eq. (23.9) repeated}$$

$$F_t = 0 \quad \text{for} \quad T = 0.55 < 0.7 \quad \text{by eq. (23.10)}$$

Table 23.11 contains the necessary calculations to evaluate the Equivalent Seismic Forces.

Table 23.11 Seismic Lateral Forces of Illustrative Example 23.1.

Level (x)	w_x (kip)	h_x (ft)	$w_x h_x$ (kip-ft)	F_x (kip)	V_x (kip)
4	645.12	48	30,965	68.2	68.2
3	789.12	36	28,408	62.6	130.8
2	789.12	24	18,939	41.8	172.6
1	789.12	12	9,469	20.7	193.3
			$\Sigma = 87,781$		

5. Shear Force at Story x:

$$V_x = F_t + \sum_{i=x}^N F_i \quad \text{eq. (23.11) repeated}$$

Calculated values for shear forces are shown in the last column of Table 23.11.

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Model the building as a plane orthogonal moment-resisting frame. (Use $E = 30,000$ ksi.)

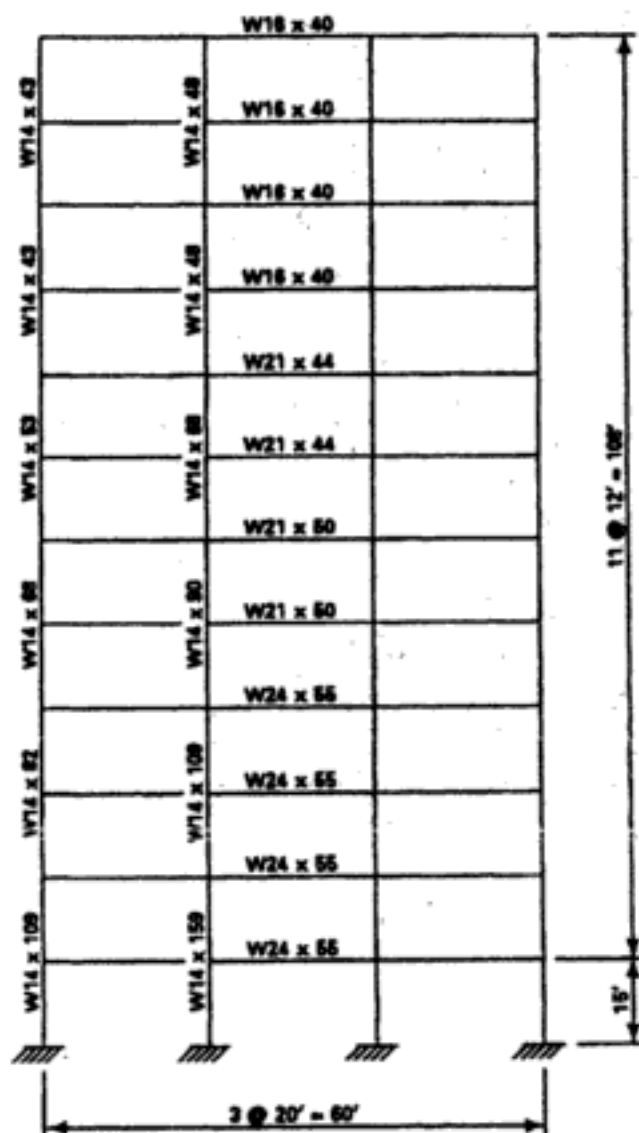


Fig. P23.2

Problem 23.3

Solve Problem 23.2 after modeling the structure as a shear building with rigid horizontal diaphragms at all levels of the building.

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24 Uniform Building Code 1997 Dynamic Method

In Chapter 5, we introduced the concept of response spectrum as a plot of the maximum response (spectral displacement, spectral velocity, or spectral acceleration) versus the natural frequency or natural period of a single degree-of-freedom system subjected to a specific excitation. In the present chapter, we will use seismic response spectra for earthquake resistant design of buildings modeled as discrete systems with concentrated masses at each level of the building.

Since response spectral charts are prepared for single degree-of-freedom system, it is necessary to perform a transformation of coordinates to obtain the modal equations of motion, and then combine their spectral responses to obtain the maximum response of the structure. In earthquake resistant design of buildings, the maximum responses include displacements, accelerations, shear forces, overturning moments, and torsional moments.

24.1. Modal Seismic Response of Building

The equations of motion of a shear building modeled with lateral displacement coordinates at the N levels and subjected to seismic excitation at the base in the same direction of the lateral displacement, may be written, neglecting damping, from eq. (8.35) as

$$[M]\{\ddot{u}_r\} + [K]\{u_r\} = -[M]\{1\}\ddot{u}_s(t) \quad (24.1)$$

In eq. (24.1), $[M]$ and $[K]$ are respectively the mass and the stiffness matrices of the system, $\{u_r\}$ and $\{\ddot{u}_r\}$ are respectively the displacement and acceleration vectors

(relative to the base), $\ddot{u}_s(t)$ is the function of the seismic acceleration at the base of the building, and $\{1\}$ a vector with all its elements equal to 1.

24.1.1 Modal Equation and Participation Factor

As presented in Chapter 8, the solution of eq. (24.1) may be found by solving the corresponding eigenproblem

$$[[K]] - \omega^2 [M]] \{ \Phi \} = \{ 0 \} \quad (24.2)$$

to determine the natural frequencies $\omega_1, \omega_2, \dots, \omega_N$ (or natural periods T_1, T_2, \dots, T_N) and the modal matrix $[\Phi]$ with its columns containing the normalized modal shapes. Then the linear transformation

$$\{u_r\} = [\Phi] \{z\} \quad (24.3)$$

introduced in eq. (24.1) yields the modal equations;

$$\ddot{z}_m + \omega_m^2 z_m = -\Gamma_m \ddot{u}_s(t) \quad (m = 1, 2, \dots, N) \quad (24.4)$$

in which Γ_m is the participation factor given by eq. (8.39) as

$$\Gamma_m = \frac{\sum_{i=1}^N W_i \phi_{im}}{\sum_{i=1}^N W_i \phi_{im}^2} \quad (24.5)$$

For normalized eigenvectors, the participation factor reduces to

$$\Gamma_m = \frac{1}{g} \sum_{i=1}^N \phi_{im} W_i \quad (24.6)$$

because for normalized eigenvectors,

$$\sum_{i=1}^N \frac{W_i}{g} \phi_{im}^2 = 1$$

where g is the acceleration due to gravity. Damping may be introduced in the modal equation (24.4) by simply adding the damping term to this equation, namely,

$$\ddot{z}_m + 2\xi_m \omega_m \dot{z}_m + \omega_m^2 z_m = -\Gamma_m \ddot{u}_s(t) \quad (24.7)$$

where ξ_m is the modal damping ratio. For convenience eq.(24.7), can be written with omission of the participation factor as

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$$S_{am} = \omega_m S_{vm} = \omega_m^2 S_{dm}$$

or in terms of the modal period $T_m = 2\pi/\omega_m$ by

$$S_{am} = \frac{2\pi}{T_m} S_{vm} = \left(\frac{2\pi}{T_m}\right)^2 S_{dm}$$

On the basis of these relations, the modal spectral acceleration S_{am} in eq. (24.10) may be replaced by the spectral displacement S_{dm} times ω_m^2 or by the spectral velocity S_{vm} times ω_m .

The modal lateral force F_{xm} at the level x of the building is then given by Newton's Law as

$$F_{xm} = a_{xm} W_x$$

or by eq. (24.10) as

$$F_{xm} = \Gamma_m \phi_{xm} S_{am} W_x \quad (24.11)$$

in which S_{am} is the modal spectral acceleration in g units and W_x is the weight attributed to the level x of the building.

The modal shear force V_{xm} at the level x of the building is equal to the sum of the seismic forces F_{im} above that level, namely,

$$V_{xm} = \sum_{i=x}^N F_{im} \quad (24.12)$$

The total modal shear force V_m at the base of the building is then calculated as

$$V_m = \sum_{i=1}^N F_{im} \quad (24.13)$$

or using eq. (24.11)

$$V_m = \sum_{i=1}^N \Gamma_m \phi_{im} W_i S_{am} \quad (24.14)$$

24.1.3 Effective Modal Weight

The effective modal weight W_m is defined by the equation

$$V_m = W_m S_{am} \quad (24.15)$$

Then, from eq. (24.14), the modal weight is

$$W_m = \Gamma_m \sum_{i=1}^N \phi_{im} W_i \quad (24.16)$$

Combining eqs. (24.5) and (24.16) results in the following important expression for the effective modal weight:

$$W_m = \frac{\left[\sum_{i=1}^N \phi_{im} W_i \right]^2}{\sum_{i=1}^N \phi_{im}^2 W_i} \quad (24.17)$$

It can be proven (Clough and Penzien 1975, pp. (559-560) analytically that the sum of the effective modal weights for all the modes of the building is equal to the total design weight of the building, that is,

$$\sum_{m=1}^N W_m = \sum_{i=1}^N W_i \quad (24.18)$$

Equation (24.18) is most convenient in assessing the number of significant modes of vibration to consider in the design. Specifically, the UBC-97 requires that, in applying the dynamic method of analysis, all the significant modes of vibration be included. This requirement can be satisfied by including a sufficient number of modes such that their total effective modal weight is at least 90% of the total design weight of the building. Thus, this requirement can be satisfied by simply adding a sufficient number of effective modal weights [eq. (24.17)] until their total weight is 90% or more of the seismic design weight of the building.

24.1.4 Modal Lateral Forces

By combining eq. (24.11) with eqs. (24.15) and (24.16), we may express the modal lateral force F_{xm} as

$$F_{xm} = C_{xm} V_m \quad (24.19)$$

where the modal seismic coefficient C_{xm} at level x is given by

$$C_{xm} = \frac{\phi_{xm} W_x}{\sum_{i=1}^N \phi_{im} W_i} \quad (24.20)$$

24.1.5 Modal Displacements

The modal displacement δ_{xm} at the level x of the building may be expressed, in view of eqs. (24.3) and (24.9), as

$$\delta_{xm} = \Gamma_m \phi_{xm} S_{dm} \quad (24.21)$$

where Γ_m is the participation factor for the m th mode, ϕ_{xm} is the component of the modal shape at level x of the building, and S_{dm} is the spectral displacement for that mode.

Alternatively, the modal displacement δ_{xm} may be calculated from Newton's Law of Motion in the form

$$F_{xm} = \frac{W_x}{g} \omega_m^2 \delta_{xm} \quad (24.22)$$

because the magnitude of the modal acceleration corresponding to the modal displacement δ_{xm} is $\omega_m^2 \delta_{xm}$. Hence, from eq. (24.22)

$$\delta_{xm} = \frac{g}{\omega_m^2} \cdot \frac{F_{xm}}{W_x} \quad (24.23)$$

or substituting $\omega_m = 2\pi/T_m$

$$\delta_{xm} = \frac{g}{4\pi^2} \cdot \frac{T_m^2 F_{xm}}{W_x} \quad (24.24)$$

where T_m is the m th natural period.

24.1.6 Modal Drift

The modal drift Δ_{xm} for the x th story of the building, defined as the relative displacement of two consecutive levels, is given by

$$\Delta_{xm} = \delta_{xm} - \delta_{(x-1)m} \quad (24.25)$$

with $\delta_{0m} = 0$.

As indicated in Article 24.7, the UBC-97 stipulates that the calculated story drift, which must include translational and torsional deflections, shall not exceed $0.04/R_w$ times the story height or 0.005 times the story height for buildings less than 65 ft in height. For buildings greater in height than 65 ft, the calculat

24.1.7 Modal Overturning Moment

The modal overturning moment M_{xm} at the level x of the building which is calculated as the sum of the moments of the seismic forces F_{im} above that level is given by

$$M_{xm} = \sum_{i=x+1}^N F_{im} (h_i - h_x) \quad (24.26)$$

where h_i and h_x are, respectively, the height of levels i and x .

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1. Seismic Zone Factor Z . This coefficient has been defined by classifying the United States in five seismic zones on the basis of past historical earthquake activity. These zones are designated as 1, 2A, 2B, 3, and 4 as shown in Fig. 23.1. The numerical values assigned to the coefficient are:

Zone	Coefficient
1	0.075
2A	0.15
2B	0.20
3	0.30
4	0.40

The seismic coefficient indicates the effective peak ground acceleration of the design earthquake.

2. The Occupancy Importance Factor I . This factor is equal to 1.25 for essential or hazardous facilities and 1.0 for special or standard occupancy structures as described in Table 23.2.
3. The Site Coefficient S . The site coefficient is defined by four soil profiles S_1 , S_2 , S_3 , and S_4 described in Table 23.3 with numerical values 1.0, 1.2, 1.5, and 2.0, respectively.
4. The Structural Factor R . Numerical values for the structural factor R are given in Table 23.8 for various structural systems. As described in Chapter 23, the R factor is related to the ductility of the structural system.
5. The Seismic Weight W . The seismic weight W includes the dead weight of the building, permanent equipment, and partitions. In specific cases, it also may include some of the design snow load and some of the design live load as indicated in Chapter 23.

The Uniform Building Code of 1997 contains provisions for the implementation of the dynamic method of analysis. The code stipulates that when this method is used it shall be implemented using accepted principles of dynamics and it shall conform to the criteria established in the code. These criteria include the following provisions:

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Illustrative Example 24.1

Model as a plane orthogonal frame the four-story reinforced building of Illustrative Example 23.1.

Solution:

Program 29, which is included in the set of educational programs is designed to model the structure as a plane frame. The program uses the stiffness method and static condensation method to determine the reduced stiffness matrix that corresponds to the lateral coordinates at the various levels of the building. The implementation of Program 29 requires numbering the joints of the frame consecutively excluding the fixed joint at the foundation (labeled zero) as shown in Fig. 24.2. In this model each vertical or horizontal member represents all nine elements or members along the building. The flexural stiffness values EI for this model are calculated as follows:

Horizontal Members:

$$\begin{aligned} EI &= 9 \times 3000 \times 12 \times 24^3 / 12 \\ &= 0.37325 \times 10^9 \text{ (kip} \cdot \text{in}^2) \end{aligned}$$

Exterior Columns:

$$\begin{aligned} \text{First and Second stories} \quad EI &= 9 \times 3000 \times 8000 \\ &= 0.216 \times 10^9 \text{ (kip} \cdot \text{in}^2) \end{aligned}$$

$$\begin{aligned} \text{Third and Fourth stories:} \quad EI &= 9 \times 3000 \times 4096 \\ &= 0.111 \times 10^9 \text{ (kip} \cdot \text{in}^2) \end{aligned}$$

Interior Columns

$$\begin{aligned} \text{First and Second stories:} \quad EI &= 9 \times 3000 \times 13,824 \\ &= 0.373 \times 10^9 \text{ (kip} \cdot \text{in}^2) \end{aligned}$$

$$\begin{aligned} \text{Third and Fourth stories:} \quad EI &= 9 \times 3000 \times 8000 \\ &= 0.216 \times 10^9 \text{ (kip} \cdot \text{in}^2) \end{aligned}$$

The following is the Input Data and Output Results using Program 29 (reduced stiffness and mass matrix)

****MODELING BUILDINGS AS ORTHOGONAL PLANES** DATA FILE=D29**

GENERAL DATA:

NUMBER OF STORIES	NST= 4
NUMBER OF JOINTS (EXCLUDING FIXED)	NJ= 12
NUMBER OF HORIZONTAL ELEMENTS	NH= 8
NUMBER OF VERTICAL ELEMENTS	NV= 12

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MASS MATRIX

```

2.044352    0      0      0
0      2.044352    0      0
0      0      2.044352    0
0      0      0      1.671295

```

OUTPUT RESULTS

EIGENVALUES:

```

77.22477    678.6013    1939.751    4052.786

```

NATURAL FREQUENCIES (C.P.S.):

```

1.398616    4.145983    7.009599    10.13204

```

EIGENVECTORS BY ROWS:

```

.112775    .2707485    .4312348    .5154034
-.3107422   -.4679007   -6.907032E-02 .4545203
-.353076    -.1149679    .5030974   -.3465219
.5051842   -.4286013    .212865    -7.766375E-02

```

□

Illustrative Example 24.2

Consider again the four-story reinforced concrete building of Illustrative Example 24.1. Model this building as a plane orthogonal frame and perform the seismic analysis by the UBC-97: dynamic method. Use the response spectra provided by the code Fig. (24.1).

Solution:

Problem Data (from Illustrative Example 24.1)

Seismic weights:

$$W_1 = W_2 = W_3 = 789.1 \text{ Kip}, W_4 = 645.1 \text{ Kip}$$

Total seismic weight:	$W = 3012.4 \text{ Kip}$
Seismic zone factor:	$Z = 0.3$ (for Zone 3)
Occupancy importance factor:	$I = 1.0$ (warehouse)
Site coefficient:	$S = 1.0$ (rock)

1. Modeling the Structure: This structure has been modeled as a plane orthogonal frame (using Program 29) in the solution of Illustrative Example 24.1. The reduced stiffness matrix in reference to the four lateral displacement coordinates of the building, from Example 24.1, is given by

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shear force calculated by the dynamic method. The base shear determined in Illustrative Example 23.2 using the equivalent lateral force method was equal to 140.75 Kip, while the value calculated for this example using the dynamic method is equal to 87.4 Kip. Therefore, the scaling ratio is

$$r = \frac{0.90 \times 140.75}{87.4} = 1.45$$

Table 24.2 shows the result of scaling (by the factor $r = 1.45$) the values in Table 24.1 for the modal effective weight W_m and the modal base shear V_m .

7. **Modal Seismic Force:** The numerical values for the seismic coefficients C_{xm} and for the seismic lateral forces F_{xm} calculated from eqs. (24.20) and (24.19) are shown respectively in Tables 24.3 and 24.4. The design seismic forces calculated by eq. (24.30) are shown in the last column of Table 24.4.
8. **Model Shear Force:** Values for the model shear force V_{xm} at level x calculated using eq. (24.12) and results from Table 24.4 are given in Table 24.5. Design values for the story shear forces V_x calculated by eq. (24.31) are given in the last column of Table 24.5. It should be noticed that the values shown in Table 24.5 for the modal shear force at level $x = 1$ are precisely the values for the base shear force calculated by eq. (24.15) and shown in Table 24.2.

TABLE 24.2 Scaled Values for Modal Effective Weight and Modal Base Shear

Mode m	Modal Effective W_m (Kip)	Weight (%)	Modal Base Shear V_m (Kip)
1	3584	82	122
2	531	12	33
3	144	3	8
4	119	3	5
$\Sigma = 4387$ Kips			

TABLE 24.3 Modal Seismic Coefficient C_{xm}

Level	Model 1	Model 2	Model 3	Model 4
4	0.340	-0.780	1.141	-0.381
3	0.348	0.145	-2.027	0.942
2	0.219	0.982	0.463	-1.897
1	0.091	0.652	1.422	2.235

TABLE 24.4 Modal Seismic Force F_{xm} (Kip)

Level	Model 1	Model 2	Model 3	Model 4	Design F_x Values
4	41	-26	9	-1	50
3	42	5	-16	5	46
2	27	32	4	-9	43
1	11	22	11	11	29

9. **Modal Lateral Displacement:** Values for the modal lateral displacement δ_{xm} at level x calculated from eq. (24.24) are shown in Table 24.6. This table also shows in the last column design values for lateral displacements δ_x calculated by eq. (24.32).
10. **Modal Story Drift:** Table 24.7 shows the values for modal story drift Δ_{xm} calculated by eq. (24.25). Design values for story drift obtained from eq. (24.33) are given in the last column of this table. The maximum story drift permitted by the UBC-1997 should not exceed $(0.04/R_w)H_x = [(0.04/12)144 = 0.48 \text{ in}]$ or $0.005H_x (0.005 \times 144 = 0.72 \text{ in})$, where H_x is the story height and R_w the structural factor from Table 23.3. The design values Δ_x calculated in Table 24.7 for this example are well below these limits for story drift.

TABLE 24.5 Modal Shear Force V_{xm} (Kip)

Level x	Model 1	Model 2	Model 3	Model 4	Design V_x Values
4	41	-26	9	-1	49
3	84	-21	-7	3	86
2	111	11	-3	-6	111
1	122	33	8	5	126

TABLE 24.6 Modal Lateral Displacement δ_{xm} (in)

Level	Model 1	Model 2	Model 3	Model 4	Design δ_x Values
4	0.321	-.023	0.002	0.001	0.320
3	0.268	0.003	-0.005	0.000	0.266
2	0.169	0.023	0.000	-0.002	0.167
1	0.070	0.015	0.002	0.001	0.063

TABLE 24.7 Modal Story Drift Δ_{xm} (in)

Story	Model 1	Model 2	Model 3	Model 4	Design Δ_x Values
4	0.053	-0.026	0.007	-0.001	0.059
3	0.099	-0.020	-0.005	0.002	0.101
2	0.099	0.008	-0.002	-0.003	0.099
1	0.070	0.015	0.003	0.001	0.071

11. **Modal Overturning Moments:** Table 24.8 shows the values for modal overturning moment calculated using eq. (24.26). The last column of this table gives the design values for overturning moments M_x calculated by eq. (24.34).
12. **Modal Torsional Moments:** Table 24.9 shows the values calculated from eq. (24.28) for the modal torsional moments assuming only accidental eccentricity e_x of 5% at each level x of the building (for this example $e_x = 0.05 \times 96 \text{ ft} = 4.8 \text{ ft}$). Design values for torsional moments M_{tx} calculated from eq. (24.35) are shown in the last column of Table 24.9.
13. **The P- Δ Effect:** The UBC-97 specifies that the P - Δ effect does not need to be considered when the ratio of the secondary moment $M_s = P_x \Delta_x$ to the overturning or primary moment M_p is less than 0.10 calculated for each level x of the building. The results of the necessary calculations to evaluate this ratio [eq. (23.13)] are shown in Table 24.10. The largest moment ratio in Table 24.10 is 0.013, which is well below the code limit of 0.1. Consequently, there is no need to account for the P - Δ effect.

TABLE 24.8 Modal Overturning Moment M_{xm} (Kip-ft)

Level x	Model 1	Model 2	Model 3	Model 4	Design M_x Values
4	—	—	—	—	—
3	497	-310	109	-17	596
2	1504	-562	24	22	1605
1	2832	-425	-17	-52	2864
Base	4293	-30	79	8	4293

TABLE 24.9 Modal Torsional Moment M_{tx} (Kip-ft)

Level x	Model 1	Model 2	Model 3	Model 4	Design M_{tx} Values
4	199	-124	43	-7	238
3	402	-101	-35	15	416
2	531	54	-17	-30	534
1	584	157	38	23	606

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25 International Building Code IBC-2000

The IBC-2000 is similar to the UBC-97, but it contains some significant differences. An important difference is that the IBC-2000 includes a set of maps to obtain seismic response spectral values that will result in the same level of risk at any given geographic location in the United States. This code also introduces the concept of Seismic Use Group, which is somewhat analogous to the Importance Factor in the UBC-97. In addition, the IBC-2000 classifies every building in a Seismic Design Category which determines the analysis procedure to be used, the maximum allowed height and drift limitations.

25.1 Response Spectral Acceleration: S_s , S_1

The reader may recall that the response spectral acceleration is defined as the maximum acceleration of a structural system, modeled as a linear single-degree-of-freedom system of mass m and stiffness k subjected to a specific time-history excitation at its base. For such a system, the maximum response (maximum displacement or maximum acceleration) is only a function of its natural period ($T = 2\pi\sqrt{m/k}$) and of damping c expressed as a fraction of the critical damping ($\xi = c/c_{cr}$ with $c_{cr} = 2\sqrt{km}$). In the IBC-2000, the earthquake time-history excitation assumed at a given geographic location is designated as "Maximum Considered Earthquake" (MCE). The response spectral acceleration of a structural system with a natural period, $T_s = 0.2$ sec (short period) and with a natural period $T_1 = 1$ sec is obtained from response acceleration of two sets of maps prepared for various the regions of the United States. These maps are available from the Federal Emergency Management Agency (FEMA). They provide contour lines for the

values of the response spectral acceleration S_S for the short period ($T_s = 0.2$ sec) and for the response spectral acceleration S_1 for period $T_1 = 1$ sec. Interpolation or the value of the next higher contour line may be used for sites located between contour lines. Because these two values, S_S and S_1 , are obtained from maps, they are usually referred to as mapped response spectral accelerations. These maps provide the response spectral acceleration for an assumed damping in the structure equal to 5% of the critical damping and for a type of soil classified as Site Class B (rock). For a different type of soil other than rock, the mapped spectral response accelerations must be modified by factors that depend on the Site Class as described in the Section 25.2.

The Spectral Acceleration Maps are provided in two sets of eight maps each; a set of maps for the short period ($T=0.2$ sec) a set for the long period ($T=1.0$ sec). These maps encompasses the following eight regions:

- | | |
|--------------------------|----------------------------|
| 1. - National | 5. - Intermountain |
| 2. - Northern California | 6.- New Madrid Region |
| 3.- Southern California | 7.- Southwest |
| 4. - Pacific Northwest | 8.- Alaska and Territories |

25.2 Soil Modified Response Spectral Acceleration: S_{MS} , S_{M1}

To obtain the soil modified response spectral acceleration S_{MS} for the short period and S_{M1} for the period $T_1 = 1$ sec. the values of the mapped response spectral accelerations S_S and S_1 , for the short period ($T_s = 0.2$ sec) and for the period $T_1 = 1$ sec must be modified by Site Class Coefficients designated, respectively, as F_a and F_v , namely,

$$S_{MS} = F_a S_S \tag{25.1}$$

and

$$S_{M1} = F_v S_1 \tag{25.2}$$

where F_a is the Site Class Coefficient for the short period and F_v is the Site Class Coefficient for period $T_1 = 1$ sec. Numerical value for the coefficients F_a and F_v are given, respectively, in Tables 25.1 and 25.2 as functions of the Site Class and the values for S_S and S_1 . Site Class classification and definition are described in Section 25.5.

Table 25.1 Site Coefficient (F_a) [IBC-2000: Table 1615.2(1)]

Site Class	Mapped Response Spectral Acceleration at Short Periods				
	$S_s \leq 0.25$	$S_s = 0.50$	$S_s = 0.75$	$S_s = 1.00$	$S_{ss} \leq 1.25$
A	0.8	0.8	0.8	0.8	0.8
B	1.0	1.0	1.0	1.0	1.0
C	1.2	1.2	1.1	1.0	1.0
D	1.6	1.4	1.2	1.1	1.0
E	2.5	1.7	1.2	0.9	*
F	*	*	*	*	*

* Site-specific geo-technical investigation and dynamic site response analysis shall be performed to determine appropriate values.

Table 25.2 Site Coefficient (F_v) [IBC-2000: Table 1615.1.2(2)]

Site Class	Mapped Response Spectral Acceleration at Period $T = 1$ sec				
	$S_s \leq 0.25$	$S_s = 0.50$	$S_s = 0.75$	$S_s = 1.00$	$S_s \leq 1.25$
A	0.8	0.8	0.8	0.8	0.8
B	1.0	1.0	1.0	1.0	1.0
C	1.7	1.6	1.5	1.4	1.3
D	2.4	2.0	1.8	1.6	1.5
E	3.5	3.2	2.8	2.4	*
F	*	*	*	*	*

* Site-specific geo-technical investigation and dynamic site response analysis shall be performed to determine appropriate values.

25.3 Design Response Spectral Acceleration: S_{DS} , S_{D1}

The Design Response Spectral Accelerations (for 5% damping), S_{DS} for the short period and S_{D1} for the period $T = 1$ sec are calculated by

$$S_{DS} = \frac{2}{3} S_{MS} \quad (25.3)$$

and

$$S_{D1} = \frac{2}{3} S_{M1} \quad (25.4)$$

where S_{MS} is the Soil Modified Response Spectral Acceleration for the short period given by eq.(25.1) and S_{M1} is the Soil Modified Earthquake Response Spectral Acceleration for the period $T = 1$ sec given by eq.(25.2).

25.4 Site Class Definition: A, B ...F

The classification of the Site Class (A, B, C, D, E or F) where the structure is located is based on the average properties in the top 100 ft of the soil profile according to the description given in Table 25.3. The Site Class is to be estimated or measured at the building site by a geotechnical engineer or by a geologist/seismologist.

Table 25.3 Site Class Definitions [IBC-2000: Table 1615.1.1]

Average Properties in Top 100 ft. of Soil Profile

Site Class	Soil Profile Name	Shear Wave Velocity (v_s , ft/sec) ⁽¹⁾	Standard Penetration Resistance (N blows/ft) ⁽²⁾	Undrained-Shear Strength (S_{u1} , psf)
A	Hard Rock	> 5000		
B	Rock	2500 to 5000		
C	Very Dense Soil and Soft Rock	1200 to 2500	> 50	> 2000
D	Profile	600 to 1200	15 to 50	1000 TO 2000
E	Profile	< 600	< 15	< 1000
E	-----	Any profile with more than 10 ft of soil having one or more of the following...		
		1. Plasticity Index $PI^{(3)}$		
		2. Moisture content > 40%		
		3. Unconfined shear strength < 500 psf		
F	-----	Any profile containing soils having one or more of the following..		
		1. Liquefiable soils/quick and highly sensitive clays/collapsible weakly cemented soils		
		2. Peats and/or highly organic clays ($h > 10$ ft of peat and/or highly organic clay, h = thickness of soil)		
		3. Very high plasticity clays		
		4. Very thick soft/medium stiff clays ($h > 120$ ft)		

(1) Standard penetration Resistance (\bar{N} blows/ft): ASTM D 1586-84

(2) Undrained Shear Strength (S_{u1} psf): ASTM D 2166-91

(3) Plastic Index (PI): ASTM D 4318

25.5 Seismic Use Group (SUG) and Occupancy Importance Factor (I_E)

Each Structure is assigned a Seismic Use Group (SUG) based on the use of the building. An Occupancy Importance Factor (I_E) is then assigned to each Seismic Use Group. Four SUG designations (I, II, III and IV) and corresponding values of the Occupancy Importance Factor (I_E) are defined and assigned as described in Table 25.4.

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given in Table 25.7 considering the exception stated in Section 25.6 and without the consideration of this exception.

Table 25.7 Design Response Spectral Acceleration and Seismic Design Category for the two-story building of Illustrative Example 25.1.

	S_S	S_1	S_{DS}	S_{D1}	Seismic Design Category
With "Exception"	1.5	0.6	1.00	0.4	D
Without "Exception"	2.05	0.911	1.367	0.607	E

25.7 Design Response Spectral Curve: S_a vs. T

The general procedure to plot the Design Response Spectrum Curve for a given geographical location results in a plot as the one shown in Fig. 25.1. This plot provides the Response Spectral Acceleration S_a as a function of the fundamental period T of the structure. The construction of the Design Response Spectral Curve requires the calculation of the following parameters:

$$T_0 = 0.2 \frac{S_{D1}}{S_{DS}} \quad \text{and} \quad T_s = \frac{S_{D1}}{S_{DS}} \quad (25.5)$$

where S_{DS} and S_{D1} are, respectively, the Design Response Spectral Acceleration for the short period [eq.(25.3)] and the Design Response Spectral Acceleration for the period $T = 1$ sec [eq.(25.4)].

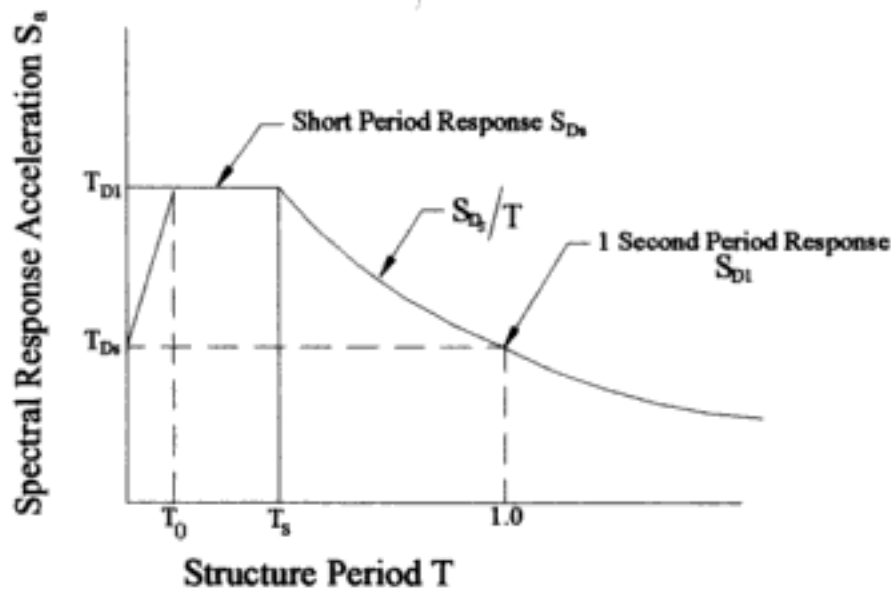


Fig. 25.1 Design Response Spectrum Plot.

The construction of the Design Response Spectrum Plot for a given geographical location such as the plot shown in Fig.25.1, can be obtained by entering the values of T_0 , T_s , as defined in eqs.(25.5) and 1.0 in the abscissa and the values of the Design Spectral Acceleration, S_{D1} and S_{Dk} , in the ordinate axis. Then, on the left, two straight lines are drawn through established points, $(0, S_{D1})$, (T_0, S_{Dk}) and (T_s, S_{Dk}) .

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Print Output Data: For a complete view of all output data click on "Print" button to print the Output Data.

Alternatively, click the button "Save to File"

Enter filename "EXD 25.2" and save in location of choice.

Then SAVE.

The Output Data may be edited using "notepad" or a similar word processor.

Results: The screen "Design Parameters", "output for all Calculations" provides the following results:

For $T_s = 0.2$ sec: $S_S = 0.426g$ and $S_{MS} = 0.341g$

For $T_l = 1.0$ sec: $S_l = 0.940g$ and $S_{Ml} = 0.075g$

25.8 Determination of the Fundamental Period

The IBC-2000 allows the determination of the fundamental period of the building using: 1) approximate empirical formulas, or 2) calculations by rational analysis using structural properties of the resisting elements in a properly substantiated analysis. In this relation, the Code provides the following information:

1. Natural Period determined using approximate formulas

The fundamental period of the building may be taken as the approximate value T_a calculated by the following formula:

$$T_a = C_t h_N^{3/4} \quad (25.6)$$

where

h_N = total height of the building in feet

$C_t = 0.035$ for steel moment-resisting frames¹

$C_t = 0.030$ for reinforced concrete moment-resisting frames and eccentrically braced frames

$C_t = 0.020$ for all other buildings.

Alternatively, the fundamental natural period of moment-resisting frames (steel or concrete) not exceeding 12 stories in height and having a minimum story height of 10 feet may be approximately determined as

$$T_a = 0.1N \quad (25.7)$$

¹ A moment-resisting frame is a structural frame in which the members and joints are capable of resisting forces primarily by flexure.

where N is the number of stories.

2. Natural Period calculated by rational analysis.

The fundamental period of the building, T , in the direction under consideration may be calculated using the structural properties of resisting elements in a properly substantiated analysis. However, the IBC-2000 requires that the calculated fundamental period T be less or equal to the coefficient C_u times the value of the period obtained by an approximate formula, that is

$$T \leq C_u T_a \quad (25.8)$$

where the upper limit coefficient C_u is given in Table 25.8 as a function of the design response spectral acceleration S_{D1} corresponding to the period $T = 1$ sec.

Table 25.8 Upper limit Coefficient C_u [IBC-2000: Table 1617.4.2]

Design Response Spectral Acceleration S_{D1}	Upper Limit Coefficient C_u
< 0.4	1.2
0.3	1.3
0.2	1.4
0.15	1.5
> 0.1	1.7

Note: Upper limit calculated for T does not apply for interstory drift determination.

25.9 Minimum Lateral Force Procedure [IBC-2000: Section 1616.4.1]

The Minimum Lateral Force Procedure is applicable to regular and irregular structures assigned to Seismic Design Category A. This procedure requires a complete lateral-force-resistant system designed to resist minimum forces, F_x , simultaneously applied at the various levels of the building as shown in Fig. 25.3. These minimum forces are calculated by

$$F_x = 0.01 w_x \quad (25.9)$$

in which w_x is the seismic weight allocated to level x of the building.

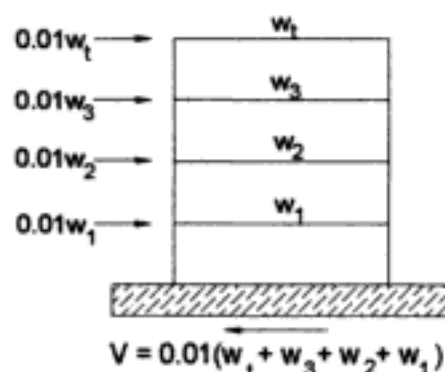


Fig. 25.3 Force distribution for Minimum Lateral Force Procedure.

In the Minimum Lateral Force Procedure, the design seismic forces may be applied independently in two orthogonal directions (orthogonal combined effects are permitted to be neglected).

25.10 Simplified Analysis Procedure [IBC-2000: Section 1617.5]

The use of the Simplified Analysis Procedure is permitted for any structure in Seismic Use Group I of no more than two-story buildings of any material and up to three story buildings of light frame construction.

25.10.1 Seismic Base Shear

The seismic base shear, V , in a given direction shall be determined with the following equation:

$$V = \frac{1.2 S_{DS}}{R} W \quad (25.10)$$

where

S_{DS} = The Design Response Spectral Acceleration for the Short Period. (eq.(25.3))

R = The Response Modification Factor from Table 25.9.

W = Seismic weight of the structure that includes the dead weight and any permanent weight on the building. It also includes (1) a minimum of 25 percent of the live load, (2) partition load or a minimum of 10 pounds per square foot where partition load is included in the floor load design, (3) total operating weight of permanent equipment, and (4) twenty percent of flat roof snow load where the design flat roof snow load exceeds 30 pounds per square foot.

Note: The design story drift (Δ) may be taken as 1% of story height.

25.10.2 Response Modification Factor R

The Response Modification Factor R used in the calculation of base shear force V and of the lateral forces F_x , respectively, given by eqs. (25.10) and (25.11) serve to reduce the design loads to account for the ductility in the structural system as well as for increase damping as the structure is subjected to large deformations beyond the elastic range. Table 25.9 contains an abbreviated set of values for R obtained from a much more detailed table provided by IBC-2000.

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The IBC-2000 stipulates that the structure should be designed for a total base shear force calculated by the formula:

$$V = C_S W \quad (25.12)$$

in which W is seismic weight of the structure that includes the dead weight and any permanent weight on the building. It also includes (1) a minimum of 25 percent of the reduced live load, (2) partition load or a minimum of 10 pounds per square foot where partition load is included in the floor load design, (3) total operating weight of permanent equipment, and (4) twenty percent of flat roof snow load where the design flat roof snow load exceeds 30 pounds per square foot. The coefficient C_S is the Seismic Response Coefficient given by

$$C_S = \frac{S_{DS}}{R/I_E} \quad (25.13)$$

in which

S_{DS} is the Design Spectral Acceleration for short period defined by eq. (25.3)

R is the response modification factor from Table 25.9

I_E is the Occupancy Importance Factor described in Section 25.8

The value of the seismic response coefficient, C_S , calculated by eq. (25.13) need not exceed the following:

$$C_S \leq \frac{S_{D1}}{(R/I_E)T} \quad (25.14)$$

but shall not be less than

$$C_S \geq 0.044 S_{DS} I_E \quad (25.15)$$

and for buildings in categories E and F and buildings for which $S_1 \geq 0.6g$, the value of C_S shall not be less than

$$C_S \geq \frac{0.5S_1}{R/I_E} \quad (25.16)$$

where

R is the Response Modification Factor from Table 25.9

I_E is the Occupancy Importance Factor given in Table 25.4

S_{D1} is the Design Response Spectral Acceleration defined by eq. (25.4)

S_1 is the mapped earthquake response spectral; acceleration at $T=1$ -sec period determined in accordance with Section 25.1

25.11.1 Distribution of Lateral Forces

The total base shear force V calculated from eq. (25.12) is distributed over the height of the structure as a lateral force, F_x at each level calculated by

$$F_x = \frac{w_x h_x^k}{\sum_{i=1}^N w_i h_i^k} \quad (25.17)$$

where

N = total number of stories above the base of the building

h_x = height of level x

w_x = seismic weight assigned to the x level of the building.

$k = 1.0$ for buildings having a period $T \leq 0.5$ sec

$k = 2.0$ for buildings having a period $T \geq 2.5$ sec

k is determined by linear interpolation for buildings having a period $0.5 < T < 2.5$ sec

The distribution of the lateral forces F_x is shown in Fig. 25.4 for a multistory building. The IBC-2000 stipulate that the force F_x at level x , be applied over the area of the building according to the mass distribution at that level.

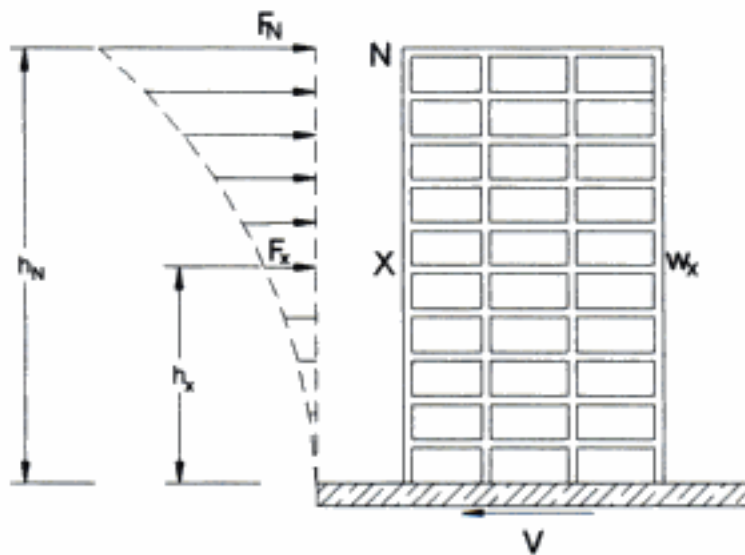


Fig. 25.4 Vertical distribution of the base shear force.

25.11.2 Overturning Moments

The IBC-2000 requires that overturning moments be determined at each level of the building. The overturning moment is calculated by static from the equivalent lateral forces F_x [eq. (25.17)] applied at the levels of the building above the level under consideration. However, the code allows for a reduction in the design overturning

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overturning moment, M'_x , and the primary moment M_x includes the Deflection Amplification Factor C_d . That is, in the IBC-2000 the ratio θ_x is calculated by

$$\theta_x = \frac{M'_x}{M_x} = \frac{P_x \Delta_x}{V_x h_{sx} C_d} \quad (25.20)$$

where:

P_x = total weight at level x and above

Δ_x = Drift of story x

V_x = shear force of story x

h_{sx} = height of story x

C_d = Deflection Amplitude Factor (Table 25.9)

25.11.5 Story Drift

The IBC-2000 specifies that the design deflection δ_x , at level x , be determined in accordance with the following equation:

$$\delta_x = \frac{C_d \delta_{xe}}{I_E} \quad (25.21)$$

where

C_d = Deflection Amplification Factor (Table 25.9)

δ_{xe} = Deflection determined by an elastic analysis of the seismic-force-resisting system

I_E = Occupancy Importance Factor determined in accordance to Section 25.5

The design story drift, Δ , is computed as the difference of the lateral deflections of the center of mass at the top and bottom levels of the story under consideration. However, for structures assigned to Seismic Design Categories C, D, E or F (see Section 25.6) with plan irregularities, the design story drift Δ shall be computed as the largest difference of the deflection along any of the edges of the structure at the top and bottom levels of the story under consideration.

The IBC-2000 stipulates a maximum allowable story drift Δ_a which depends on the type of building and on the Seismic Use Group (SUG). Table 25.11. provides values for Δ_a , the allowable story drift. Therefore, the limitation on the story drift, Δ_x may be expressed as:

$$\Delta_x = (\delta_x - \delta_{x-1}) \leq \Delta_a \quad (25.22)$$

where

δ_x is the lateral displacement at level x calculated by eq.(25.21)

Δ_a is the allowable story drift from Table 25.10

Table 25.10 Allowable Story Drift (Δ_a) [IBC-2000: 1617.3]

Building	Seismic Use Group (SUG)		
	I	II	III
Buildings $\delta \leq 4$ stories in height (other than masonry)	$0.025 h_{sx}$	$0.020 h_{sx}$	$0.015 h_{sx}$
Masonry cantilever shear wall bldgs.	$0.010 h_{sx}$	$0.010 h_{sx}$	$0.010 h_{sx}$
Other masonry shear wall buildings	$0.007 h_{sx}$	$0.007 h_{sx}$	$0.007 h_{sx}$
Masonry wall frame buildings	$0.013 h_{sx}$	$0.013 h_{sx}$	$0.010 h_{sx}$
All other buildings	$0.020 h_{sx}$	$0.015 h_{sx}$	$0.010 h_{sx}$

h_{sx} = story height below level x

25.12 Redundancy/Reliability Factor

The Redundancy/Reliability Factor in IBC-2000 is almost identical to the Redundancy/Reliability Factor defined by eq.(24.19) in Chapter 24 for IBC-97. The only difference is that the area at the base of the building, A_B used in the UBC-97 is replaced in the IBC-2000 by the area A_i at the floor level i of the building. That is, the Redundancy/Reliability Factor, ρ_i , corresponding to level i is now defined as

$$1 \leq \rho_i = 2 - \frac{20}{r_{max} \sqrt{A_i}} \leq 1.5 \quad (25.23)$$

where

A_i is the floor area at level i of the building in square feet, and
 r_{max} is the maximum value of r_i , which is defined as the ratio of the force on a resisting element to the total shear at a story in the building. The maximum ratio, r_{max} , is then defined as the largest value of r_i in the lower two-thirds of the building.

The code provides the following guidance for determining r_{max} .

- For special moment-resisting frames $r_{max} = 70\%$ of the maximum shear in two internal columns/total story shear.
- For shear walls, $r_{max} = \text{maximum wall shear times } (10/L_w) \text{ divided by the total story shear force } (L_w = \text{length of the shear walls}).$

The code also specifies that for special moment-resisting frames, if ρ exceeds 1.25, additional bays must be added. For the purpose of calculating lateral displacements or drifts in seismic zones 0, 1 and 2, $\rho = 1.0$.

25.13 Earthquake Load Effect

The IBC-2000 specifies that the Earthquake Load, E , on a structural element be calculated as the sum of the effects due to the lateral seismic forces amplified by the redundancy factor ρ plus the effect of the vertical component of the earthquake ground motion. Namely,

$$E = \rho Q_E + 0.2 S_{DS} D \quad (25.24)$$

where

- Q_E = the effect of horizontal seismic forces
- ρ = Redundancy/Reliability Factor to be taken as the largest of the values for ρ_i obtained in accordance to Section 25.1.
- S_{DS} = Design Spectral Response Acceleration for short periods calculated by eq.(25.4).
- D = vertical seismic load on an element

25.14 Building Irregularities

The Equivalent Lateral Force Procedure is based on the assumptions and characteristics of regular structures. Building irregularities are the cause of stress concentrations leading to structural damage and poor performance. If the structure has irregularities, it must comply with additional code requirements. Figures 25.5 and 25.8 show, respectively, examples of plan irregularities and of vertical irregularities.

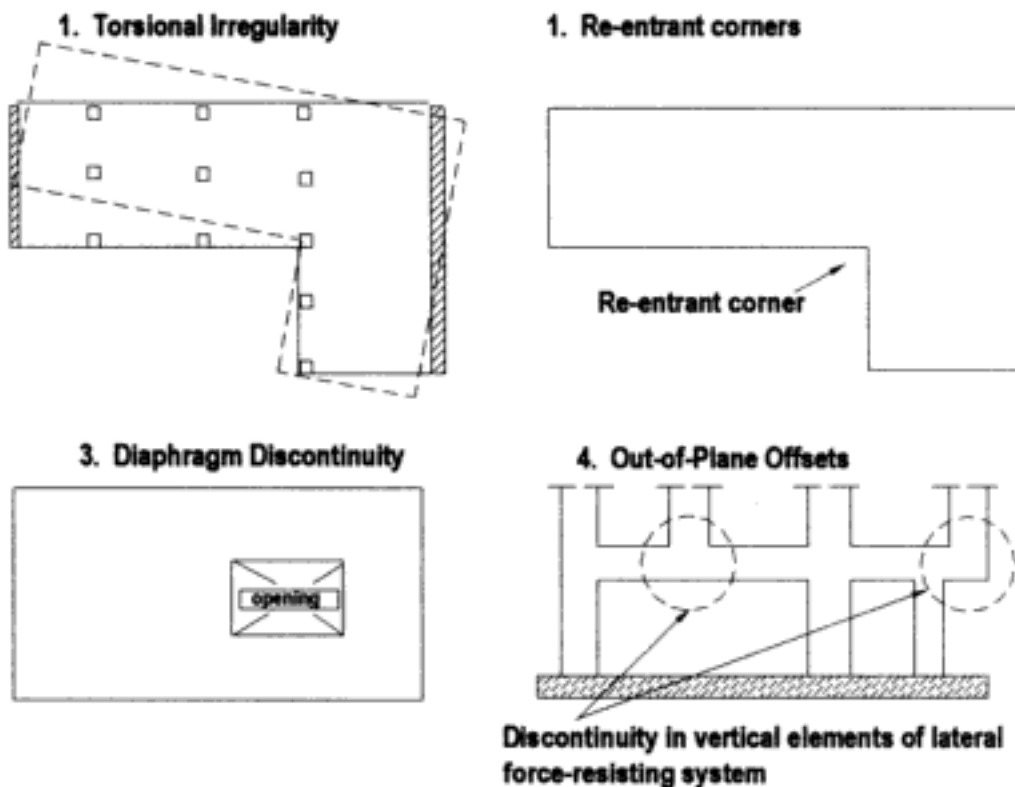


Fig. 25.5 Examples of structural plan irregularities.

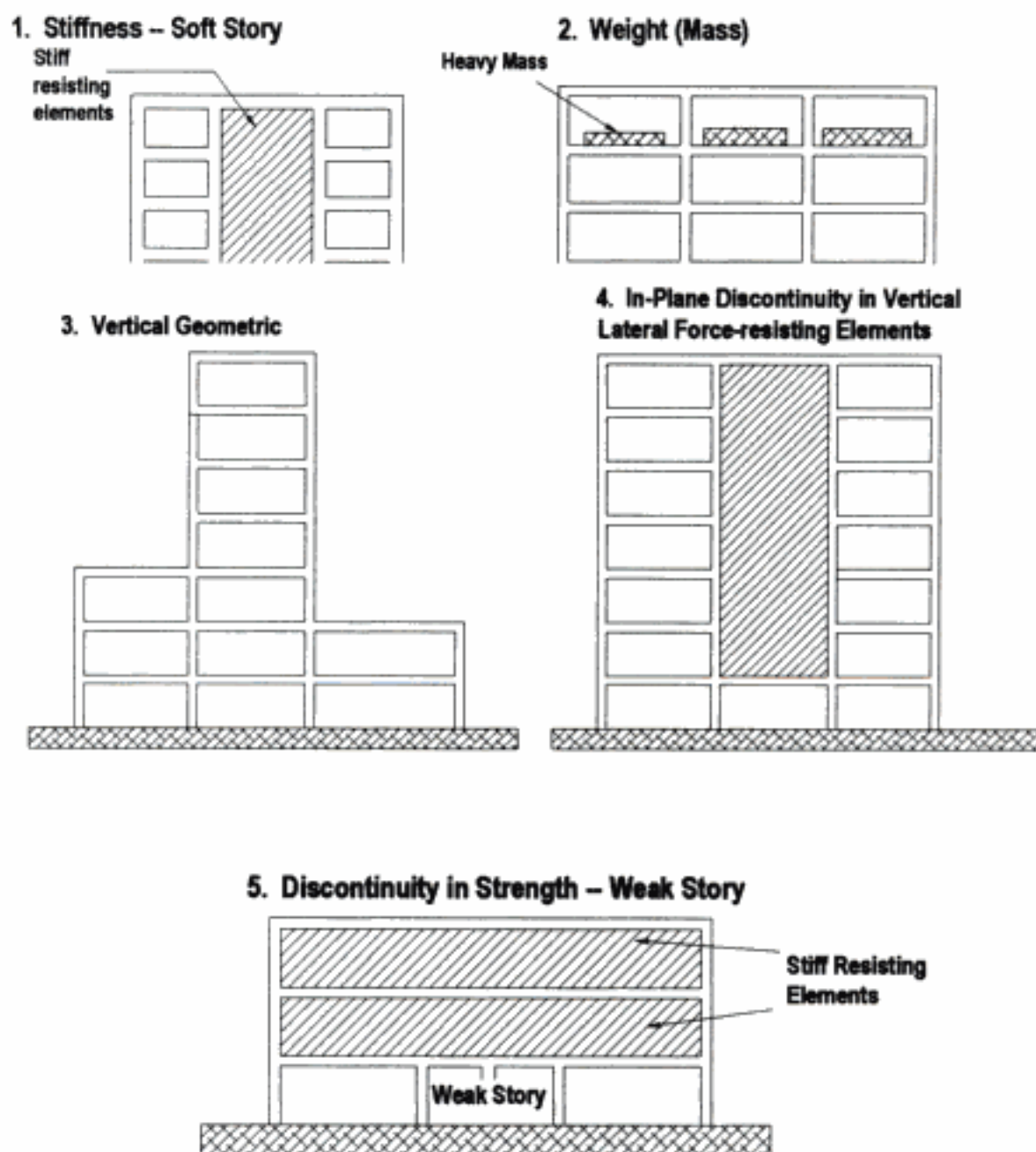


Fig. 25.6 Examples of Vertical Structural Irregularities

Illustrative Example 25.3

Using the Equivalent Lateral Force Method of the IBC-2000, perform the seismic analysis of four-story concrete building of Illustrative Example 24.1 presented in Chapter 24. The building site is in Seattle, Washington with Zip Code 94704.

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where

V_x is the story shear force calculated in Table 25.11

K_x is the stiffness of the story calculated in the solution of Illustrative Example 24.1, now listed in Table 25.12.

Lateral Displacement and Story Drift:

The elastic lateral displacement δ_{ex} may be calculated at each level of the building by adding story drifts for the story at that level and those below as shown calculated in Table 25.12.

The design lateral displacement, δ_x , is then determined by

$$\delta_x = \frac{C_D \delta_{xe}}{I_E} \quad \text{eq.(25.21) repeated}$$

$C_D = 5.5$ (Deflection Amplification Factor) (Table 25.9)

$I_E = 1.0$ (Occupancy Importance Factor) (Table 25.4)

Calculated values for the design lateral displacements, δ_x at the various levels of the building are shown in Table 25.12.

Table 25.12 Lateral Displacement and Story Drift for Illustrative Example 25.3

Level x	Story Shear V_x (kip)	Story Stiffness K_x (kip/in)	Story Drift (Elastic) Δ_x (in)	Lateral Displ. (Elastic) δ_{ex} (in)	Lateral Displ. (Inelastic) δ_x (in)	Story Drift (Inelastic) Δ_{Mx} (in)	Allowable Story Drift (Inelastic) Δ_a (in)
4	91.83	1,757.7	0.052	0.301	1.66	0.29	3.6
3	174.62	1,757.7	0.099	0.249	1.37	0.54	3.6
2	229.25	3,236.0	0.071	0.150	0.83	0.50	3.6
1	256.10	3,236.0	0.079	0.079	0.43	0.43	3.6

The inelastic story drift is then given by

$$\Delta_{Mx} = \delta_x - \delta_{x-1} \leq \Delta_a \quad \text{eq.(25.22) repeated}$$

where the allowable story drift Δ_a is given by

$$\Delta_a = 0.025 h_x = 0.025 \times 144 = 3.6 \text{ in} \quad \text{(Table (25.10))}$$

Values calculated for the inelastic story drift Δ_{Mx} shown in the last column of Table 25.12 are well below the allowable limit $\Delta_a = 3.6$ in.

Redundancy/Reliability Factor:

$$1 \leq \rho_i = 2 - \frac{20}{r_{\max} \sqrt{A_i}} \leq 1.5 \quad \text{eq.(25.23) repeated}$$

$$A_i = 48 \times 96 = 4608 \text{ ft}^2$$

$$r_i = 0.7 \frac{\frac{V_s}{24} + \frac{V_s}{24}}{V_s} = 0.058 \quad (\text{Code guidance to calculate } r_i, \text{ Sec. 25.12})$$

$$\rho_i = -3.08$$

Then by eq.(25.23):

$$\underline{\rho_{\max} = 1.0}$$

25.15 Summary

The International Building Code was prepared by the International Code Council (ICC), whose members are representatives of BOCA (Building Officials and Code Administrators), ICBO (International Conference of Building Officials) and SBCCI (Southern Building Code Congress International). The unified effort of all three agencies resulted in the International Building Code, which contains provisions for earthquake resistant design specified in the latest versions of several building codes. These codes are in current use in different regions of the country. The IBC-2000 includes a set of maps to obtain seismic response spectral values that will result in the same level of risk at any location in the country.

Appendices

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$$3.12 \quad \omega = 49.2 \text{ (rad/sec)}$$

$$Y = 0.387 \text{ (cm)}$$

$$\sigma_{\max} = 110.7 \text{ (Mpa)}$$

$$3.13 \quad f = f_r \sqrt{1 + \frac{m_t}{m}}$$

$$3.14 \quad \omega_p = \omega \sqrt{1 - 2\xi^2} \text{ for } \xi < \frac{1}{\sqrt{2}}$$

$$U_p = \frac{u_H}{2\xi\sqrt{1-\xi^2}}$$

$$3.15 \quad (a) f = 18.58 \text{ cps}$$

$$(b) \xi = 0.0735$$

$$(c) F_0 = 4825 \text{ lb}$$

$$(d) F_0 = 4840 \text{ lb}$$

$$3.16 \quad \xi = \frac{U_1(1-r_1^2)}{2r_1\sqrt{U_r^2 r_1^2 - U_1^2}}$$

$$F_r = \frac{U_r U_1(1-r_1^2)k}{r_1\sqrt{U_r^2 r_1^2 - U_1^2}}$$

$$3.17 \quad (a) M\ddot{u} + c\dot{u} + ku = \frac{m_1 F_0}{m_1 + m_2} \sin \omega t$$

$$\text{where } M = \frac{m_1 m_2}{m_1 + m_2}$$

$$(b) u = \frac{m_1 F_0 \sin(\omega t - \theta)}{k(m_1 + m_2)\sqrt{(1-r^2)^2 + (2r\xi)^2}}$$

Chapter 4

$$4.1 \quad (a) u(t = 0.5) = -0.407 \text{ in}$$

$$(b) u_{\max} = 1.37 \text{ in}$$

$$4.2 \quad (a) u(t = 0.5) = -0.102 \text{ in}$$

$$(b) u_{\max} = 1.17 \text{ in}$$

$$4.3 \quad DLF = \frac{t}{t_d} - \frac{\sin \omega t}{\omega t_d} \text{ for } t \leq t_d$$

$$DLF = 1 + \frac{1}{\omega t_d} (\sin \omega t - \sin \omega(t + t_d)) \text{ for } t \geq t_d$$

$$4.4 \quad V_{\max} = 18,093 \text{ lb for left column}$$

$$V_{\max} = 1908 \text{ lb for right column}$$

$$4.5 \quad V_{\max} = 15,640 \text{ lb for left column}$$

$$V_{\max} = 1649 \text{ lb for right column}$$

$$4.8 \quad u_1(t) = -\frac{V}{\omega} \cos \omega t + vt$$

- 4.9 $u(t) = \frac{u_{st}}{t_d} \left[t - \frac{\sin \omega t}{\omega} \right]$ for $t \leq t_d$
 $= \frac{u_{st}}{t_d} \left\{ t_d - \frac{\sin \omega t_d}{\omega} \right\} \cos \omega(t - t_d) + \frac{1}{\omega} \sin \omega(t - t_d)$ for $t \geq t_d$
- 4.10 $u_{\max} = 1.348$ (in)
 $\sigma_{\max} = 11,906$ (psi)
- 4.11 $u_{\max} = 0.72$ (in)
 $\sigma_{\max} = 16,282$ (psi)
- 4.12 $u(t=0.5) = -1.903$ in
- 4.13 $u(t=0.5) = -0.060$ in
- 4.14 $u(t=1) = -2.809$ in
- 4.15 $u(t=1) = -2.397$ in
- 4.16 $u_{\max} = 6.03$ in (undamped system)
 $u_{\max} = 4.59$ in (with 20% damping)
- 4.17 $u_{\max} = 1.51$ in
- 4.18 $u_{\max} = 1.42$ in
- 4.19 $u_{\max} = 0.58$ in
- 4.20 $u_{\max} = 1.00$ in
- 4.21 $u_{\max} = 0.76$ in
- 4.22 $u_{\max} = 0.66$ in
- 4.23 $u_{\max} = 0.34$ in
- 4.24 $\sigma_{1\max} = 3842$ psi
 $\sigma_{2\max} = 6831$ psi
- 4.25 $u_{\max} = 0.71$ in
- 4.26 (a) $\sigma = \pm 6193$ psi
 (b) $F_{\max} = 11,376$ lb

Chapter 5

- 5.1 $u_{\max} = 0.374$ in
- 5.2 $\sigma_{\max} = 7.246$ ksi
- 5.3 $u_{\max} = 0.418$ in
- 5.4 $\sigma_{\max} = 12.788$ ksi
- 5.5 $S_D = 1.9$ in
 $S_v = 22.4$ in/sec
 $S_a = 0.68$ g
- 5.6 $S_D = 1.28$ in
 $S_v = 15.36$ in/sec
 $S_a = 0.48$ g
- 5.7 $S_D = 11.0$ in
 $(F_s)_{\max} = 88.0$ Kip
- 5.8 $(F_s)_{\max} = 36.0$ Kip
- 5.9 $(F_T)_{\max} = 44.0$ Kip
- 5.10 $S_D = 8.0$ in
- 5.11 $\mu = 1.8$
- 5.12 (a) $S_D = 40$ in

$$S_v = 50.3 \text{ in/sec}$$

$$S_a = 1.63 \text{ g}$$

$$(b) S_D = 4.8 \text{ in}$$

$$S_v = 60.0 \text{ in/sec}$$

$$S_a = 1.96 \text{ g}$$

$$5.13 \quad (a) S_D = 0.46 \text{ in}$$

$$S_v = 5.8 \text{ in/sec}$$

$$S_a = 0.19 \text{ g}$$

$$(b) S_D = 6.0 \text{ in}$$

$$S_v = 18 \text{ in/sec}$$

$$S_a = 0.6 \text{ g}$$

$$5.14 \quad S_D = 0.78 \text{ in (at } f = 0.5 \text{ cps)}$$

$$S_v = 2.46 \text{ in/sec}$$

$$S_a = 7.72 \text{ in/sec}^2$$

$$5.15 \quad S_D = 8.03 \text{ in (at } f = 1.00 \text{ cps)}$$

$$S_v = 50.45 \text{ in/sec}$$

$$S_a = 317.00 \text{ in/sec}^2$$

$$5.16 \quad S_D = 3.26 \text{ in (at } f = 1.00 \text{ cps)}$$

$$S_v = 20.38 \text{ in/sec}$$

$$S_a = 127.40 \text{ in/sec}^2$$

Chapter 6

$$6.1 \quad u_{\max} = -10.27 \text{ in}$$

$$6.2 \quad u_{\max} = 2.56 \text{ in}$$

$$6.3 \quad u_{\max} = 6.47 \text{ in}$$

$$6.4 \quad u_{\max} = 1.03 \text{ in}$$

$$6.5 \quad u_{\max} = 5.19 \text{ in}$$

$$6.6 \quad u_{\max} = 2.38 \text{ in}$$

$$6.7 \quad \mu = 1.7$$

$$6.8 \quad u(t = 0.5) = 0.4477 \text{ in}$$

$$6.9 \quad u(t = 0.5) = 0.2654 \text{ in}$$

$$6.10 \quad u(t = 0.5) = 0.1423 \text{ in}$$

$$6.11 \quad u(t = 0.5) = 0.1340 \text{ in}$$

$$6.12 \quad a_0 = 2.78$$

Chapter 19

$$19.1 \quad F(t) = \frac{120}{\pi} \left[\sin 2\pi t + \frac{1}{3} \sin 6\pi t + \frac{1}{5} \sin 10\pi t \dots \right]$$

$$19.2$$

$$F(t) = 10^{-6} [357 \sin 2\pi t - 26 \cos 2\pi t + 36 \sin 6\pi t - 532 \cos 6\pi t - 35 \sin 10\pi t - 7 \cos 10\pi t + \dots]$$

$$19.3 \quad u(t = 0.5) = 0.3518 \text{ in}$$

- 19.4 (a) $a_n = \frac{720}{\pi(1-n^2)}, n = 2, 4, 6, \dots$
 $a_n = 0, n = 1, 3, 5, \dots$
 $b_n = 0, n = 1, 2, 3, \dots$
- 19.5 $u(t = 0.05) = -0.2065$ in
 19.6 $u(t = 0.05) = -0.2064$ in
 19.7 $u(t = 0.05) = 0.1295$ in
 19.8 (a) $a_0 = 0.0350$
 $a_1 = 0.0069, b_1 = -0.0361$
 $a_2 = -0.0724, b_2 = -0.0402$
 (b) $u(t = 0.35) = 0.2570$ in
 19.9 $u(t = 0.35) = 0.2327$ in
 19.10 $u(t = 0.5) = 0.027$ in
 19.11 $u(t = 0.5) = (0.02842 - 0.00011i)$ in
 19.12 (a) $a_0 = \frac{P_0}{\pi}$
 $a_n = 0, n = 1, 3, 5, \dots; b_1 = \frac{P_0}{2}$
 $a_n = \frac{P_0}{\pi} \cdot \frac{2}{1-n^2}, n = 2, 4, 6, \dots; b_n = 0, n > 1$
- 19.13 $u(t = 0.5) = 0.0731$ in
 19.14 $u(t = 0.5) = 0.0543$ in
 19.15 $u_1(t) = 0.229(\sin \pi t - 0.75 \sin 4.19t)$
 $u_2(t) = 0.229[\sin \pi(t-1) - 0.75 \sin 4.19(t-1)]$
 $u(t) = u_1(t)$ for $0 \leq t \leq 1.0$ sec
 $u(t) = u_1(t) + u_2(t)$ for $t \geq 1.0$ sec
 19.16 $u(t = 1 \text{ sec}) = 0.2419$ (in)
 19.17 $u(t = 1 \text{ sec}) = 0.0532$ (in)
 19.18 $u(t = 1 \text{ sec}) = 0.0035$ (in)
 19.19 $u(t = 1 \text{ sec}) = 0.0034$ (in)
 19.20 $u(t = 1 \text{ sec}) = 0.0028$ (in)

Chapter 21

- 21.1 $M^* = 4.48 \text{ lb} \cdot \text{sec}^2 / \text{in}$
 $C^* = 2250 \text{ lb} \cdot \text{sec} / \text{in}$
 $K^* = 45,000 \text{ lb} / \text{in}$
 $F^*(t) = 625 \cdot f(t) \text{ lb}$
- 21.2 $M^* = \frac{5}{6} m$
 $C^* = c$
 $K^* = k$
 $F^*(t) = \frac{M(t)}{L}$

$$21.3 \quad M^* = \frac{2}{3} \bar{m} L^2$$

$$C^* = cL$$

$$K^* = kL$$

$$F^* = \frac{P_0 L}{6} f(t)$$

$$21.4 \quad M^* = \frac{m}{2\pi} (5\pi - 8)$$

$$K^* = \frac{EI\pi^4}{32L^3}$$

$$F^*(t) = 0.2929 F_0 f(t)$$

$$21.5 \quad K_G^* = -\frac{N\pi^2}{8L}$$

$$21.6 \quad M^* = 0.1237 \frac{\gamma d}{g}$$

$$K^* = \frac{E_c \pi d^4}{128L^3}$$

$$F^*(t) = -0.1807 P_0(t) L d$$

$$21.7 \quad \omega = \sqrt{\frac{48EI}{L^3(m + \frac{17}{35}m_b)}} \text{ rad/sec}$$

$$21.8 \quad \omega = 7.825 \sqrt{\frac{EI}{m_b L^3}} \text{ rad/sec}$$

$$21.9 \quad f = 3.51 \text{ cps}$$

$$21.10 \quad \omega = 2.15 \sqrt{\frac{gEI}{WL^3}} \text{ rad/sec}$$

$$21.11 \quad \omega = 0.62 \sqrt{\frac{gAG}{WL}} \text{ rad/sec}$$

$$21.12 \quad f = 0.496 \text{ cps}$$

$$21.13 \quad f = 0.492 \text{ cps}$$

Appendix II

Computer Programs

Two sets of computer programs are used through out this textbook: (1) A set of computer programs in Structural Dynamics, which are identified as Educational Program, and (2) The professional program SAP2000.

Installation

SAP2000:

1. - Insert in the CD-drive the diskette provided with this book.
2. - From the "start" menu click on RUN.
3. - Select SETUP.EXE followed by OK.
4. - Follow the prompts to complete the installation of SAP2000.

Educational Programs:

1. - Create a new folder in the C-drive, which could be named DYNAQ.
2. - Copy to this new folder, DYNAQ just created, the folders QB-EXE and QB-SOURCE. (The folder QB-EXE contains the executable programs and the folder QB-SOURCE the source programs which are provided to allow the interested user to introduce modifications in the programs as desired)

The Educational Programs are provided in the QUICK BASIC language. They are menu driven and they can be executed directly by simply entering the name of the program (P2, P3,.....P29) or alternatively they can be accessed through the execution of the program named MASTER that contains the commands to perform various tasks in Structural Dynamics. These programs were developed by the author almost 30 years ago and the present time they may be considered passé. Initially, the author had planned to delete them from this new edition of his textbook on Structural Dynamics. However, as the revision of the previous edition progressed, he realized that it would still be helpful to the student to have these programs during the course of learning the material. Finally, it was decided to provide in this volume a CD-ROM containing the files for both sets of programs – SAP2000 (Introductory version) and the Educational Programs developed by the author –.

Table II-1 presents a list of the Educational Programs with a reference to the pertinent chapter in which the program is described and used.

TABLE II-1 Educational Computer Programs in Structural Dynamics

Program	File	Chapter	Purpose
1	MASTER		Menus with options and commands
2	P2	4	Response by direct integration
3	P3	4	Response simple oscillator to impulses
4	P4	19	Response in the frequency domain
5	P5	6	Response for elastoplastic behavior
6	P6	5	Development of seismic response spectra
7	P7	7	Modeling structures as shear buildings
8	P8	7	Natural frequencies and normal modes
9	P9	8	Response by modal superposition
10	P10	8	Response to harmonic excitation
11	P11	20	Absolute damping from modal damping
12	P12	9	Reduction of the dynamic problem
13	P13	10	Modeling structures as beams
14	P14	11	Modeling structures as plane frames
15	P15	12	Modeling structures as grid frames
16	P16	13	Modeling structures as space frames
17	P17	14	Modeling structures as plane trusses
18	P18	14	Modeling structures as space trusses
19	P19	16	Response by step integration
20	P20	*	Maximum member's end-forces
21	P21	22	Response to random vibrations
24	P24	24	Earthquake Analysis UBC 97
29	P29	24	Modeling Orthogonal Frames
X1	X1	*	File for modeled structures
X2	X2	*	File for eigensolution

* Programs not described in this book

Notes: The output or result of problems solved using the Educational Programs will be stored in a file whose name is provided by the user. In fact, during the execution of any of the Educational Programs, the user is requested to provide the name of a file to store the output results. The user is also asked to select between a "new file" or to "append" to an existing file.

This file containing the output obtained in the execution of a program, can be open using Notepad or any other editor or word processor selected by the user. Once the file containing the output with results has been open, it can, of course, be edited and printed as desired

There are several alternatives in SAP2000 to execute commands (use of the Menus, combination of key strokes or click on an ICON). The use of Menus has been adopted for all the Illustrative Examples presented in this textbook. The user, as he gains experience, he may very well have other preferences.

Appendix III

Glossary

Accelerometer. An instrument for measuring ground acceleration as a function of time.

Aliasing. The phenomenon in which higher harmonics introduce spurious low frequency components. This occurs when the number of sampled points of a function is insufficient to describe the function. (See Nyquist Frequency.)

Amplitude. Maximum value of a function as it varies with time. If the variation with time can be described by either a sine or cosine function, it is said to vary harmonically.

Angular Frequency /Circular Frequency. The frequency of periodic function in cycles per second (Hertz) multiplied by 2π , expressed in rad/sec.

Autocorrelation of a Random Function $x(t)$. Correlation between the function $x(t)$ and the out-of-phase function $x(t + \tau)$ as defined by eq. (22.19).

Base Shear Force. The total lateral force on the structure equivalent to the earthquake excitation at the base of the structure.

Basic Design Spectra. Smooth or average plots of maximum response of single degree-of-freedom systems used in seismic design of structures.

Boundary Condition. A constraint applied to the structure independent of time.

Braced Frame. An essentially vertical truss system of the concentric or eccentric type which is provided to resist lateral forces.

Building Frame System. An essentially complete space frame which provides support for gravity loads.

Characteristic Equation. An equation whose roots are the natural frequencies.

Circular Frequency. See Angular Frequency.

Complementary Solution. The solution of a homogeneous differential equation (no external excitation).

Complete Quadratic Combination (CQC). A method of combining maximum values of modal contributions which is based on random vibration theory and includes cross correlation terms.

Concentric Braced Frame. A brace frame in which the members are subjected primarily to axial forces.

Consistent Mass. Mass influence coefficients determined by assuming that the dynamic displacement functions are equal to the static displacement functions.

Correlation between Random Variables $x_1(t)$ and $x_2(t)$. The time average of the product of the functions $x_1(t)$ and $x_2(t)$.

Coupled Equations. A system of differential equations in which the equations are not independent from each other.

Critical Damping. Minimum amount of viscous damping for which the system will not vibrate.

D' Alembert Principle. This principle states that a dynamic system may be assumed to be in equilibrium provided that the inertial forces are considered as external forces.

Damped Frequency. The frequency at which a viscously damped system oscillates in free vibration.

Damping. The property of the structure to absorb vibrating energy.

Damping Ratio. The ratio of the viscous damping coefficient to the critical damping.

Degrees of Freedom. The number of independent coordinates required to completely define the position of the system at any time.

Deterministic Vibration. A process which can be predicted by an exact mathematical expression.

Dirac's Delta Function. A generalized function having the properties described in eq. (22.52).

Direct Stiffness Method. The method of assembling the system stiffness matrix by proper summation of the stiffness coefficients of the elements in the system.

Discrete Fourier Transform. A summation of harmonic terms to express the Fourier transform for a function defined by a finite number of points.

Dual System. A combination of a special or intermediate moment-resisting space frame and shear walls or braced frames.

Ductility Ratio. The ratio between the maximum displacement for elastoplastic behavior and the displacement corresponding to yield point.

Dynamic Condensation. A method of reducing the dimension of the eigenproblem by establishing the dynamic relationship between primary and secondary coordinates.

Dynamic Magnification Factor. The ratio of the maximum displacement of a single degree of freedom excited by a harmonic force to the deflection that would result if a force of that magnitude were applied statically.

Earthquake. The vibrations of the Earth caused by the passage of seismic waves radiating from some source of elastic energy.

Eigenproblem. The problem of solving a homogeneous system of equation: containing a parameter which should be determined to provide nontrivial solutions.

Elastic Rebound Theory. The theory of earthquake generation proposing that faults remain locked while strain energy slowly accumulates in the surrounding rock, and then suddenly slip, releasing this energy.

Elastoplastic. A system which behaves elastically for a force that does not exceed a maximum value and plastically above this maximum.

Ensemble. A set of samples or records of a random process.

Epicenter. The point on the Earth's surface directly above the focus.

Ergodic Process. A stationary random process for which the time average of any record is equal to the average across the ensemble.

Fast Fourier Transform (FFT). A very efficient algorithm implemented in a computer program for the calculation of the response in the frequency domain.

Flexibility Coefficient. f_{ij} is the displacement at coordinate i due to a unit force (or unit moment) applied at coordinate j .

- Forced Vibration.** Vibration in which the response is due to external excitation of the system.
- Fourier Analysis.** Method of determining the response by superposition of the responses to the harmonic components of the excitation.
- Fourier Transform.** The Fourier transform $C(\omega)$ of a function $F(t)$ is defined by eq. (22.23).
- Fourier Transform Pair.** In reference to the function $F(t)$, the Fourier transform pair is given by eqs. (22.23) and (22.24).
- Free Body Diagram.** A sketch of the system, isolated from all other bodies, in which all the forces external to the body are shown.
- Free Vibration.** The vibration of a system in absence of external excitation.
- Frequency Analyzer or Spectral Density Analyzer.** An instrument that measures electronically the spectral density function of a signal.
- Frequency Ratio.** The ratio between the forcing frequency to the natural frequency for a system excited by a harmonic load.
- Fundamental Frequency.** The lowest natural frequency of a multidegree-of freedom vibrating system.
- Gauss-Jordan Reduction or Elimination.** A computational technique in which elementary row operations are applied systematically to solve a linear system of equations.
- Generalized Coordinates.** A set of independent quantities which describe the dynamic system at any time. These quantities are generally functions of the geometric Coordinates.
- Geometric Stiffness Coefficient.** k_{Gij} is the force at coordinate i due to a unit displacement at coordinate j and resulting from the axial forces in the structure.
- Harmonic.** A sinusoidal function having a frequency that is an integral multiple of the fundamental frequency.
- Harmonic Force.** A force expressed by a sine, cosine (or equivalent exponential) function.
- Hypocenter or Focus.** The Point in the interior of the earth at which rupture is initiated during an earthquake.
- Impulsive Load.** A load that is applied during a relatively short time interval producing an instantaneous change in velocity.
- Initial Conditions.** The initial values of specific functions such as displacement, velocity, or acceleration evaluated at time $t = 0$.
- Intensity (of Earthquakes).** A measure of ground shaking obtained from the damage done to man-made structures, changes in the Earth's surface, and witness reports.
- Intermediate Moment-Resisting Space Frame (IMRSF).** A concrete space frame designed in conformance with Section 1921.8 (k) of UBC-97.
- Isolation.** The reduction of severity of the response, usually attained by proper use of a resilient support.
- Lateral Force Method.** A method of analysis in which lateral horizontal forces at various levels of the structure are considered equivalent to seismic excitation at the base of the structure.
- Lateral Force-Resisting System.** That part of the structural system assigned to resist lateral forces.

Linear Acceleration Method. A step-by-step method for the integration of the differential equations of motion in which the acceleration is assumed to be a linear function during each time step.

Linear System. A system of differential equations in which no term contains products (or exponents) of the dependent variables or their derivatives.

Logarithmic Decrement. The natural logarithm of the ratio of any two successive amplitudes of the same sign obtained in the decay curve in a free vibration test.

Lumped Mass. A method of discretization in which the distributed mass of the elements is lumped at the nodes or joints.

Mathematical Model. The idealization of a system including all the assumptions imposed on the physical problem.

Mean-Square Value. The time average of the square of a random function as defined by eq. (22.2).

Mean Value. The time average of a random function defined by eq. (23.1).

Modal Shapes (also Normal Modes). The relative amplitude of the displacements at the coordinates of a multidegree-of-freedom system vibrating at one of the natural frequencies.

Modal Superposition Method. A method of solution of multidegree-of-freedom systems in which the response is determined from the solution of independent modal (or normal) equations.

Modified Mercalli Intensity (MMI). A measure of the effect of an earthquake at a particular location.

Moment-Resisting Space Frame. A space frame in which the members and joints are capable of resisting forces primarily by flexure.

Narrow-Band Process. A random process whose spectral density function has nonzero values only in a narrow frequency range.

Natural Frequency. The number of cycles per second at which a single degree of-freedom system vibrates freely or a multidegree-of-freedom system vibrates in one of the normal modes.

Natural Period. The time interval for a vibrating system in free vibration to do one oscillation.

Newmark Beta Method. A numerical method to calculate the response of a structure subjected to external excitation (force or motion).

Node or Joint: A point joining elements of the structure and at which displacements are known or to be determined.

Normal Distribution or Gaussian Distribution. A function whose probability density function is given by eq. (22.12).

Normal Modes. See modal Shapes.

Nyquist Frequency. The maximum frequency component that can be detected from a function sampled at time spacing Δt [$N_y = 1/2\Delta t$ (Hz)].

Occupancy Factor. A numerical factor in the calculation of the base shear force that depends on the intended use of the structure.

Ordinary Moment-Resisting Space Frame (OMRSF). A moment-resisting not meeting special detailing requirements for ductile behavior.

P-Delta Effect. The secondary effect on shears, axial forces, and moments of frame members induced by the vertical loads on the laterally displaced building frame.

Periodic Function. A function that repeats itself at a fixed time interval known as the period of the function.

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requirements given in Chapter 19 or 22 of UBC-97 (International, Conference of Building Officials 1997).

Spectral Analysis or Spectrum. A description of contributions of the frequency components to the mean-square value of a random function.

Spectral Density Function. A function that describes the intensity of random vibration in terms of the mean-square value per unit of frequency.

Square Root Sum of Squares (SRSS). A method of combining maximum values of modal contributions by taking the square root of the sum of the squared modal contributions.

Standard Deviation. The square root of the variance. It may be calculated by eq.(22.6).

Static Condensation. A method of reducing the dimensions of the stiffness and of the mass matrices by establishing the static relation between primary and secondary coordinates.

Stationary Process. A random process for which the average across the ensemble has the same value at any selected time.

Steady-State Vibration. The motion of the system that remains after the transient motion existing at the initiation of the motion has vanished.

Stiffness Coefficient. k_{ij} is the force at coordinate i due to a unit displacement at coordinate j .

Story Drift. The relative lateral displacements of consecutive levels of a building.

Spring Constant. The change in load on a linear elastic structure required to produce a unit increment of deflection.

Strong Motion Accelerograph. An instrument to register seismic motions higher than a specified amplitude.

Structure. An assemblage of framing members designed to support gravity loads and resist lateral forces. Structures may be categorized as building structures or non-building structures.

Structural Factor. A numerical factor in the calculation of the base shear force that depends on the type of structural system.

Tectonic Earthquakes. Earthquakes resulting from the sudden release of energy stored by a major deformation of the earth.

Time History Response. The response (motion or force) of the structure evaluated as a function of time.

Transient Vibration. The initial portion of the motion which vanishes due to the presence of damping forces in the system.

Transmissibility. The non-dimensional ratio, in the steady-state condition, of the response motion to the input motion. Or the non-dimension the amplitude of the force transmitted to the foundation to the amplitude of the force exciting the system.

Variance of $x(t)$. The average of the squares of the deviations of $x(t)$ values from the mean value \bar{x} .

Viscous Damping. Dissipation of energy such that the motion is resisted by a force proportional to the velocity but in the opposite direction.

White Noise. A wide-band random process for which the spectral density function is constant over the whole frequency range.

Wide-band Process. A random process whose spectral function has nonzero value over a large range of frequencies.

Wiener-Kinchin Equations. These are equations that relate the autocorrelation function and the spectral density function [eqs. (22.49) and (22.50)].

Wilson's θ Method. A modification of the step-by-step linear acceleration method in which the time step is multiplied by a factor necessary to render the method unconditionally stable.

Wood-Anderson Seismograph. An instrument used to register seismic motions.

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