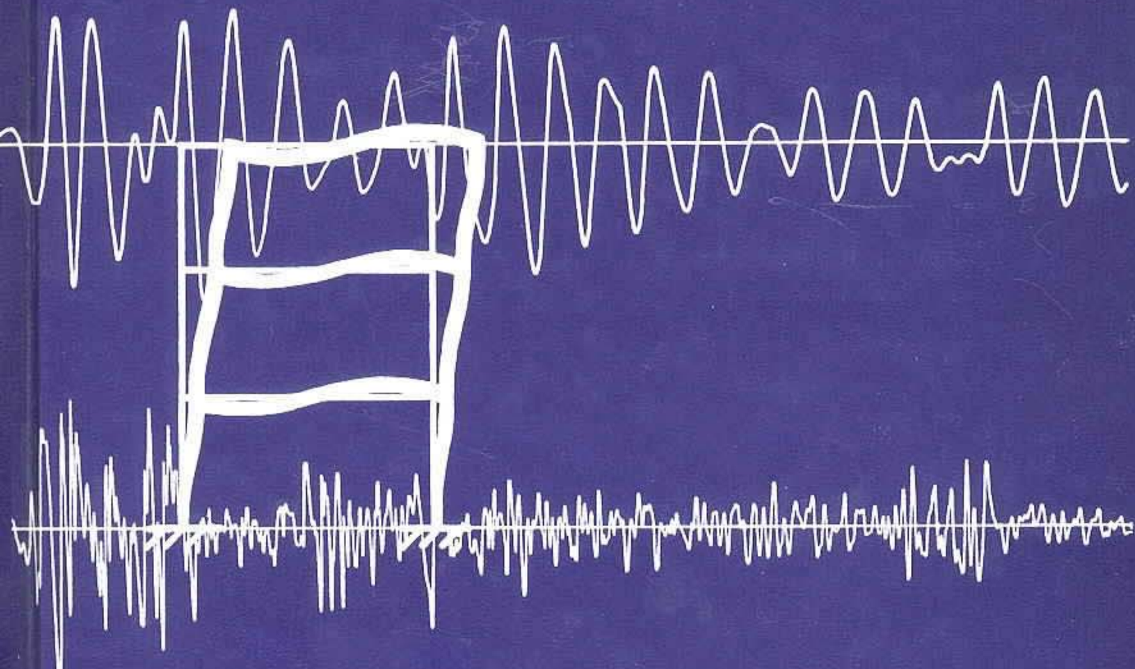
DYNAMICS OF STRUCTURES

A Primer



ANIL K. CHOPRA

EARTHQUAKE ENGINEERING RESEARCH INSTITUTE

DYNAMICS OF STRUCTURES
A PRIMER

by ANIL K. CHOPRA

University of California, Berkeley

EARTHQUAKE ENGINEERING RESEARCH INSTITUTE

Published by

The Earthquake Engineering Research Institute, whose objectives are the advancement of the science and practice of earthquake engineering and the solution of national earthquake engineering problems.

This is volume two of a series titled: Engineering Monographs on Earthquake Criteria, Structural Design, and Strong Motion Records.

The publication of this monograph was supported by a grant from the *National Science Foundation*.

Library of Congress Catalog card number 81-65052

This monograph may be obtained from:
Earthquake Engineering Research Institute
2620 Telegraph Avenue
Berkeley, California 94704

Dedicated to my Parents

FOREWORD

The occurrence of earthquakes poses a hazard to cities that can lead to disaster unless appropriate engineering countermeasures are employed. Recent earthquake disasters with high death tolls, in Guatemala, 1976 (20,000); Iran, 1978 (19,000); Algiers, 1980 (10,000); Italy, 1978 (4,000), demonstrate the great advantages that could be gained by earthquake resistant construction. To provide an adequate degree of safety at an affordable cost requires a high level of expertise in earthquake engineering and this in turn requires an extensive knowledge of the properties of strong earthquakes and of the dynamics of structures that are excited by ground shaking. To achieve this it is necessary for relevant information to be published in an appropriate form.

This monograph by A. K. Chopra, on dynamics of structures, is the second in a projected series of monographs on different aspects of earthquake engineering. Each monograph is authored by an expert especially qualified to prepare an exposition of the subject; and each monograph covers a single topic, with more thorough treatment than would be given to it in a textbook on earthquake engineering.

The monograph series grew out of the seminars on earthquake engineering that were organized by the Earthquake Engineering Research Institute and were presented in Los Angeles, San Francisco, Washington, D.C., Seattle, Chicago, Puerto Rico, St. Louis, and Houston, and were aimed at acquainting engineers, building officials, and members of government agencies with the basics of earthquake engineering. In the course of these seminars it became apparent that a more detailed written presentation of each seminar topic would be of value to the members of the audience, and this led to the monograph project. The seminar presented by Dr. Chopra set forth the basic elements of vibration theory that must be known in order to discuss technical features of earthquake engineering, and the following seminar speakers utilized this information in presenting more advanced features of the theory. Dr. Chopra was requested to prepare a monograph that would set forth the elements of earthquake engineering vibrations (in a form suitable for practicing engineers who had not studied earthquake engineering or vibration theory at the university) and to indicate those aspects of earthquake engineering dynamics that underlie the seismic requirements of the building code. It should be noted, however, that this presentation of the elements of earthquake vibrations also has applications to more advanced problems, for most structures will vibrate elastically under earthquake excitations and these vibrations can be decomposed into a sum of the vibrations of the normal modes, each of which responds to earthquake ground shaking in a manner similar to the single-degree-of-freedom structure. The first monograph in this series, by D. E. Hudson on reading and interpreting strong motion accelerograms, provides detailed information on the ground motions recorded during earthquakes and thus provides information relevant to Dr. Chopra's monograph.

The EERI monograph project, and also the seminar series, were funded by the National Science Foundation. EERI member, M. S. Agbabian, served as Coordinator of the seminar series and is also serving as Coordinator of the monograph project. Each monograph is reviewed by the members of the Monograph Committee, M. S. Agbabian, G. V. Berg, R.W. Clough, H. J. Degenkolb, G. W. Housner, and C. W. Pinkham, with the objective of maintaining a high standard of presentation.

GEORGE W. HOUSNER
Chairman, Monograph Committee

Pasadena, California
December 1980

PREFACE

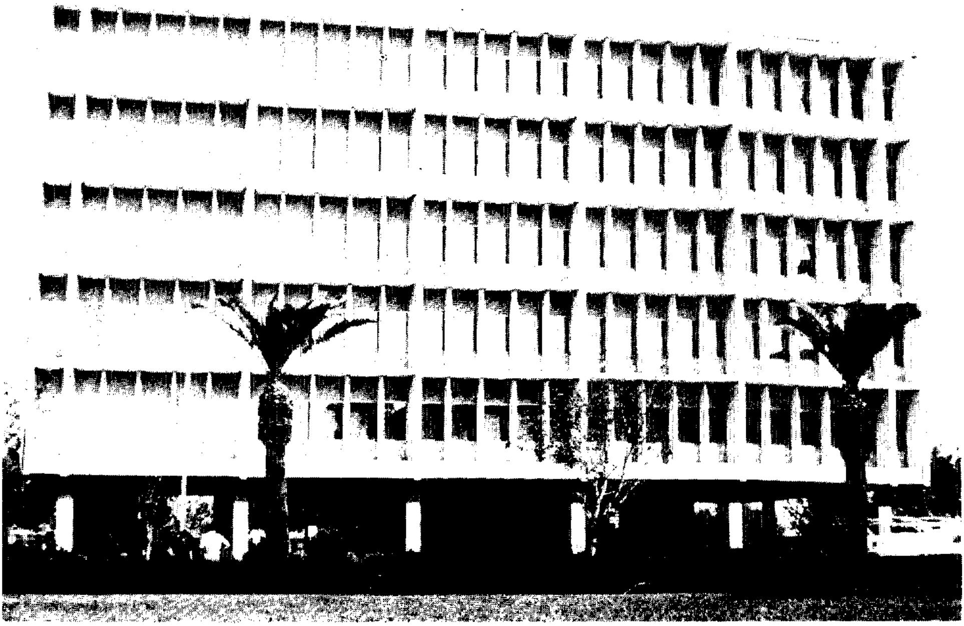
The study of dynamics of structures excited by ground shaking has now been incorporated into the graduate curriculum at many universities, and even in the undergraduate program at some universities. Excellent textbooks are available to a student of this subject. The formal, detailed treatment of the subject in textbooks draws upon courses in mathematics, mechanics, and static structural analysis that are standard in university curricula. For those who wish to study the subject in depth, there are no shortcuts. The only way to master the subject is to study the textbook material and practice its application to example problems, over a period of several months. However, the available textbooks may be overwhelming for the reader who is interested only in the fundamentals of the dynamics of structures excited by ground shaking and does not have the time or need for a formal, detailed treatment and comprehensive coverage of the subject.

This monograph is intended for such a reader. Its purpose is to provide the non-specialist in dynamics of structures with the basic concepts and knowledge needed to understand the response of structures excited by ground shaking. Much of the material treated is classical and has simply been presented without the formalism and detail characteristic of textbooks. The treatment is of an introductory nature in that it is concerned primarily with basic concepts and is limited almost entirely to the simplest idealization of buildings. However, many of the concepts and the analysis procedures presented can be generalized so that they apply to refined idealizations of buildings and to other classes of structures.

The selection of topics and the extent of coverage of each topic included in this monograph has been dictated by the fact that it is one in a projected series of monographs and by the desire to avoid unnecessary overlap with other monographs to come. The work is divided into three chapters that treat the dynamics of simple structures, the dynamics of multistory buildings, and the relation between dynamic analysis and building code design procedures.

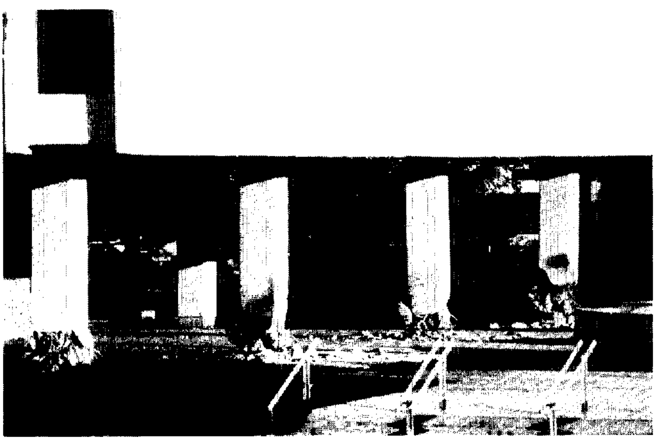
ANIL K. CHOPRA
University of California, Berkeley

Berkeley, California
August, 1980



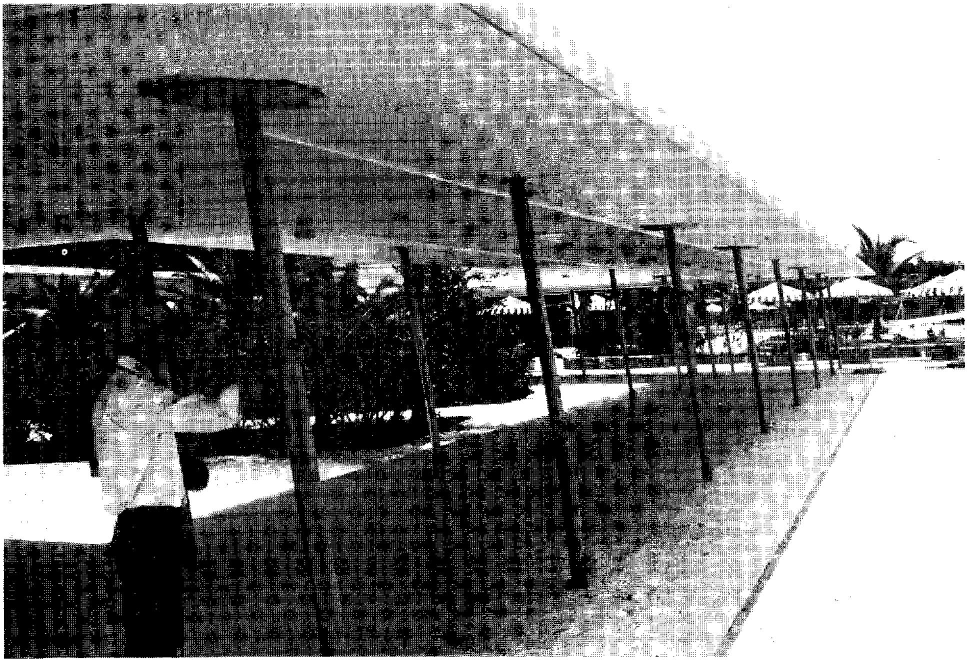
Courtesy of G. W. Housner

The six-story Imperial County Services Building was overstrained by the Imperial Valley Earthquake of 15 October 1979. The building is located in El Centro, California, nine kilometers from the causative fault of this magnitude 6.5 earthquake; the peak ground acceleration near the building was .23 g. The first-story reinforced concrete columns were overstrained top and bottom with partial hinging. The four columns at the right end were shattered at ground level, which dropped the end of the building about six inches. (See detail below.)



Courtesy of V. V. Bertero

CONTENTS



Courtesy of G. W Housner

This pergola at the Macuto-Sheraton Hotel near Caracas, Venezuela was damaged by the earthquake of 29 July 1967. The magnitude 6.5 event, which was centered about 15 miles from the hotel, overstrained the steel pipe columns. The heavy concrete deck supported by flexible steel columns is an ideal single-degree-of-freedom system.

1. Dynamics of Simple Structures

SIMPLEST POSSIBLE STRUCTURE

It is desirable to begin this introduction to dynamics of structures with the simplest possible structural system, so for this purpose we consider the one-story structure idealized as shown in Fig. 1. In this idealization we assume that the columns supporting the roof are massless and the entire mass of the structure is concentrated at the roof; the roof is rigid whereas the columns are flexible to lateral deformation but rigid in the vertical direction. The structure is assumed to be supported on rigid ground.

It will be seen later that if the roof of this structure is displaced laterally through some distance u_0, then released and permitted to oscillate freely, the structure will oscillate around its initial equilibrium position. These oscillations would continue forever with the same amplitude u_0 and the structure would never come to rest, as shown in Fig. 2. This, of course, is unrealistic. Intuition suggests that an actual structure in free vibration should oscillate with ever decreasing amplitude and eventually come to

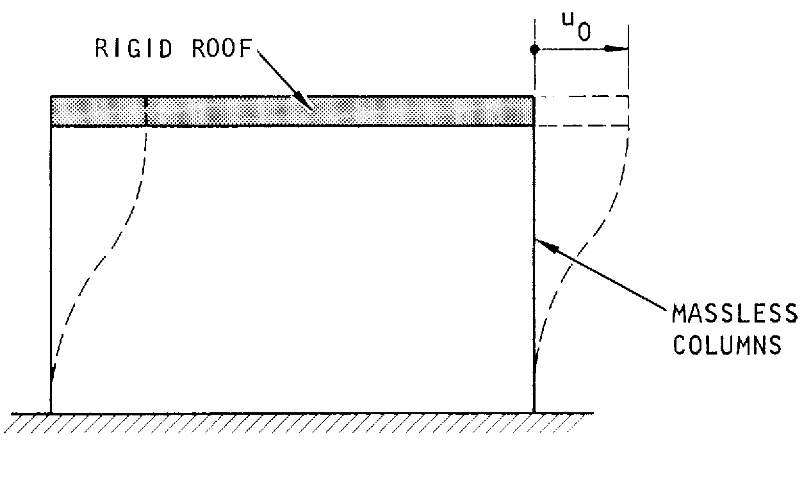


Figure 1. Idealized one-story structure without damping

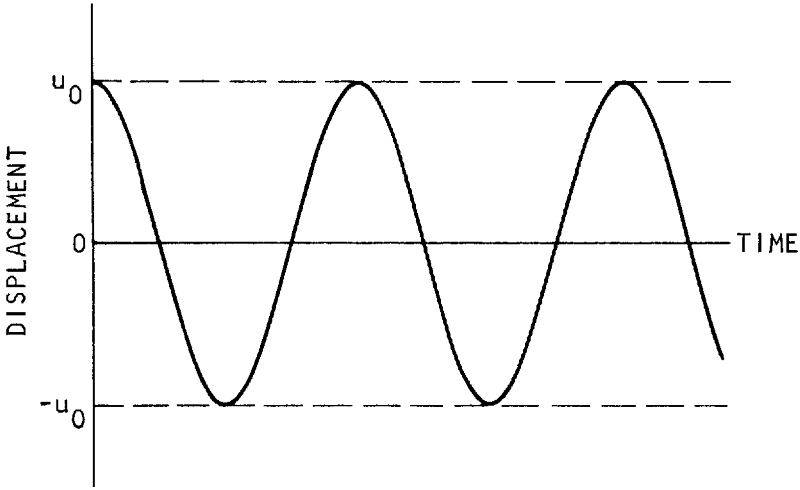


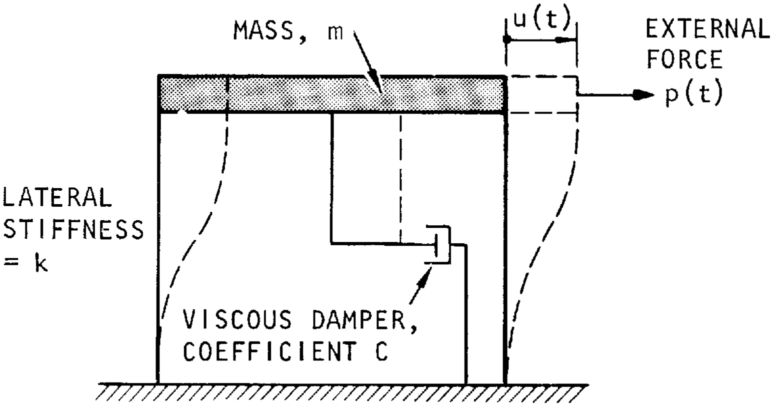
Figure 2. Free vibration of the idealized one-story structure without damping

rest. In order to incorporate this feature into the dynamics of the structure, an energy absorbing element should be introduced. The viscous damper included in the structure of Fig. 3 is the most commonly used element of this type.

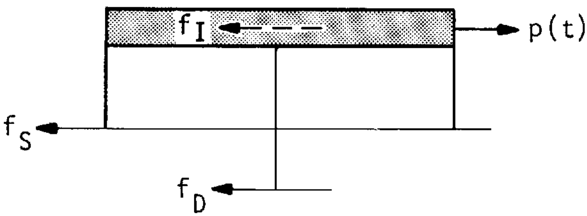
The basic concepts can be most conveniently developed by studying the dynamics of the structure of Fig. 3. The results obtained are applicable to simple structures such as the one-story building shown in Fig. 4. As we shall see later, such results are also useful in the modal analysis of multistory buildings.

EQUATIONS OF MOTION

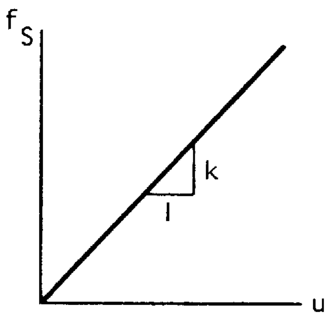
The motion of the idealized one-story structure due to dynamic excitation will be governed by an ordinary differential equation. The governing equation, or *equation of motion*, is derived for two types of dynamic excitation: external force and earthquake ground motion.



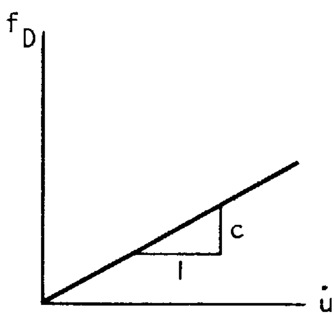
(a) Idealized one-story structure



(b) Free-body diagram



(c) Elastic force-deformation relation



(d) Damping force-velocity relation

Figure 3.

External Force

Figure 3a shows a linear structure of mass m, lateral stiffness k, and viscous damping c subjected to an externally applied dynamic force p(t). This notation indicates that the force p varies with time t. Under the influence of such a force, the roof of the structure displaces in the lateral direction by an amount u(t), which is also the deformation in the structure (displacement of roof relative to base). Because the force p varies with time, so does the displacement u.

The various forces acting on the mass at some instant of time are shown in a free-body diagram of the mass (Fig. 3b). These include the external force p(t), the elastic resisting force f_S, the damping force f_D, and the inertia force f_I. The elastic and damping forces act to the left because they resist the deformation and velocity, respectively, which are taken as positive to the right. The inertia force also acts to the left, opposite to the direction of positive acceleration. At each instant of time, the mass is in equilibrium under the action of these forces at that time. From the free body diagram, this condition of dynamic equilibrium is

$$f_I + f_D + f_S = p(t) \tag{1}$$

The inertia, damping, and elastic forces are next expressed in terms of u(t) and related quantities. For a linear structure, the elastic force is

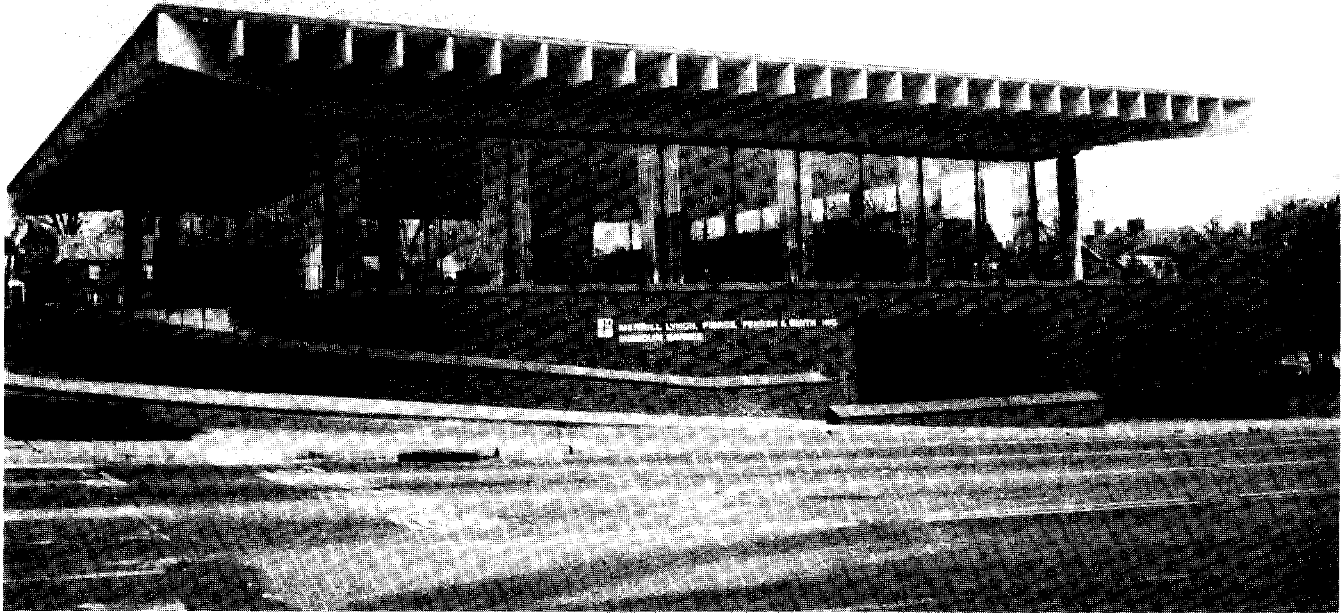
$$f_S = ku \tag{2a}$$

where k is the lateral stiffness of the structure and u is the interfloor (or relative) displacement; the damping force is

$$f_D = c\dot{u} \tag{2b}$$

where c is the damping coefficient for the structure and $\dot{u}$ is the interfloor (or relative) velocity. As shown in Figs. 3c and 3d the elastic force is proportional to the relative displacement and the damping force to the relative velocity. The inertia force associated with the mass m undergoing an acceleration $\ddot{u}$ is

$$f_I = m\ddot{u} \tag{2c}$$



Courtesy of G. W. Housner

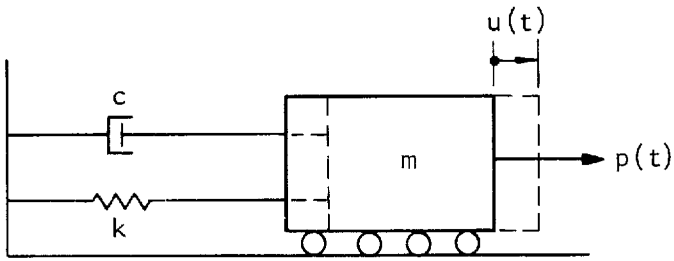
Figure 4. A one-story building. Most of the mass is concentrated at the roof level and the roof is essentially rigid compared to the lateral-force resisting system

Substitution of Eqs. 2 into Eq. 1 results in

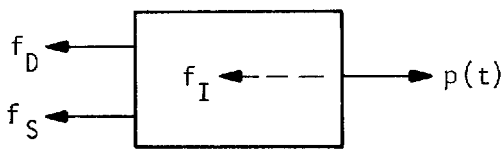
$$m\ddot{u} + c\dot{u} + ku = p(t) \tag{3}$$

This is the equation of motion governing the deformation u(t) of the idealized structure of Fig. 3a subjected to an external dynamic force p(t).

Dynamics of the system shown in Fig. 5a is studied in text-books on mechanical vibrations (see for example Thomson, 1965). It includes a mass m connected to a fixed support by two elements in parallel, a spring of stiffness k, and a viscous damper having a damping coefficient c. At any instant of time the forces acting on the mass (Fig. 5b) satisfy Eq. 1. Thus, the equation of motion derived above for the idealized one-story structure of Fig. 3a is also valid for the mass-spring-damper system of Fig. 5a.



(a) Mass-spring damper system



(b) Free-body diagram

Figure 5.

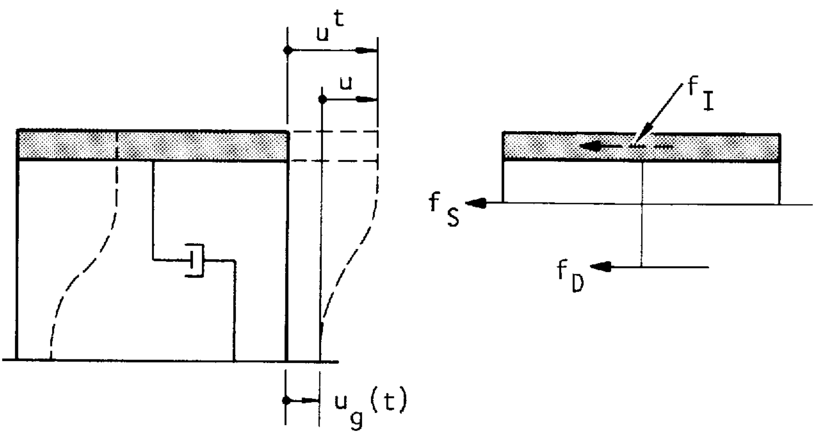
Earthquake Ground Motion

No external dynamic force is applied to the roof in the idealized one-story structure shown in Fig. 6a. The excitation in this case is the earthquake-induced motion of the base of the structure, presumed to be only a horizontal component of ground motion, with displacement $u_g(t)$, velocity $\dot{u}_g(t)$, and acceleration $\ddot{u}_g(t)$. Under the influence of such an excitation, the base of the structure is displaced by an amount $u_g(t)$ if the ground is rigid, and the structure undergoes deformation (displacement of roof relative to base) $u(t)$. The total displacement of the roof of the structure is

$$u^t(t) = u_g(t) + u(t) \tag{4}$$

From the free-body diagram of the mass shown in Fig. 6b, the equation of dynamic equilibrium is

$$f_I + f_D + f_S = 0 \tag{5}$$



(a) One-story structure subjected to earthquake ground motion

(b) Free-body diagram

Figure 6.

Eqs. 2a and 2b still apply because the elastic and damping forces depend only on the relative displacement and velocity, not on the total quantities. However, the mass in this case undergoes acceleration $\ddot{u}^t$, and the inertia force therefore is

$$f_I = m\ddot{u}^t$$

which with the aid of Eq. 4 can be expressed as

$$f_I = m(\ddot{u}_g + \ddot{u}) \tag{6}$$

Eq. 5 after substitution of Eqs. 2a, 2b, and 6 can be expressed as

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t) \tag{7}$$

This is the equation of motion governing the deformation u(t) of the idealized structural system of Fig. 6a subjected to earthquake ground acceleration $\ddot{u}_g(t)$.

Comparison of Eqs. 3 and 7 shows that the equations of motion for the structure subjected to two excitations—ground acceleration = $\ddot{u}_g(t)$ and external force = $-m\ddot{u}_g(t)$—are one and the same. The deformation response u(t) of the structure to ground acceleration $\ddot{u}_g(t)$ will be identical to the response of the structure on fixed base due to an external force equal to mass times the ground acceleration, acting opposite to the sense of acceleration. As shown in Fig. 7, the ground motion can therefore be replaced by an *effective force* = $-m\ddot{u}_g(t)$.

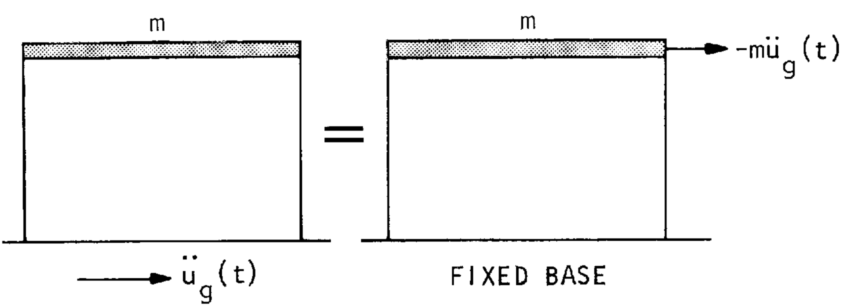


Figure 7. Effective earthquake force

Problem Statement

Given the mass m, stiffness k, damping c, and the excitation force p(t) or ground acceleration $\ddot{u}_g(t)$, a fundamental problem in structural dynamics is to determine the deformation response u(t) of the idealized one-story structure. Other response quantities of interest, such as base shear, can subsequently be determined from the deformation response.

We will examine the response of the system in free vibration to harmonic force and to earthquake ground motion. The idealized one-story structure is a *single-degree-of-freedom (SDF) system*, because its motion is governed by one differential equation (Eq. 3 or 7) containing only one unknown u(t).

FREE VIBRATION RESPONSE

Free vibration takes place when a structure vibrates under the action of forces inherent in the system itself and in the absence of external force or ground motion.

Undamped Structures

Consider first the idealized one-story structure of Fig. 3 without any damping. If the mass is disturbed from its equilibrium position by imparting to it some displacement u(0) and/or velocity $\dot{u}(0)$, the system will vibrate (or oscillate) as shown in Fig. 8 about the equilibrium position. This is a graphical representation of

$$u(t) = \frac{\dot{u}(0)}{\omega} \sin \omega t + u(0) \cos \omega t \tag{8}$$

which can be obtained as a solution of the equation of motion (Eq. 3 or 7 without the damping term and without any excitation, i.e., $p(t) = \ddot{u}_g(t) = 0$). The displacement *vs* time plot starts with the ordinate u(0) and slope $\dot{u}(0)$.

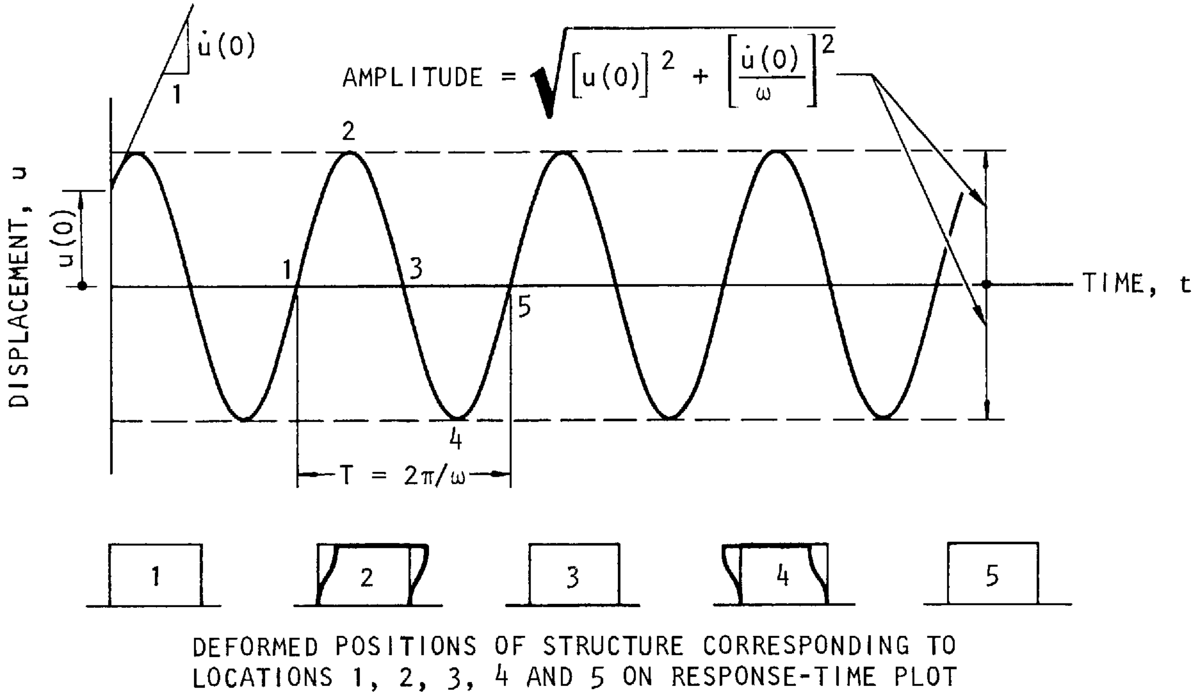


Figure 8. Free vibration of an undamped structure

Let us follow one cycle of vibration of the structure in Fig. 8. The mass in its equilibrium position at 1 moves to the right reaching the maximum (positive) displacement at 2, at which time the displacement begins to decrease and it returns back to its equilibrium position 3, continues moving to the left reaching the maximum (negative) displacement at 4, and then the displacement decreases again with the mass returning to its equilibrium position 5. One cycle of motion is described by the portion 1−2−3−4−5 of the displacement-time curve. At point 5, the state (displacement and velocity) of the mass is the same as it was at point 1, and the mass is ready to begin another cycle of vibration.

The amplitude of the simple harmonic motion, as shown in Fig. 8, depends on the initial displacement and velocity. Because the structure is undamped, the motion does not decay, i.e., the displacement amplitude is the same in all vibration cycles.

The *natural period of vibration* T (sec) of the structure is the time required for one cycle of free vibration. It is related to the *natural circular frequency of vibration* ω (rads/sec) and the *natural cyclic frequency of vibration* f (cycles/sec or Hz) as follows:

$$T = \frac{2\pi}{\omega} \tag{9a}$$

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{9b}$$

The term *natural* is used to qualify each of the above quantities to emphasize the fact that these are natural properties of the structure when it is allowed to vibrate freely without any external excitation. Because the structure is linear, these properties are independent of the initial displacement and velocity.

If we had mathematically solved the equation of motion governing free vibration of an undamped structure (Eq. 1 with $c = 0$ and $p(t) = 0$) we would have shown that

$$\omega = \sqrt{k/m} \tag{10}$$

Thus the free vibration properties ω, T, and f depend only on the mass and stiffness of the structure. The stiffer of two struc-

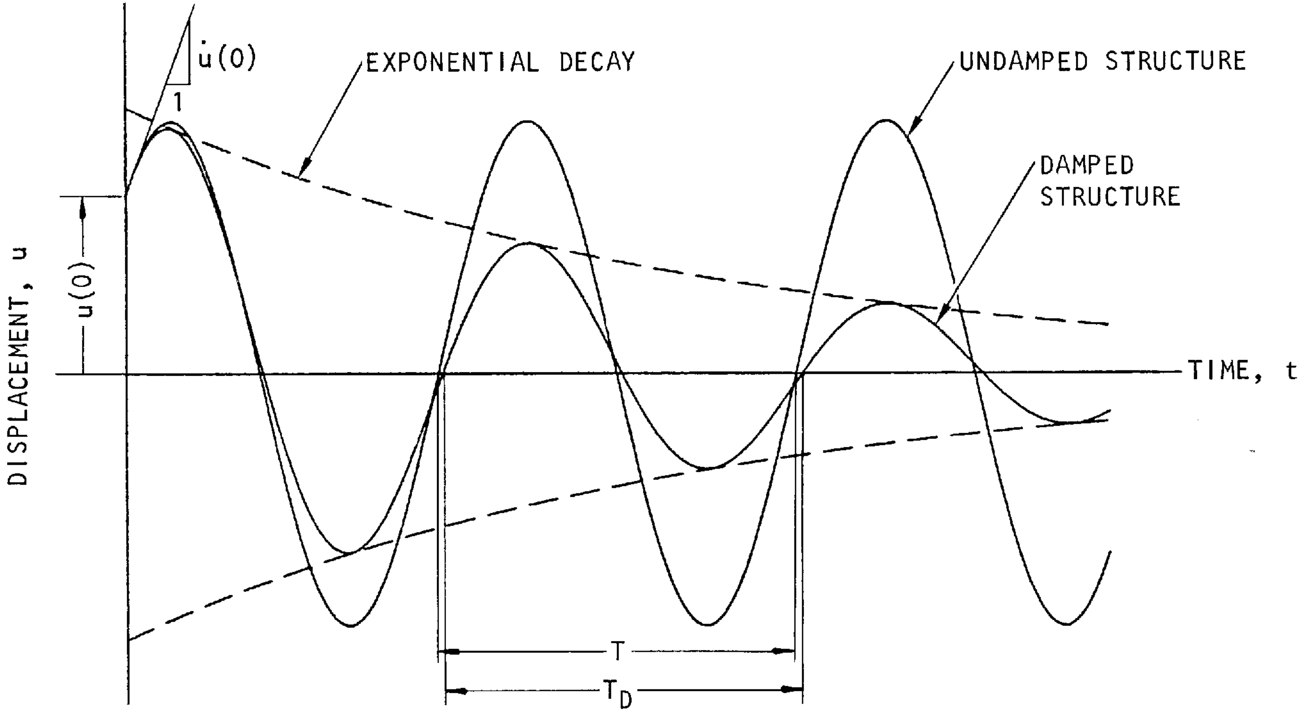


Figure 9. Effect of damping on free vibration

tures having the same mass will have the higher vibration frequency and the shorter vibration period. Similarly, the lighter (less mass) of two structures having the same stiffness will have the higher vibration frequency and the shorter vibration period.

Damped Structures

Figure 9 shows the free vibration response of two one-story structures, identical in all respects except that one is undamped and the other includes a viscous damper. Free vibration of both systems results from an initial displacement $u(0)$ and velocity $\dot{u}(0)$ imparted to the mass. The displacement-time plots for both systems start at $t = 0$ with the same ordinate and slope. The displacement amplitude of the undamped structure is the same in all vibration cycles, but the damped structure oscillates with amplitude decreasing with every cycle of vibration.

The natural period T_D, circular frequency ω_D, and cyclic frequency f_D of vibration of the damped structure are interrelated as

$$T_D = \frac{2\pi}{\omega_D} \tag{11a}$$

$$f_D = \frac{1}{T_D} \tag{11b}$$

in the same manner as for the undamped structure. Furthermore, the natural circular frequency and period of vibration are influenced by damping as follows:

$$\omega_D = \omega\sqrt{1-\xi^2} \tag{12a}$$

$$T_D = T/\sqrt{1-\xi^2} \tag{12b}$$

where $\xi = c/2m\omega$ is the *fraction of critical damping coefficient*; it is a dimensionless measure of the damping coefficient c for the system. For brevity, we will refer to ξ as the *damping ratio*. Damping has the effect of decreasing the natural circular frequency of vibration and increasing the natural period of vibration. Fig. 10 shows that for damping ratios less than 0.2, a range which includes most structures, these effects are negligible, i.e., ω_D is approximately equal to ω and hence T_D is approximately equal to T.

The displacement amplitude decreases progressively because of damping. As shown in Fig. 9, this decay in amplitude is ex-

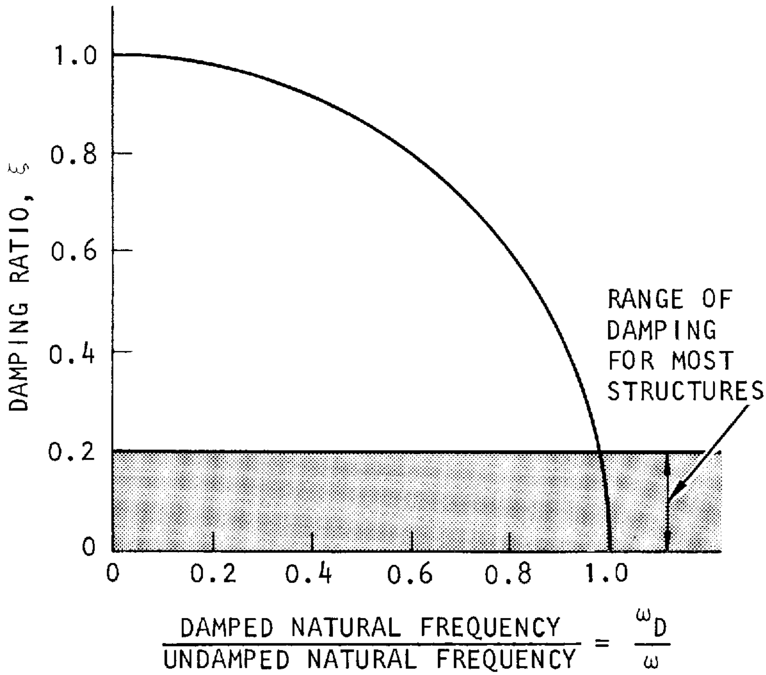


Figure 10. Effect of damping on natural frequency of vibration

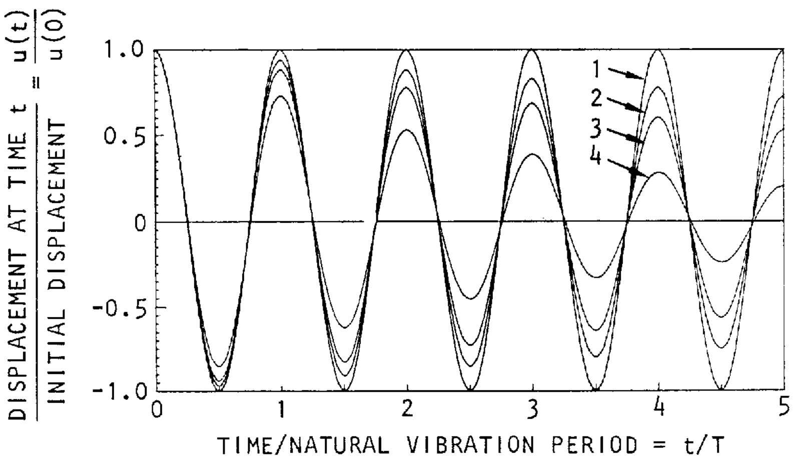


Figure 11. Effect of damping on free vibration. Curves 1, 2, 3, and 4 are for damping ratios of 0, 1, 2, and 5 percent, respectively

onential with time and it is apparent from Fig. 11 that the rate of decay strongly depends on the damping ratio ξ. It can be shown that for systems with low damping, the ratio of any two successive peaks (both positive or both negative) is

$$\frac{u_i}{u_{i+1}} \approx e^{2\pi\xi} \tag{13}$$

where the symbol $\approx$ represents "approximately equal." The ratio of any two successive peaks is the same, i.e. u_i/u_{i+1} does not depend on i. The *logarithmic decrement* is defined as

$$\delta = \ln(u_i/u_{i+1}) \tag{14}$$

Combining Eqs. 13 and 14 leads to

$$\delta \approx 2\pi\xi \tag{15}$$

Considering response peaks which are several cycles apart, say j cycles, it can be shown that

$$\ln(u_i/u_{i+j}) = j\delta \approx 2j\pi\xi \tag{16}$$

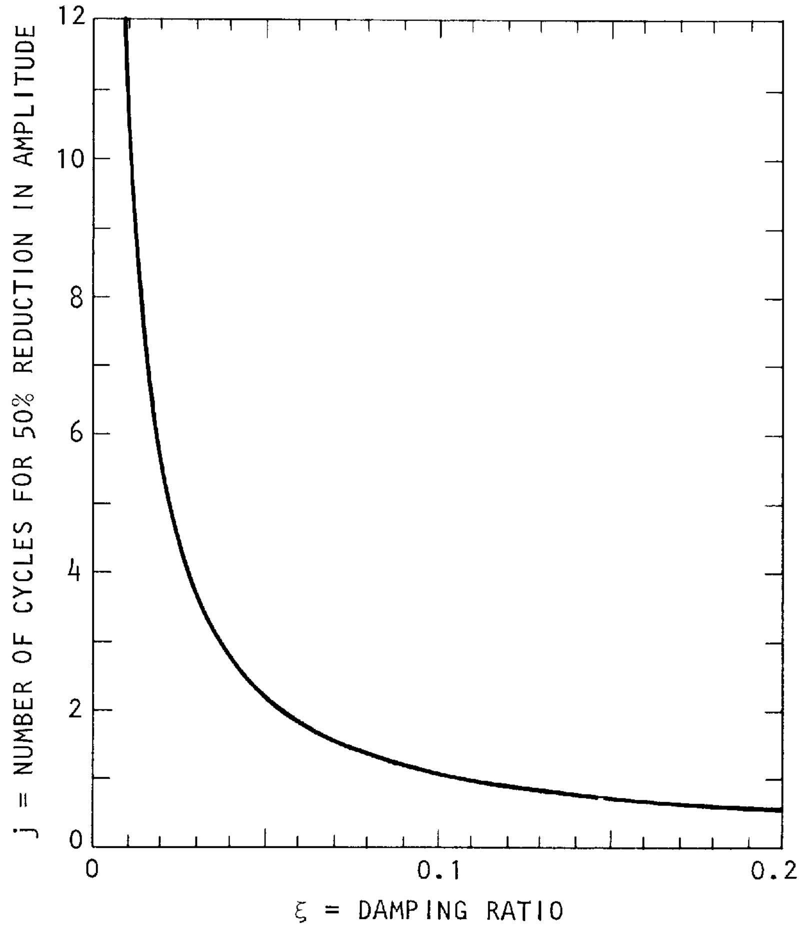


Figure 12. Number of cycles required to reduce the free vibration amplitude by 50 percent plotted as a function of damping ratio

From this equation, the number of cycles j required to reduce the amplitude by 50% can be obtained; this is plotted against the damping ratio in Fig. 12.

It is implicitly assumed in the preceding presentation of free vibration of damped systems that the damping in the structure was less than critical damping, i.e., ξ is less than 1. This assumption is appropriate because most structures are lightly damped at much less than critical damping; typically the damping ratio is in the range 0.02 to 0.10 (2 to 10 percent). If the damping coefficient for the structure is equal to or greater than the critical damping coefficient, i.e., $\xi \geq 1$, the motion is non-oscillatory. The mass of such a system, upon being disturbed and then released, will simply tend to creep back to its equilibrium position.

Large damping is sometimes built into a system to obtain the desired performance. For example, most of the accelerographs that record strong earthquake ground motion are designed to possess damping equal to 60% of critical damping (Hudson, 1979—Appendix D).

Free Vibration Tests

Because it is not possible to analytically determine the damping coefficient c or damping ratio ξ for a structure, there is considerable interest in evaluation of damping from experiments. The results of the preceding section provide a basis for evaluating damping from free vibration experiments; the natural period of vibration T of the structure can also be determined from these experiments.

Vibration period and damping of a one-story structure can be determined by the following procedure:

1. Disturb the structure from its equilibrium position through some displacement u_0 and release the structure.

2. Record the free vibration of the structure to obtain the displacement-time plot shown in Fig. 13.

3. Measure the time required to complete one cycle of vibration to obtain the vibration period T.

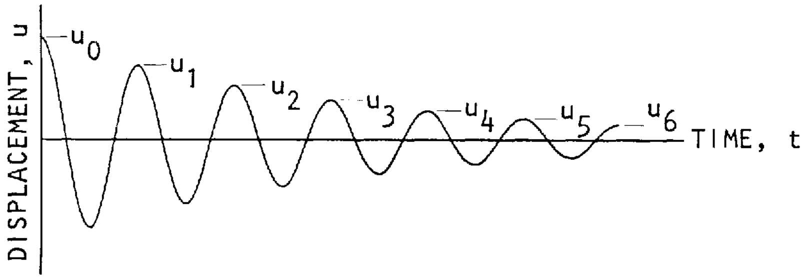


Figure 13. Measured displacement response from a free-vibration test

4. Measure u_i and u_{i+j}
5. Compute $\delta = \frac{1}{j}\ \ell n(u_i/u_{i+j})$
6. Compute $\xi = \delta/2\pi$

Although the procedure is applicable with j = 1, it may lead to erroneous results for a lightly damped system, in which case two successive peaks would have very similar ordinates. The procedure provides better results by considering two peaks that are several cycles apart.

Accelerations can be measured more conveniently than displacements. For lightly damped systems, the acceleration-time plot would have an appearance similar to the displacement-time plot of Fig. 13. The procedure summarized above can also be used to compute the vibration period and damping from acceleration-time plots.

RESPONSE TO HARMONIC EXCITATION

The theory of steady state response of structures to harmonic excitation has several applications, including forced vibration tests on structures, accelerograph design and vibration isolation. We shall examine selected results of this theory considering harmonic forces of two types: external force of constant amplitude and external force due to vibration generator.

External Force of Constant Amplitude

Consider an external force varying harmonically with time: $p(t) = p_0 \sin \bar{\omega}t$, where the amplitude, or maximum value, of the load is p_0, its period is $\bar{T}$ ($\bar{T} = 2\pi/\omega$), and the circular frequency is $\bar{\omega}$ (Fig. 14). The equation of motion

$$m\ddot{u} + c\dot{u} + ku = p_0 \sin \bar{\omega}t \tag{17}$$

can be solved by standard procedures to obtain the response of the structure in two parts: free vibration response plus steady state response. In a damped structure, the free vibration response decays, eventually becoming insignificant, and usually only the steady state response is considered. Figure 14 shows that the steady state motion occurs at the forcing frequency $\bar{\omega}$

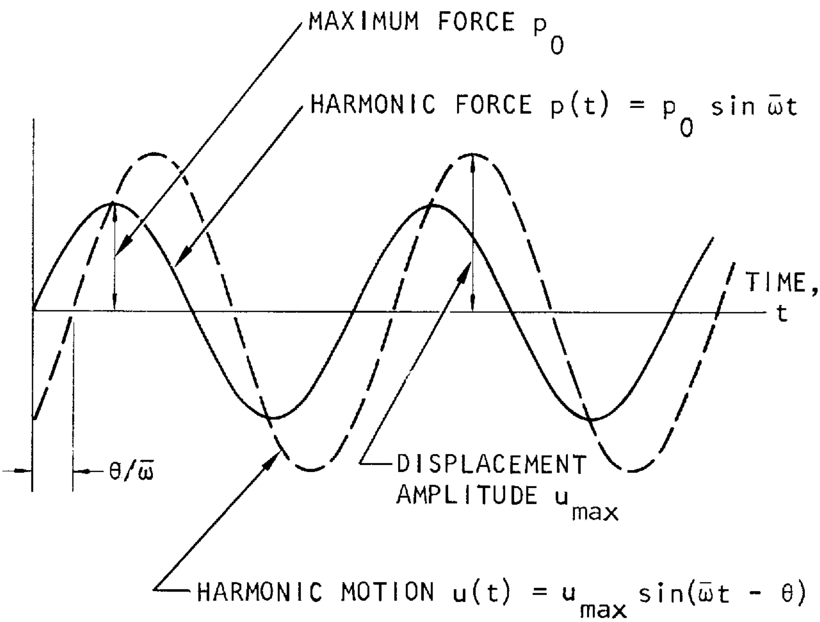


Figure 14. Steady state motion due to harmonic force

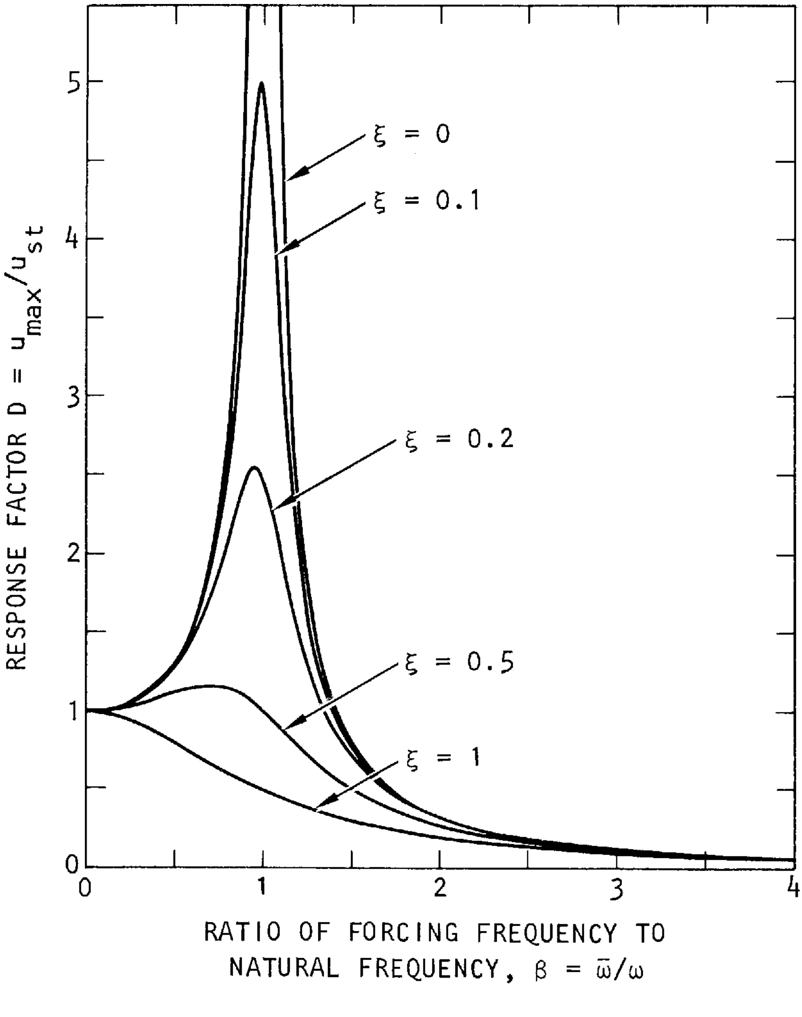


Figure 15. Response factor for a one-story structure subjected to harmonic force

with a time shift $\theta/\bar{\omega}$ (where θ is the phase angle, or the angular phase shift). Thus the steady state response may be written as

$$\frac{u(t)}{u_{st}} = D \sin(\bar{\omega}t - \theta) \tag{18}$$

in which

$$u_{st} = \frac{p_0}{k} \tag{19a}$$

$$D = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \tag{19b}$$

$$\theta = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right) \tag{19c}$$

where $\beta = \bar{\omega}/\omega$.

The amplitude of the dynamic displacement u_{max} is given by

$$\frac{u_{max}}{u_{st}} = D \tag{20}$$

where u_{st} as defined in Eq. 19a is the displacement of the structure that would occur if the maximum force p_0 were applied as a static force. Thus D is a dimensionless response factor, equal to the ratio of the dynamic to the static displacement response amplitudes. The response factor D depends only on two parameters: (1) $\beta = \bar{\omega}/\omega$, the ratio of the frequency of the external force to the natural frequency of vibration of the structure; and (2) damping ratio ξ. Figure 15 is a plot of Eq. 19b showing the variation of D with β for several values of ξ. Equations 18–20 and Fig. 15 permit several important observations.

For β close to zero, u_{max} is about the same as u_{st}; that is, the dynamic effects are negligible if the forcing frequency is much smaller than the natural frequency of the structure. For small values of β, the maximum displacement is controlled by the stiffness of the system with little effect of mass or damping. When $\beta = 1$, $D = 1/2\xi$; that is, the response factor is inversely proportional to the damping ratio if the forcing frequency is the same as the natural frequency of the structure. For β close to 1, the response factor is controlled by the damping ratio ξ with negligible

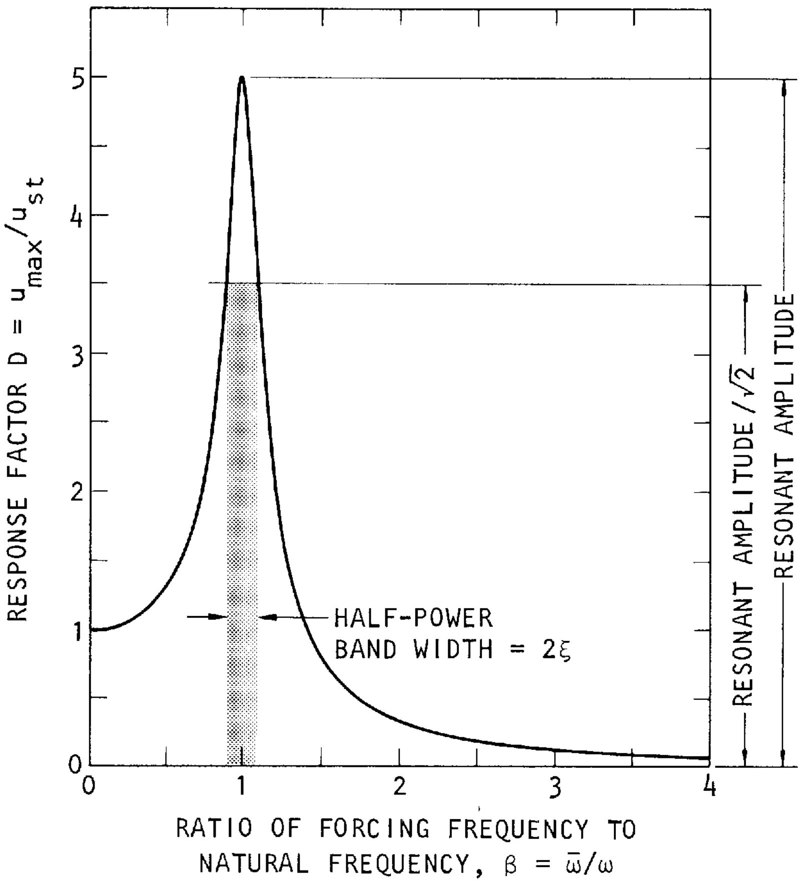


Figure 16. Evaluation of damping from forced vibration tests

influence of mass or stiffness. The response factor is essentially independent of damping and approaches zero as the forcing frequency $\bar{\omega}$ becomes much higher than the natural frequency ω of the structure. It can be shown that at high forcing frequencies, the maximum displacement depends primarily on the mass.

A *resonant frequency* is defined as the frequency for which the response is a maximum. At very low values of $\bar{\omega}$ (β close to zero) the response factor D is approximately equal to 1; it rises to a peak near $\bar{\omega} = \omega$ ($\beta = 1$) and approaches zero as $\bar{\omega}$ (or β) becomes large. It can be shown that the peak value of D occurs at $\beta = \sqrt{1\text{-}2\xi^2}$ and the corresponding frequency is the *displacement resonant frequency*. Thus, the relations among the displacement resonant frequency, the damped natural frequency ω_D, and the undamped natural frequency ω are

$$\text{Displacement resonant frequency} = \omega\sqrt{1\text{-}2\xi^2}$$

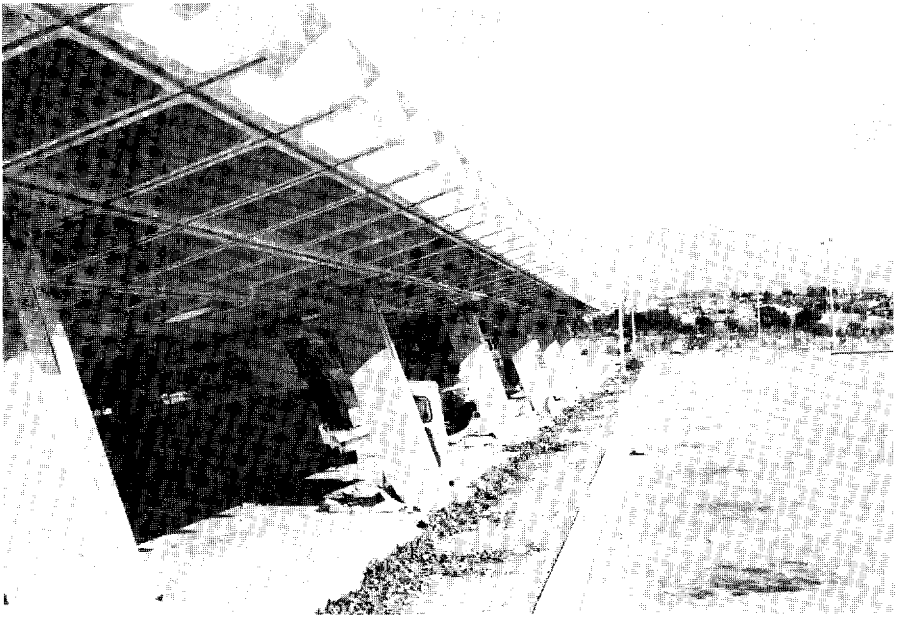
$$\text{Damped natural frequency } \omega_D = \omega\sqrt{1\text{-}\xi^2}$$

Although the displacement resonant frequency is different from the damped or undamped natural frequencies, the difference is negligible for the degree of damping typical of structures—less than 4% if the damping ratio does not exceed 20%.

Figure 15 shows that the sharpness or width of the response curve in the vicinity of the resonant frequency depends on the damping in the system. The width at the "half-power point" (i.e. at a value of $D = D_{max} \div \sqrt{2}$) is 2ξ, as illustrated in Fig. 16.

External Force Due to Vibration Generator

Vibration generators (or shaking machines) have been developed to conduct forced vibration tests on full-scale structures (Hudson, 1970). Two equal eccentric weights rotating in opposite directions generate a unidirectional force varying sinusoidally with time. Two weights each of mass m rotating at a radius (eccentricity) r at a frequency $\bar{\omega}$ (in rad /sec) would produce a force $= 2m_0r\bar{\omega}^2 \sin \bar{\omega}t$. The amplitude of this force is proportional to $\bar{\omega}^2$, in contrast to the constant amplitude of the external force considered in the preceding section. Thus, based on



Courtesy of G. W. Housner

The parking structure for ambulances at the Olive View Hospital was a heavy roof supported on columns. The vibrations were so severe during the San Fernando earthquake of 9 February 1971 that the columns failed.

Eqs. 19a and 20, the amplitude of the displacement response u_{max} due to the vibration-generator force is given by the equation

$$u_{max} = \frac{2m_0 r}{k} \bar{\omega}^2 D \tag{21}$$

where D is as defined in Eq. 19b. The amplitude of the acceleration response is

$$\ddot{u}_{max} = \bar{\omega}^2 u_{max} = \frac{2m_0 r}{k} \bar{\omega}^4 D \tag{22}$$

Forced Vibration Tests

The development of vibration generators has provided an effective approach to forced vibration tests on structures (Hudson, 1970). The vibration properties of a structure are determined by varying the frequency of the vibration generators through an appropriate range. The amplitude of the steady state acceleration of the structure at each forcing frequency is measured. Frequency-response curves, in the form of acceleration amplitude versus forcing frequency, may be plotted directly from the measured data. However, the curves are for a force with its amplitude proportional to the square of the forcing frequency, and each acceleration amplitude should be divided by the square of the corresponding frequency to obtain acceleration frequency-response curves for constant amplitude force. If the original accelerations are divided by the forcing frequency to the fourth power, displacement frequency-response curves for constant amplitude force would be obtained.

The natural frequency of vibration and damping ratio for a one-story structure (such as that of Fig. 4) can be determined from any one of the frequency-response curves by the following procedure:

1. Determine natural frequency of vibration as the forcing frequency at resonance
2. Measure half-power band width
3. Compute damping ratio ξ = half-power band width ÷ 2

For lightly-damped structures (damping ratio less than 5%) there is little difference among the values obtained for the natural frequency from the different frequency-response curves; also, essentially the same values are obtained for the damping ratio from the various frequency-response curves.

RESPONSE TO EARTHQUAKE GROUND MOTION

Response History

The motion of the one-story structure subjected to earthquake ground motion (Fig. 6a) is governed by Eq. 7, which can be written as

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = -\ddot{u}_g(t) \tag{23}$$

The solution to this equation leads to the deformation response u(t), which depends on (1) the characteristics of the ground acceleration $\ddot{u}_g(t)$, (2) $\omega = \sqrt{k/m}$, the natural circular frequency of vibration (or equivalently the natural period of vibration T) of the structure without damping, and (3) the damping ratio ξ of the structure.

The solution to Eq. 23 can be written as

$$u(t) = -\frac{1}{\omega_D}\int_0^t \ddot{u}_g(\tau)\exp[-\xi\omega(t-\tau)]\sin[\omega_D(t-\tau)]d\tau \tag{24}$$

where $\omega_D = \omega\sqrt{1-\xi^2}$ is the natural circular frequency of vibration of the damped structure. For a given ground acceleration function $\ddot{u}_g(t)$ and system properties ω (or $T = 2\pi/\omega$) and ξ, the Duhamel integral in Eq. 24 can provide the deformation response history u(t). Earthquake ground accelerations vary irregularly (see for example Fig. 17) to such an extent that analytical evaluation of this integral must be ruled out. Of the various other approaches available, numerical methods implemented on digital computers are most effective.

The earthquake accelerogram is digitized and appropriately filtered to control accelerogram errors and baseline distortions, and accelerograph transducer corrections are introduced to obtain the corrected ground accelerogram (Hudson, 1979). The

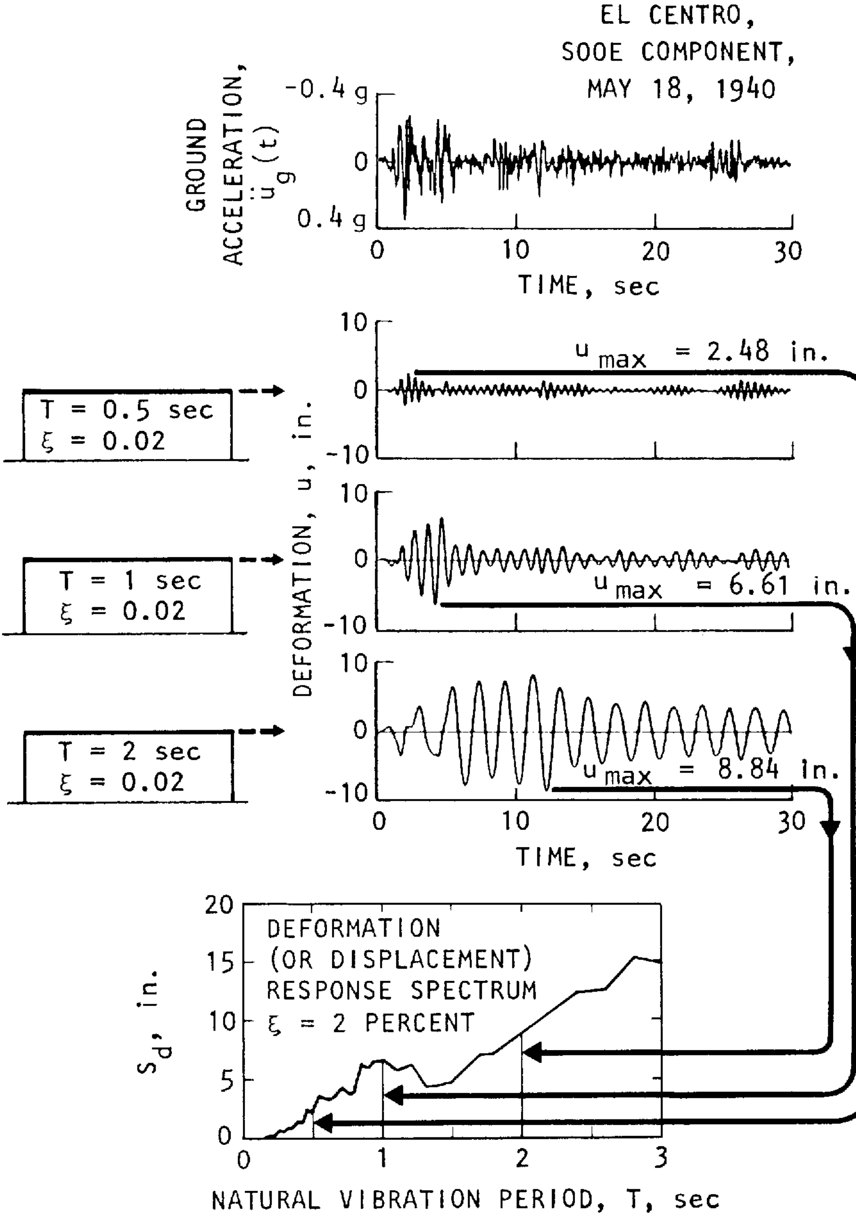


Figure 17. Computation of deformation (or displacement) response spectrum

function $\ddot{u}_g(t)$ in Eqs. 23 and 24 is then defined by the numerical coordinates of the corrected accelerogram at time intervals spaced closely enough to accurately define the accelerogram. In the California Institute of Technology (Caltech) strong motion data program, the corrected accelerograms were defined at 0.02 second time intervals (Hudson, 1979).

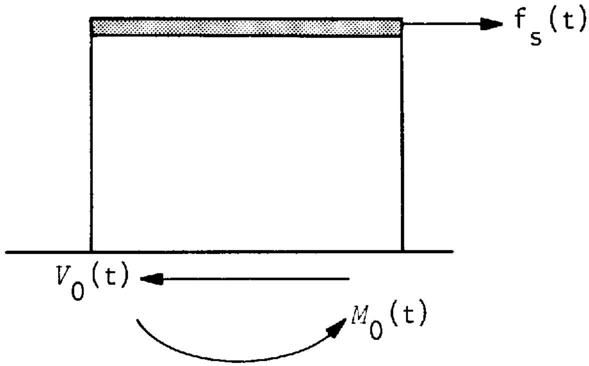
With the ground acceleration $\ddot{u}_g(t)$ defined in this manner and substituting numerical values for ω and ξ of the structure in Eq. 24, the response history could be determined by numerical evaluation of the Duhamel integral (Newmark and Rosenblueth, 1971—Section 1.5; Clough and Penzien, 1975—Sections 7-2, 7-3). The more common approach, however, is to directly solve the equation of motion (Eq. 23) by numerical procedures. Various procedures have been developed for this purpose (Newmark and Rosenblueth, 1971—Section 1.5; Clough and Penzien, 1975—Chapter 8; Hudson, 1979—page 57). When properly implemented, both approaches—the numerical evaluation of the Duhamel integral and the numerical solution of the equation of motion—provide equivalent results.

Figure 17 shows the results of such computations for three structures subjected to the same ground motion. The damping ratio $\xi = 2\%$ is the same for the three structures, so that the differences in their deformation responses are associated with their natural period of vibration. The time required for the structure to complete a cycle of vibration in response to typical earthquake ground motion is very close to the natural period of vibration of the structure.

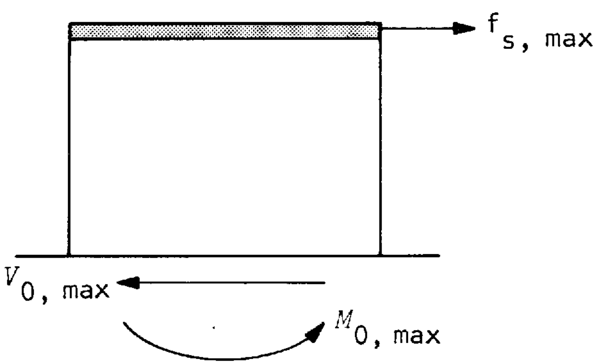
Mathematical expressions can also be obtained for other response quantities such as relative velocity $\dot{u}(t)$ and total acceleration $\ddot{u}^t(t)$ (Newmark and Rosenblueth, 1971—Section 1.4; Hudson, 1979—pages 62-63).

Once the deformation response history u(t) has been evaluated, the shear and moment at the base of the building can be conveniently determined by introducing the concept of *equivalent lateral force*. This is an external force f_s that, if applied as a static force, would cause deformation u (Fig. 18). Thus, at any instant of time the equivalent lateral force is

$$f_s(t) = ku(t) \qquad (25a)$$



(a) At time t



(b) Maximum during earthquake

Figure 18. Equivalent lateral force

which can be expressed in terms of the mass as

$$f_s(t) = m\omega^2 u(t) \tag{25b}$$

The base shear V_0 and base moment M_0 can be determined by static analysis of the structure subjected to the equivalent lateral force. Thus,

$$V_0(t) = f_s(t) \tag{26a}$$

$$M_0(t) = h f_s(t) \tag{26b}$$

where h is the height of the roof above the base. After substitution of Eq. 25b, the base shear and base moment can be expressed as

$$V_0(t) = m\omega^2 u(t) \tag{27a}$$

$$M_0(t) = h V_0(t) \tag{27b}$$

Response Spectrum

The complete history of any response quantity, namely deformation, velocity, acceleration, base shear, or base moment, can be determined by the numerical procedures outlined above. However, for design purposes, it is generally sufficient to know only the maximum value of the response due to the earthquake. The subscript *max* will generally be used to designate the maximum value of the response, without regard to algebraic sign. Thus, for any response quantity r,

$$r_{max} = max \mid r(t) \mid$$

A plot of the maximum value of a response quantity as a function of the natural vibration frequency of the structure, or as a function of a quantity which is related to the frequency such as natural period, constitutes the *response spectrum* for that quantity. The *deformation (or displacement) response spectrum* is such a plot of the quantity S_d defined as

$$S_d = u_{max} \tag{28}$$

Figure 17 shows the basic concept underlying computation of the deformation response spectrum. The time variations of deformation responses of three structures to a selected ground motion

Courtesy of G. W. Housner

This elevated water tank survived the 15 October 1979 Imperial Valley earthquake even though the rod bracing was stretched beyond the yield point. When this tank is full of water and the rods are unstretched, the structure can be analyzed as a single-degree-of-freedom system.

are presented. For each structure the maximum value of the deformation, without regard to algebraic sign, during the earthquake is determined from its response history. The u_{max} so determined for each structure provides one point on the deformation response spectrum. Repeating such computations for a range of values of T, while keeping the damping ratio ξ constant, provides the deformation response spectrum for the ground motion. As will be seen later, the complete response spectrum includes such spectrum curves for several values of damping.

Alternatively the maximum deformation may be expressed in terms of the quantity S_v defined as

$$S_v = \omega S_d \tag{29a}$$

or equivalently as

$$S_v = \frac{2\pi}{T} S_d \tag{29b}$$

where ω is the natural circular frequency of vibration of the structure and T is the natural vibration period of the structure. The quantity S_v has units of velocity and is related to maximum strain energy E_{max} stored in the structure during the earthquake by the equation

$$E_{max} = \frac{1}{2} m S_v^2 \tag{30}$$

which can be derived as follows:

$$E_{max} = \frac{1}{2} k u_{max}^2 = \frac{1}{2} k S_d^2 = \frac{1}{2} k \left(\frac{S_v}{\omega}\right)^2 = \frac{1}{2} m S_v^2$$

The *pseudo-velocity response spectrum* is a plot of S_v as a function of the natural frequency or period of vibration of the system.*

For the ground motion of Fig. 17, the S_v quantity corresponding to any vibration period T can be determined from Eq. 29b and the S_d value for the same T, computed as illustrated in Fig. 17 and plotted in Fig. 19a. The resulting values of S_v are plotted

* The prefix "pseudo" is intended to emphasize the fact that this spectrum is not the same as the relative velocity response spectrum. The latter is a plot of $\dot{u}_{max}$ (the maximum value of $\dot{u}(t)$ for a SDF system during the earthquake) as a function of the natural frequency or period of vibration of the system (Hudson, 1979—pages 55–64).

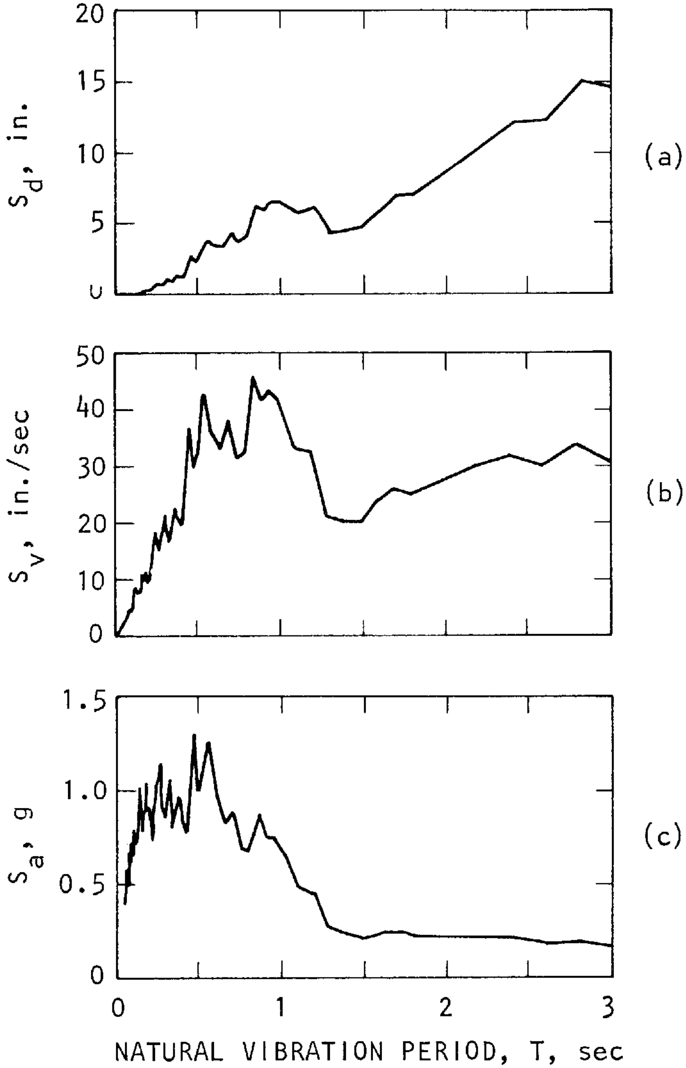


Figure 19. (a) Deformation (or Displacement), (b) pseudo-velocity and (c) pseudoacceleration response spectra. El Centro ground motion—S00°E component. Damping ratio $\xi = 2$ percent

in Fig. 19b as a function of vibration period T, for a fixed value of damping ratio, to provide the pseudo-velocity response spectrum for the ground motion of Fig. 17.

Another convenient measure of the maximum deformation is the quantity S_a, defined as

$$S_a = \omega S_v = \omega^2 S_d \tag{31a}$$

or equivalently as

$$S_a = \frac{2\pi}{T} S_v = \left(\frac{2\pi}{T}\right)^2 S_d \tag{31b}$$

The quantity S_a has units of acceleration and is related to the maximum value of the base shear as follows:

$$V_{0,max} = kS_d = m\omega^2 S_d = mS_a \tag{32}$$

The maximum base shear may be written in the form

$$V_{0,max} = \frac{S_a}{g} w \tag{33}$$

where w is the weight of the system and g is the acceleration of gravity. When written in this form, S_a /g may be interpreted as the so-called base shear coefficient in building codes. The *pseudo-acceleration response spectrum* is a plot of S_a as a function of the natural frequency or the period of vibration of the system.†

For the ground motion of Fig. 17, the S_a value corresponding to any value of T can be determined using Eq. 31b and the S_d value for the same T, computed as illustrated in Fig. 17 and plotted in 19a. The resulting values of S_a are plotted in Fig. 19c as a function of vibration period T, for a fixed value of damping ratio, to provide the pseudo-acceleration response spectrum for the ground motion of Fig. 17.

The deformation, pseudo-velocity, and pseudo-acceleration response spectra for an earthquake ground motion are interre-

† The prefix "pseudo" is used to distinguish this spectrum from the absolute acceleration response spectrum. The latter is a plot of $\ddot{u}^t_{max}$ (the maximum value of $\ddot{u}^t(t)$ for a SDF system during the earthquake) as a function of the natural frequency or period of vibration of the system (Hudson, 1979—pages 55-64).

lated through Eq. 31. Any one of these spectra can be obtained from one of the other two, and each of the three spectra contains the same information, no more and no less. The three spectra are simply different ways of presenting the same information on structural response.

Because the S_d, S_v, and S_a quantities are simply related by powers of the vibration period T as given in Eq. 31b, the response spectrum can be presented on a so-called tripartite or four-way logarithmic plot from which all three spectral quantities can be read. The S_v−T data in the linear plot of Fig. 19b is replotted with logarithmic scales for S_v and T as shown in Fig. 20. The S_d and S_a values can be read from the logarithmic scales

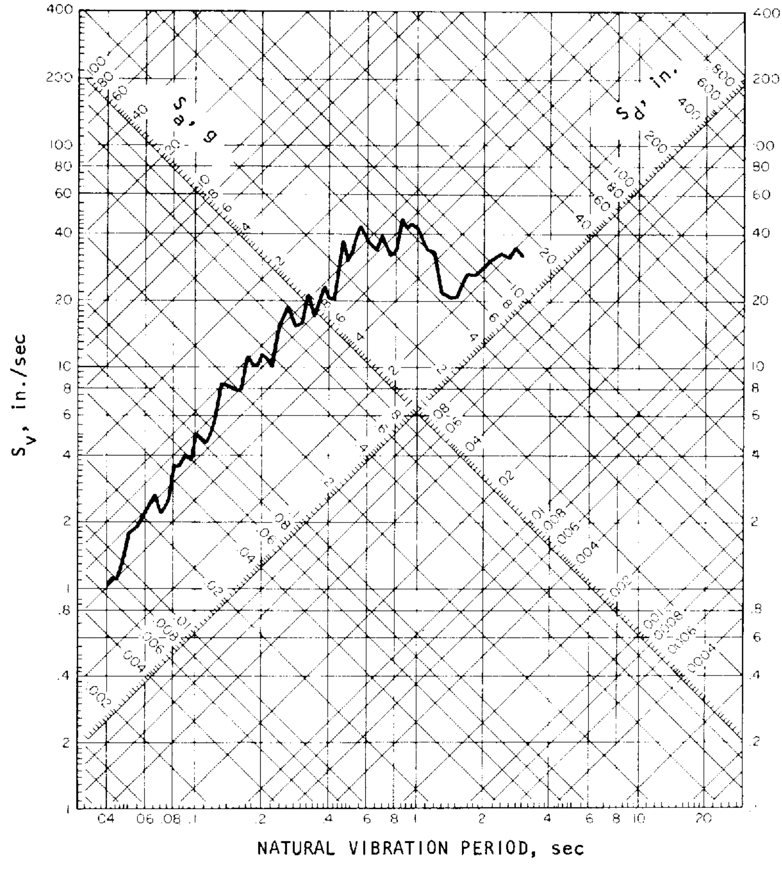


Figure 20. Four-way logarithmic plot of response spectrum. El Centro ground motion—S00°E component. Damping ratio ξ = 2 percent

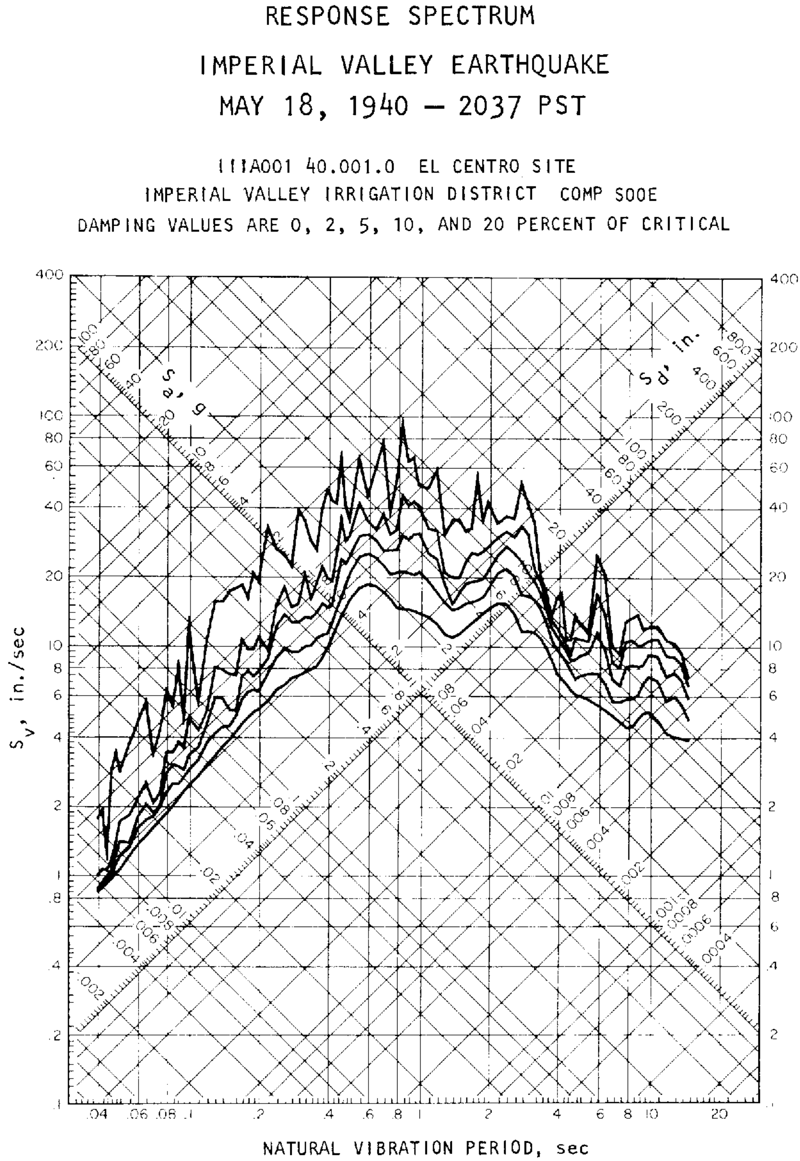


Figure 21. Four-way logarithmic plot of response spectrum. El Centro ground motion—S00°E component (after Hudson, 1979)

oriented at 45° to the period scale. This four-way plot is a compact presentation of the three response spectra, for a single plot of this form replaces the three linear plots of Fig. 19. Spectrum curves for several damping values are usually plotted on the same graph, as shown in Fig. 21. This is one of the standard spectrum plots prepared by the Caltech strong motion data program including vibration periods up to 15 sec (Hudson, 1979—Appendices A and C).

The El Centro ground motion (S00°E component) shown in Fig. 17 had a maximum acceleration of 0.348 g, maximum velocity of 13.15 in/sec, and maximum displacement 4.29 in. (Hudson, 1979). With this data and from the shape of the four-way logarithmic plot of the response spectrum, it becomes apparent that the maximum response of short-period (or high frequency) structures is controlled by the ground acceleration, that of long-period (or low frequency) structures by the ground displacement, and that of intermediate-period structures by the ground velocity (Veletsos, Newmark and Chelapati, 1965; Veletsos, 1969).

Standard data processing procedures were developed for the routine treatment of accelerograms for the Caltech strong motion data program, and tables and graphs for the response spectra of all past earthquake ground motions were prepared (Hudson, 1979*). Figure 21 is a typical graph wherein the spectrum ordinates are plotted against natural vibration period. Alternatively, the data can be plotted as a function of natural cyclic frequency f (Veletsos, Newmark and Chelapati, 1965).

The maximum response of a one-story structure with known natural vibration period T (or circular frequency ω or cyclic frequency f) and damping ratio ξ to earthquake ground motion, for which the response spectrum is available, can be readily deter-

* For convenience of the reader, the notation used in this monograph for the response spectrum quantities is compared with that in the monograph by Hudson (1979).

Response Spectrum	*This Monograph*	*Hudson (1979)*
Pseudo-Velocity	S_v	PSV
Deformation (or Displacement)	S_d	SD
Pseudo-Acceleration	S_a	PSA

mined without computing the response history. The maximum deformation is related to the response spectrum ordinates by the equations:

$$u_{max} = S_d = \frac{S_v}{\omega} = \frac{S_a}{\omega^2} \tag{34a}$$

or

$$u_{max} = S_d = \left(\frac{T}{2\pi}\right) S_v = \left(\frac{T}{2\pi}\right)^2 S_a \tag{34b}$$

Similarly, the maximum base shear

$$V_{0,max} = kS_d = m\omega S_v = mS_a \tag{35a}$$

The maximum base moment is related to the maximum base shear by the equation

$$M_{0,max} = hV_{0,max} \tag{35b}$$

Corresponding to the T (or ω or f) and ξ for the structure, any one of the S_d, S_v, or S_a values is read from the response spectrum, such as the one shown in Fig. 21, and substituted in Eqs. 34 and 35 to determine the maximum value of the deformation and base shear in the structure due to the earthquake.

The deformation, pseudo-velocity, and pseudo-acceleration response spectra are sufficient for computing the maximum deformations and forces needed in structural design. These spectra are related to other types of spectra—relative velocity response spectrum, absolute acceleration response spectrum and Fourier spectrum—that have been introduced in the literature for different types of studies (Newmark and Rosenblueth, 1971—Chapter 1; Hudson, 1979—pages 55-64).

2. *Dynamics of Multistory Buildings*

SIMPLEST IDEALIZATION OF MULTISTORY BUILDINGS

It is desirable to begin the study of dynamics of multistory buildings with their simplest possible idealization, as shown in Fig. 22. In this idealization, we assume that the columns supporting and interconnecting the floor systems are massless and the entire mass of the structure is concentrated at the floor levels; the floor systems and beams are rigid whereas the columns are flexible to lateral deformation but rigid in the vertical direction. The structure is assumed to be supported on rigid ground. This so-called shear building model is useful in developing the basic concepts of multistory building dynamics. However, refined idealizations are usually necessary to accurately determine the dynamic response of buildings. Such idealizations are briefly discussed later.

The masses concentrated at the floor levels are denoted by m_1, m_2, . . . m_N where m_j = mass at the jth floor. The stiffness properties of the linear structure are characterized by the lateral stiffness k_1, k_2, . . . k_N of individual stories, where k_j = lateral stiffness of the jth story, i.e. the story shear force required to cause unit deformation in the story (Fig. 22).

It will be convenient to first develop the equations of motion for systems with no damping; the damping terms will subsequently be included.

EQUATIONS OF MOTION

The motion of the idealized multistory building due to dynamic excitation will be governed by ordinary differential equations, as many as the number of stories in the building. We derive the equations of motion for two types of dynamic excitation: external forces and ground motion. For each type of excitation it will be

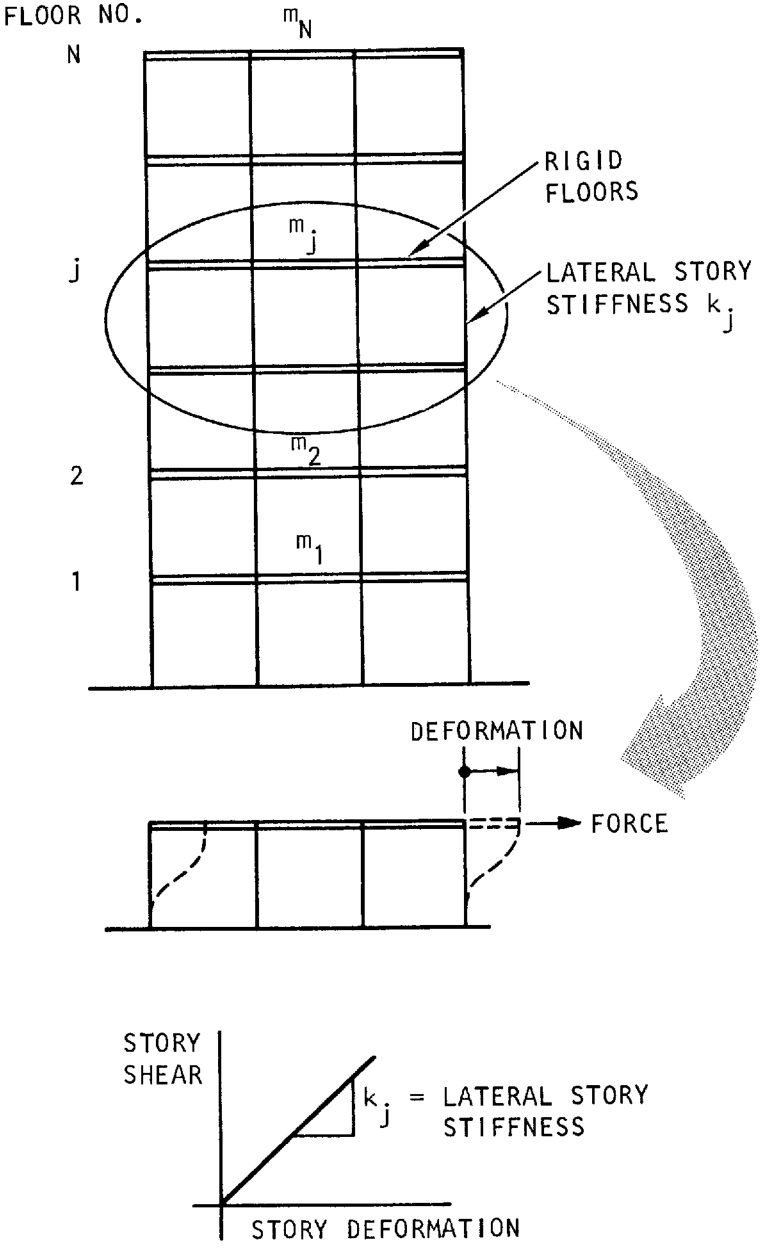


Figure 22. Idealized multistory building

useful to first derive the equations of motion for a two-story building. Subsequently, these will be generalized to obtain the equations of motion in general form, valid for a building with any number of stories.

External Forces

Figure 23a shows a two-story building subjected to externally applied dynamic forces $p_1(t)$ and $p_2(t)$ at the first and second floors, respectively. Under the influence of these forces, the structure is displaced in the lateral direction. At any instant of time, the displaced configuration of the structure can be specified by the displacements $u_1(t)$ and $u_2(t)$ of the first and second floors, respectively.

The various forces acting on the floor masses are shown in the free-body diagrams of Fig. 23b. These include the external forces $p_j(t)$, the inertia forces f_{I_j}, and the elastic resisting forces f_{S_j}. The inertia forces act to the left, opposite to the direction of positive acceleration. The elastic resisting forces are shown acting to the left, opposite to the direction of positive deformation. At each instant of time, each mass of the structure is in equilibrium under the action of these forces at that time. From the free-body diagrams, this condition of dynamic equilibrium for the first floor mass is

$$f_{I_1} + f_{S_1} = p_1(t) \tag{36a}$$

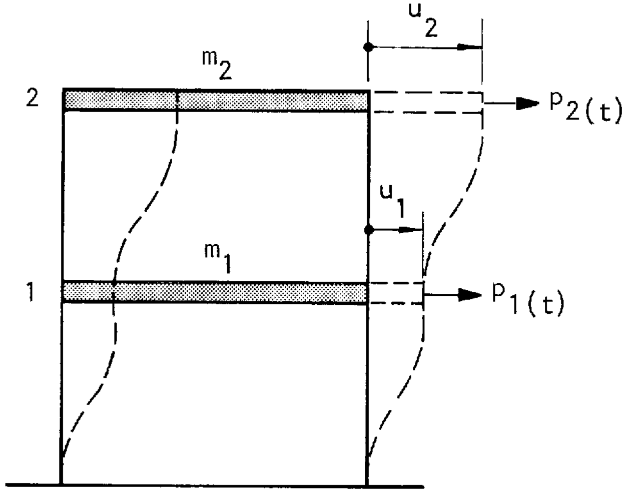
and for the second floor mass it is

$$f_{I_2} + f_{S_2} = p_2(t) \tag{36b}$$

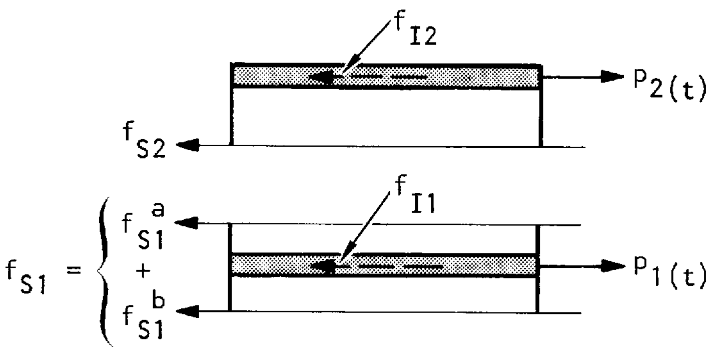
The inertia and elastic forces are next related to the accelerations and displacements of the masses. For a linear structure, the elastic resisting forces are related to the floor displacements through the story stiffnesses:

$$\begin{aligned} f_{S_1} &= f^b_{S_1} + f^a_{S_1} \\ &= k_1 u_1 + k_2(u_1 - u_2) \end{aligned} \tag{37a}$$

$$f_{S_2} = k_2(u_2 - u_1) \tag{37b}$$



(a) Two-story building subjected to external forces



(b) Free-body diagrams

Figure 23.

The inertia forces associated with the masses m_1 and m_2 undergoing accelerations $\ddot{u}_1$ and $\ddot{u}_2$, respectively, are

$$f_{I_1} = m_1\ddot{u}_1 \tag{38a}$$

$$f_{I_2} = m_2\ddot{u}_2 \tag{38b}$$

Substitution of Eqs. 37 and 38 into Eq. 36 results in

$$m_1\ddot{u}_1 + k_1u_1 + k_2(u_1 - u_2) = p_1(t) \tag{39a}$$

$$m_2\ddot{u}_2 \qquad\qquad + k_2(u_2 - u_1) = p_2(t) \tag{39b}$$

These two differential equations govern the motion defined by displacements $u_1(t)$ and $u_2(t)$ of the two-story structural system (Fig. 23) subjected to external dynamic forces $p_1(t)$ and $p_2(t)$. Note that the equations are not independent; they are coupled and in their present form they must be solved simultaneously to determine the displacement response.

The *equations of motion* can be written in matrix notation as follows:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix} \tag{40}$$

Using the following notation

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \ddot{\mathbf{u}} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} \qquad \mathbf{p}(t) = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Eq. 40 can be written as

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \tag{41}$$

where $\mathbf{u}$ and $\ddot{\mathbf{u}}$ are the displacement and acceleration vectors, $\mathbf{p}(t)$ is the vector of external dynamic forces, $\mathbf{m}$ is the mass matrix, and $\mathbf{k}$ the stiffness matrix for the two-story structure.

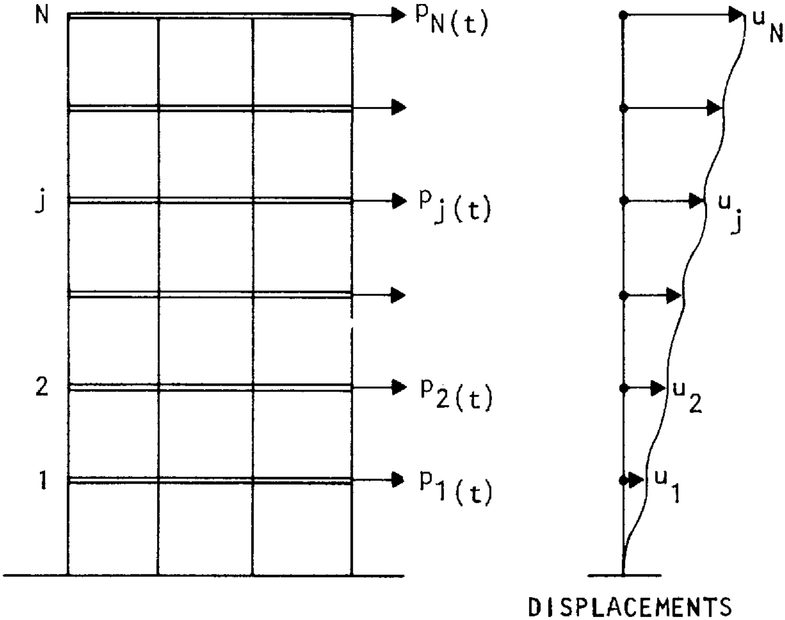


Figure 24. Multistory building subjected to external forces

We now return to the N-story building (Fig. 24) subjected to externally applied dynamic forces at the floor levels, with the force applied at the jth floor denoted by $p_j(t)$. Under the influence of these forces, the displaced configuration at any instant of time can be specified by the displacements $u_j(t)$ ($j = 1, 2, \ldots N$) of the floors. The N equations of motion for this structure can also be expressed in the form of Eq. 41, provided the various terms are appropriately generalized. For the N-story building, the displacement and external force vectors are

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ \vdots \\ u_N \end{Bmatrix} \qquad \mathbf{p}(t) = \begin{Bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_j(t) \\ \vdots \\ p_N(t) \end{Bmatrix}$$

the mass matrix is

$$\mathbf{m} = \begin{bmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & \ddots & & & \\ & & & m_j & & \\ & & & & \ddots & \\ & & & & & m_N \end{bmatrix}$$

and the stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} (k_1 + k_2) & -k_2 & & & & \\ -k_2 & (k_2 + k_3) & -k_3 & & & \\ & -k_3 & (k_3 + k_4) & -k_4 & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & -k_N \\ & & & & -k_N & k_N \end{bmatrix}$$

Equation 41 is the multistory building equivalent of Eq. 3 with $c = 0$; each term of the equation of motion for a one-story structure is represented by a matrix in the equations of motion for the multistory structure. The order of the matrices corresponds to the number of stories in the building. In principle, the damping terms can be included in Eq. 41 by analogy with the damping term in Eq. 3, leading to

$$\mathbf{m\ddot{u}} + \mathbf{c\dot{u}} + \mathbf{ku} = \mathbf{p}(t) \tag{42}$$

where $\mathbf{c}$ is the damping matrix of the structure and $\mathbf{\dot{u}}$ the velocity vector.

No external dynamic forces are applied to the idealized two-story structure shown in Fig. 25. The excitation in this case is the earthquake induced motion at the base of the structure, presumed to be only a horizontal component of ground motion with displacement $u_g(t)$, velocity $\dot{u}_g(t)$, and acceleration $\ddot{u}_g(t)$. Under the influence of such an excitation, the base of the structure displaces by an amount $u_g(t)$ if the ground is rigid, and the structure undergoes deformation resulting in floor displacements $u_1(t)$ and $u_2(t)$, relative to the ground. The total displacements of the floors are:

$$u_1^t(t) = u_g(t) + u_1(t) \tag{43a}$$

$$u_2^t(t) = u_g(t) + u_2(t) \tag{43b}$$

From the free-body diagrams for the two floor masses shown in Fig. 25, the equations of dynamic equilibrium are

$$f_{I_1} + f_{S_1} = 0 \tag{44a}$$

$$f_{I_2} + f_{S_2} = 0 \tag{44b}$$

Equations 37 still apply because the elastic forces depend only on the displacements relative to the ground displacement, not on the total displacement. However, the masses m_1 and m_2 undergo accelerations $\ddot{u}_1^t$ and $\ddot{u}_2^t$, respectively, and the inertia forces therefore are

$$f_{I_1} = m_1\ddot{u}_1^t \tag{45a}$$

$$f_{I_2} = m_2\ddot{u}_2^t \tag{45b}$$

which with the aid of Eq. 43 can be expressed as

$$f_{I_1} = m_1\,(\ddot{u}_g + \ddot{u}_1) \tag{46a}$$

$$f_{I_2} = m_2\,(\ddot{u}_g + \ddot{u}_2) \tag{46b}$$

The equations of dynamic equilibrium (Eqs. 44), after substitution of Eqs. 37 and 46, can be expressed as

$$m_1\ddot{u}_1 + k_1u_1 + k_2(u_1 - u_2) = -m_1\ddot{u}_g(t) \tag{47a}$$

$$m_2\ddot{u}_2 + k_2(u_2 - u_1) = -m_2\ddot{u}_g(t) \tag{47b}$$

These two differential equations govern the motion, defined by displacements $u_1(t)$ and $u_2(t)$, of the two-story structure subjected to earthquake ground motion $\ddot{u}_g(t)$. They can be expressed in matrix notation as

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\mathbf{1}\ddot{u}_g(t) \tag{48}$$

with the displacement vector **u**, acceleration vector $\ddot{\mathbf{u}}$, mass matrix **m** and stiffness matrix **k** as defined earlier for the two-story building, and **1** is a vector of two elements both equal to unity.

Having derived the equations of motion for a two-story building, we now return to the N-story building subjected to earthquake ground motion (Fig. 26). Under the influence of such an excitation, the base of the structure displaces an amount $u_g(t)$ if the ground is rigid, and the deformed configuration of the structure is specified by the floor displacements $u_j(t)$ ($j = 1, 2, \ldots N$), relative to the base. The N equations of motion for

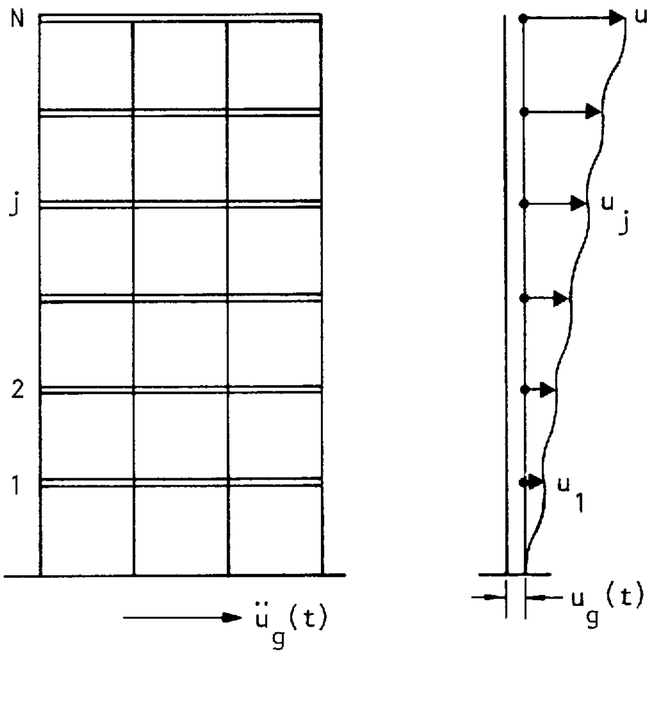
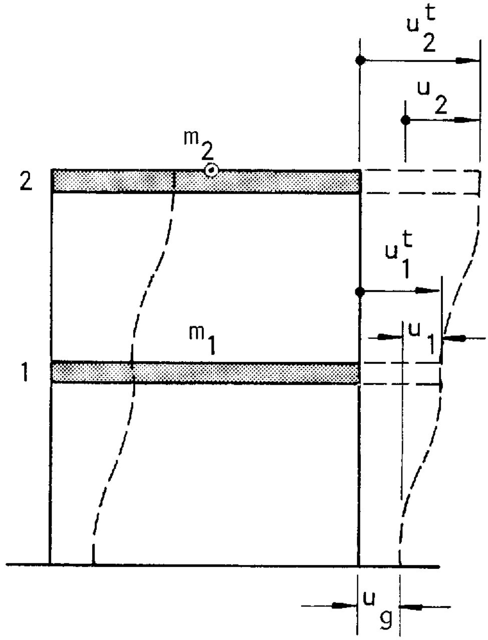
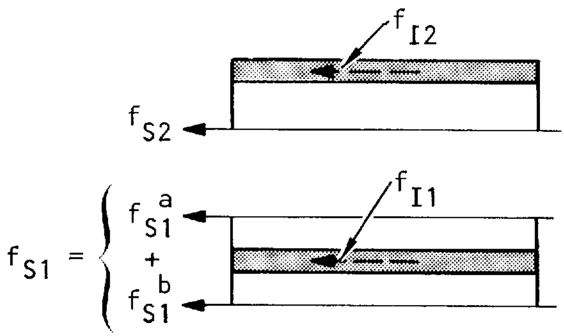


Figure 26. Multistory building subjected to earthquake ground motion



(a) Two-story building subjected to earthquake ground motion



(b) Free-body diagrams

Figure 25.

this structure can also be expressed in the form of Eq. 48, with the displacement vector **u**, the mass matrix **m**, and stiffness matrix **k** as defined earlier for the N-story building, and **1** is now a vector of N elements each equal to unity. Including the damping forces in Eq. 48 in terms of the damping matrix **c** and velocity vector $\dot{\mathbf{u}}$ leads to

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\mathbf{1}\ddot{u}_g(t) \tag{49}$$

Comparison of Eqs. 42 and 49 shows that the equations of motion for the structure subjected to two different excitations—ground acceleration = $\ddot{u}_g(t)$ and external forces = $-m_j\ddot{u}_g(t)$—are one and the same. The deformation response **u**(t) of the structure to ground acceleration $\ddot{u}_g(t)$ will be identical to the response of the structure on fixed base subjected to external forces equal to floor masses times the ground acceleration, acting opposite to the sense of ground acceleration. As shown in Fig. 27, the ground motion can therefore be replaced by *effective forces* = $-m_j\ddot{u}_g(t)$, j = 1, 2, . . . N.

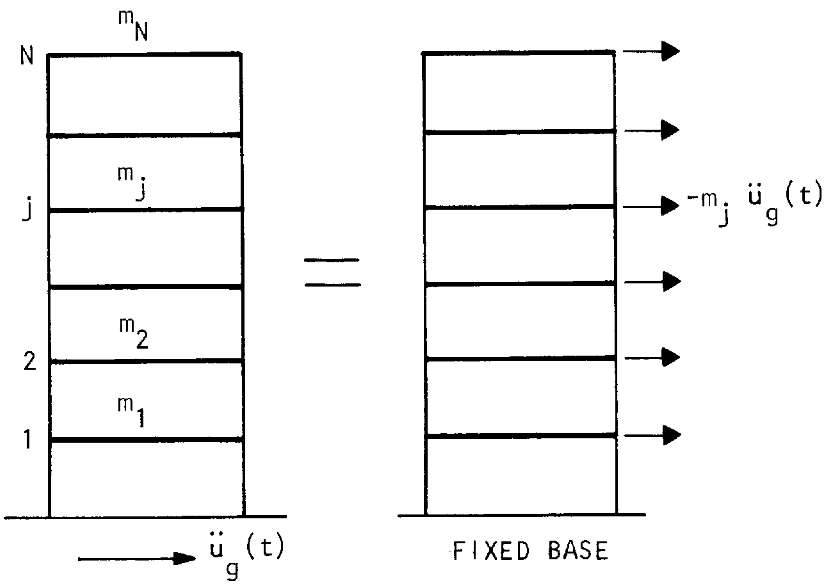


Figure 27. Effective earthquake forces

Problem Statement

Given the mass matrix **m**, stiffness matrix **k**, damping matrix **c**, and the excitation forces **p**(t) or ground acceleration $\ddot{u}_g(t)$, a fundamental problem in structural dynamics is to determine the displacement response **u**(t) of the structure. Internal forces and other response quantities of interest can subsequently be determined from the displacement response.

Whereas the mass and stiffness matrices of a structure can be computed from the dimensions and sizes of structural and non-structural elements, it is impractical to compute the damping matrix in a similar manner. Energy dissipation in a multistory building is due to the combined effects of a number of mechanisms such as friction at structural joints, friction between structural and non-structural elements, material damping, micro-cracking of concrete, etc. In general, it is not possible to quantitatively define these local energy dissipating mechanisms. For this reason, the damping matrix cannot be analytically evaluated in a manner similar to the mass and stiffness matrices. Damping in a structure is therefore usually specified on a global basis in terms of modal damping ratios, with values obtained from experiments on similar structures serving as a guide. Computation of damping ratios from vibration tests on structures and from records of their motion during earthquakes will be discussed subsequently.

We will examine the response of the structure in free vibration, to harmonic forces and to earthquake ground motion. The idealized N-story building is a *multi-degree-of-freedom* (*MDF*) *system*, having N degrees of freedom because its motion is governed by N differential equations (Eqs. 42 or 49) containing N unknowns $u_j(t)$, j = 1, 2, . . . N.

FREE VIBRATION RESPONSE

Undamped Structures

Consider first the multistory building of Fig. 22 without any damping. If the structure is disturbed from its equilibrium position by imparting to the various masses some displacements and velocities, defined by the vectors $\mathbf{u}(0)$ and $\dot{\mathbf{u}}(0)$, respectively, the structure will oscillate about its equilibrium position. This free vibration response can be described by the time-dependent displacement vector $\mathbf{u}(t)$. A mathematical description of $\mathbf{u}(t)$ can be obtained by solving the equations of motion (Eq. 42 or 49 without the damping term and without any excitation, i.e. $\mathbf{p}(t) = \ddot{u}_g(t) = 0$). The resulting displacements at the three floors of a three-story building in free vibration are displayed graphically in Fig. 28. The displacement-time plot for the jth floor starts with the ordinate $u_j(0)$ and slope $\dot{u}_j(0)$; in this particular case $\dot{u}_j = 0$. Contrary to what we observed in Fig. 8 for SDF systems, the motion of each mass of a MDF system is *not* a simple harmonic motion, and we cannot define the frequency of motion. Furthermore, not only does the value of displacement at each floor change with time, the deflected shape varies with time (see Fig. 28), i.e. the ratios u_1/u_3 and u_2/u_3 vary with time.

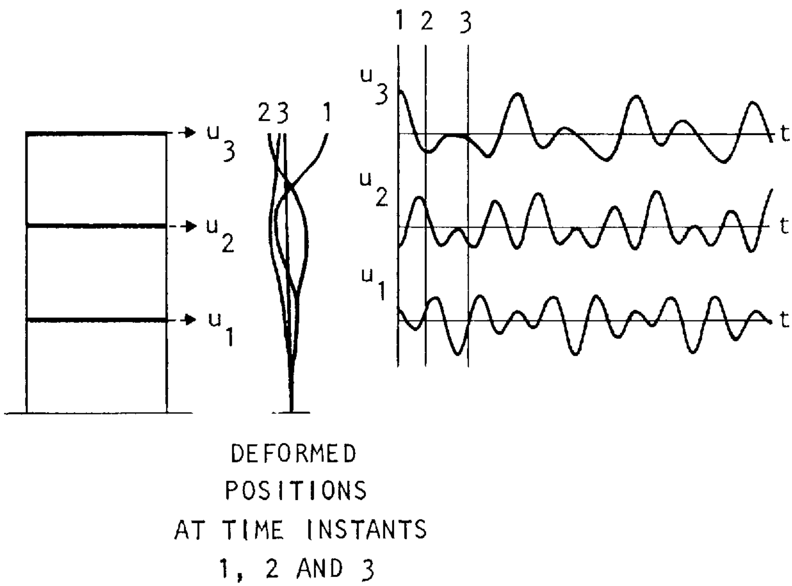


Figure 28. Free vibration with arbitrary initial displacement

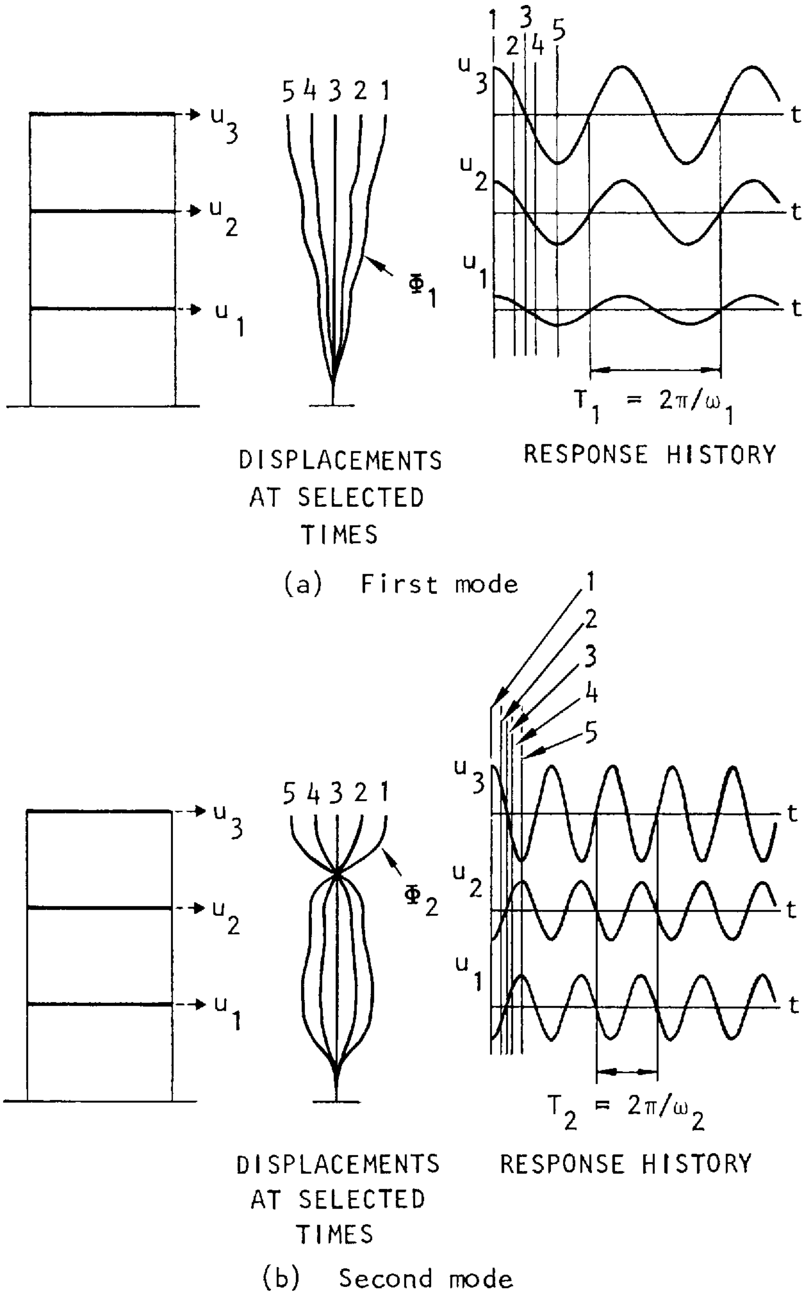


Figure 29. Free vibration in natural modes of vibration

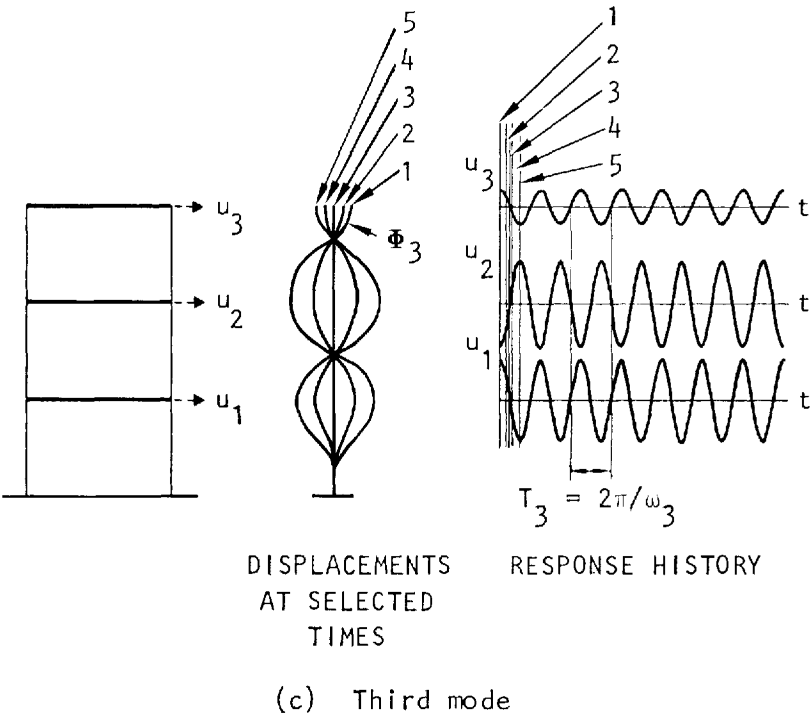


Figure 29. (concluded) Free vibration in natural modes of vibration

However, the undamped, idealized multistory building would undergo simple harmonic motion without change of deflected shape if free vibration is initiated by appropriate distributions of displacements and/or velocities over the height of the building. As shown in Fig. 29, three characteristic deflected shapes exist for an idealized three-story building, such that if it is displaced in any one of these shapes and released, it will vibrate in simple harmonic motion, maintaining the original deflected shape. All floors reach their extreme displacements at the same time and pass through the original equilibrium position at the same time.

A *natural period of vibration* T of the system is the time required for one cycle of the simple harmonic motion in one of these characteristic deflected shapes, each of which is called a *natural mode of vibration* of the structure. The corresponding *natural circular frequency* is ω, where

$$T = 2\pi/\omega \tag{50a}$$

and the *natural cyclic frequency*

$$f = 1/T \tag{50b}$$

Figure 29 shows the three natural periods T_n (n = 1, 2, 3) of the three-story building vibrating in its natural modes of vibration defined by vectors $\boldsymbol{\phi}_n$. The smallest of the three circular frequencies is denoted as ω_1, the largest as ω_3, and the intermediate frequency as ω_2. Correspondingly the longest of three vibration periods is denoted as T_1, the shortest as T_3, and the intermediate period as T_2. The vector $\boldsymbol{\phi}_n$ defines only the deflected shape of the structure vibrating in its nth natural mode of vibration, i.e. it does not define the floor displacements v_1, v_2, and v_3 but only their ratios, say v_1/v_3 and v_2/v_3. The vibration mode can be normalized by multiplying the vector $\boldsymbol{\phi}_n$ by any quantity. For example, the multiplier may be chosen so that the normalized vector $\boldsymbol{\phi}_n$ contains a unit value for the top floor. Modes may be normalized simply for convenience. How they are normalized does not affect the final results of dynamic response analysis.

An idealized N-story building possesses N natural circular frequencies of vibration ω_n (n = 1, 2, . . . N) arranged in sequence from smallest to largest, corresponding natural periods T_n, and natural modes of vibration $\boldsymbol{\phi}_n$. The term *natural* is used to qualify each of the above vibration quantities to emphasize the fact that these are natural properties of the structure, depending on its stiffness and mass, when it is allowed to vibrate freely without any external excitation. We refer to ω_n as the *nth natural circular frequency of vibration*, T_n as the *nth natural period of vibration*, and ϕ_n as the *nth natural mode of vibration*. The quantities ω_1, T_1, and ϕ_1 are also referred to as the fundamental frequency, period, and mode of vibration of the structure.

Damped Structures

Consider an idealized multistory building with damping, disturbed from its equilibrium position by displacing it in a natural mode of vibration of the corresponding undamped structure, a system with stiffness and mass properties identical to the building but with no damping. For certain forms of damping that are reasonable models for many buildings, the initial deflected shape will be maintained during the free vibration and the motion of any floor will be similar to that of the undamped structure shown in Fig. 29; except that, because of damping, the amplitude of motion at each floor would decrease with every cycle of vibration as shown in Fig. 9. The period T_{nD}, circular frequency ω_{nD}, and cyclic frequency f_{nD} of the nth mode of vibration of the damped structure are interrelated in the same manner as for the undamped structure (Eq. 50). Damping influences the natural frequencies and periods of vibration of the multistory structure in the same manner as for the SDF system (Eq. 12). Thus,

$$\omega_{nD} = \omega_n\sqrt{1-\xi_n^2} \tag{51a}$$

$$T_{nD} = T_n/\sqrt{1-\xi_n^2} \tag{51b}$$

where ξ_n is the damping ratio for the multistory building in its nth natural mode of vibration. However, for damping ratios less than 0.2, a range which includes most structures, the effects of damping on vibration frequencies are negligible (see Fig. 10). The natural frequencies and modes of vibration of a structure can therefore be computed under the assumption that the structure is undamped.

In a multistory structure undergoing free vibration in its nth natural mode of vibration, the displacement amplitude at any floor decreases with each vibration cycle. The rate of decay depends on the damping ratio ξ_n in that mode, in a manner similar to one-story structures. Thus the ratio of two response peaks separated by j cycles of vibration is related to the damping ratio by Eq. 16, with appropriate change in notation.

Consequently, the damping ratio in a natural mode of vibration of a multistory building can be determined from a free vibration test following the procedure presented earlier for one-story structures. In such a test, the structure would be deformed by pulling on it with a cable that is then suddenly released, thus causing the structure to perform free vibrations about its static equilibrium position. A difficulty in such tests is to apply the pull and release in such a way that the structure will vibrate in one of its natural modes of vibration. But if this difficulty can be overcome, the damping ratio can be computed from the decay rate of vibration amplitudes.

Computation of Natural Frequencies and Modes of Vibration

We shall see later that the vibration properties of a structure, i.e. the natural frequencies and modes of vibration, play a central role in the analysis of dynamic response of the structure in its linear range of behavior. Computation of the vibration properties requires solution of the matrix equation

$$\mathbf{k}\boldsymbol{\phi} = \omega^2\, \mathbf{m}\boldsymbol{\phi} \tag{52}$$

which in mathematical terminology defines an eigen-problem. For a N-DOF system, such as the idealized N-story building, the mass and stiffness matrices are of order N. Solution of the eigen-problem leads to the N natural frequencies and modes of vibration: ω_n, $\boldsymbol{\phi}_n$, $n = 1, 2, \ldots N$.

Many methods have been developed for numerical solution of the eigen-problem and are available in textbooks (Newmark and Rosenblueth, 1971—Chapter 4; Clough and Penzien, 1975—Chapter 14; Bathè and Wilson, 1976—Chapters 10–12). Some of these methods, such as the Stodola method, can be conveniently implemented on a pocket calculator while others have been developed for computer analysis of complex structures with a large number (up to several hundred) of degrees of freedom. Choice of the most efficient computer method depends on the properties of mass and stiffness matrices and on the number of natural frequencies and modes of vibration that need to be computed; just the first few usually suffice for earthquake response analysis.

RESPONSE TO HARMONIC EXCITATION

Consider external forces varying harmonically with time applied at the various floors of the idealized multistory building. The external force at the jth floor $p_j(t) = p_{0j} \sin \bar{\omega}t$, where the amplitude or maximum value of the force is p_{0j}, its period $\bar{T} = 2\pi/\bar{\omega}$, and circular frequency is $\bar{\omega}$. Starting with the floor masses and story stiffnesses, the mass and stiffness matrices of the system can be formed as described earlier, and then the natural frequencies and modes of vibration can be computed. For specific values of damping ratios ξ_n in the natural modes of vibration, the dynamic response of the structure to the harmonic forces can be determined by the mode superposition method (Newmark and Rosenblueth, 1971—Chapter 2; Clough and Penzien, 1975—Chapter 13).

The response of the structure consists of two parts: free vibration response plus steady state response. In a damped structure, the free vibration response decays, eventually becoming insignificant, and usually only the steady state response is considered. The frequency of the steady state response is the same as the forcing frequency $\bar{\omega}$, and the phase between the force and the response is different from zero. The steady state displacement at the jth floor can be expressed as

$$\frac{u_j(t)}{u_{j,st}} = D_j \sin(\bar{\omega}t - \theta_j) \tag{53}$$

where $u_{j,st}$ = displacement at the jth floor of the structure if the maximum forces p_{0j} are applied as static forces; and θ_j is the

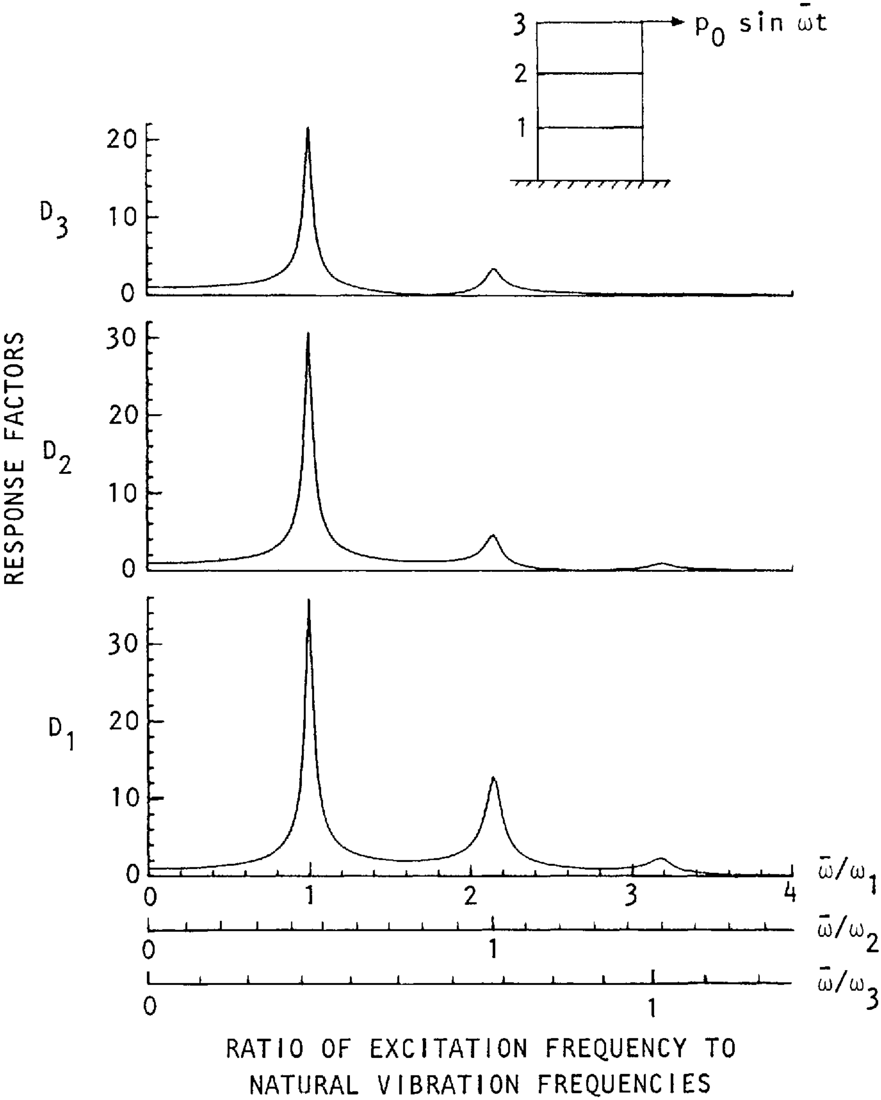


Figure 30. Response factors for a 3-story building subjected to harmonic force

phase angle, or the angular phase shift. The amplitude of the dynamic displacement at the jth floor $u_{j,max}$ is given by

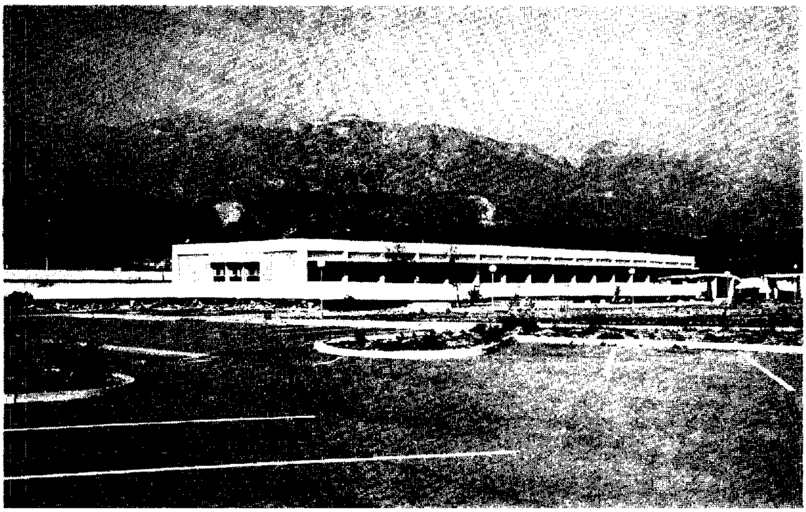
$$\frac{u_{j,max}}{u_{j,st}} = D_j \tag{54}$$

Thus D_j is a dimensionless factor, equal to the ratio of the dynamic to the static displacement response amplitudes. The response factor D_j depends on the floor location, the forcing frequency, natural frequencies and modes of vibration of the structure, and modal damping ratios.

Figure 30 shows the variation of the response factors with forcing frequency for a three-story building (with 2 percent damping ratio in each of the three modes of vibration) subjected to harmonic force at the roof. The response factor at each floor now displays resonance at three forcing frequencies that are essentially the same (except for the slight effect of damping discussed previously) as the natural vibration frequencies of the structure. In contrast, the response of a one-story structure displayed only one resonant frequency (Fig. 15).

If the natural vibration frequencies of the structure are well separated and the structure is lightly damped, the deformed configuration of the structure vibrating at forcing frequency $\bar{\omega} = \omega_n$, the nth natural vibration frequency, will be essentially the same as the shape of the nth vibration mode; furthermore the shape of the response curve of Fig. 30 in the vicinity of each of the resonant frequencies is similar to that of the response curve for a one-story structure of Fig. 15 in the vicinity of its resonant frequency. In particular, the half-power bandwidth of the response curve (see Fig. 16) near the nth resonant frequency is $2\xi_n$, where ξ_n is the damping ratio for the nth natural vibration mode. The procedure outlined earlier to determine the damping ratio from results of forced vibration tests on one-story structures is therefore applicable separately to each resonant frequency of a multistory building. While there are many factors to be considered in testing of complex structures (Hudson, 1970), the basic approach outlined above leads to the damping ratios ξ_n for the various modes of vibration.

Courtesy of V. V. Bertero



Courtesy of G. W. Housner

Psychiatric Day Care Center before and after the San Fernando earthquake of 9 February 1971. Before the earthquake, this two-story reinforced concrete building was essentially a massive second story supported on relatively flexible first-story columns. When subjected to very strong ground shaking the tied columns made of lightweight concrete were subjected to large deformations and they slowly disintegrated, depositing the superstructure on the ground.

Given the mass, stiffness, and damping matrices of the structure and the earthquake ground acceleration, the displacement response of a structure can be determined by solving the equations of motion:

$$\mathbf{m\ddot{u}} + \mathbf{c\dot{u}} + \mathbf{ku} = -\mathbf{m1}\ddot{u}_g(t) \tag{55}$$

For the idealized N-story building, this matrix equation contains N ordinary differential equations in N unknown floor displacements $u_j(t)$, $j = 1, 2, \ldots N$. In general, each of these equations contains more than one unknown, and cannot be solved independently of the other equations; therefore the set of N equations must be solved simultaneously by available computational methods (Newmark and Rosenblueth, 1971—Chapter 2; Clough and Penzien, 1975—Chapter 15; Bathè and Wilson, 1976—Chapter 8).

Simultaneous solution of the coupled equations is avoided by the modal method or the mode superposition method (Newmark and Rosenblueth, 1971—Chapter 2; Clough and Penzien, 1975—Chapter 13). This method is generally applicable to analysis of dynamic response of complex structures in their linear range of behavior, in particular to analysis of forces and deformations in multistory buildings due to medium intensity ground shaking causing moderately large but essentially linear response of the structure. The method provides "exact" results for the complete history of dynamic response during the earthquake and can be adapted to provide estimates of maximum response directly from the earthquake response spectrum.

Response History Analysis

The modal method, or mode superposition method, is based on the fact that, for certain forms of damping that are reasonable models for many buildings, the response in each natural mode of vibration can be computed independently of the others, and the modal responses can be combined to determine the total response. Each mode responds with its own particular pattern of deformation, the natural mode of vibration ϕ_n; with its own frequency, the natural frequency of vibration ω_n; and with its own modal damping, the damping ratio ξ_n. The time-history of each modal response can be computed by analysis of a single-degree-of-freedom (SDF) system with properties chosen to be representative of the particular mode and the degree to which it is excited by the earthquake motion.

The equation of motion for the nth natural vibration mode of the idealized multistory building can be expressed as

$$\ddot{Y}_n + 2\xi_n\omega_n\dot{Y}_n + \omega_n^2 Y_n = -\frac{L_n}{M_n}\ddot{u}_g(t) \tag{56}$$

where $L_n = \sum_{j=1}^{N} m_j\phi_{jn}$ and the modal mass $M_n = \sum_{j=1}^{N} m_j\phi_{jn}^2$.

This modal equation is also the equation of motion for a SDF system (Eq. 23) with natural vibration frequency ω_n and damping ratio ξ_n excited to the degree L_n/M_n by the ground acceleration $\ddot{u}_g(t)$. Thus, by analogy with Eq. 24, the modal displacement is

$$Y_n(t) = -\frac{L_n}{M_n}\frac{1}{\omega_{nD}}\int_0^t \ddot{u}_g(\tau)\exp[-\xi_n\omega_n(t-\tau)]\sin[\omega_{nD}(t-\tau)]d\tau \tag{57}$$

with ω_{nD} as defined in Eq. 51a.

The contribution of the nth mode to the displacement $u_j(t)$ at the jth floor is given by

$$u_{jn}(t) = Y_n(t)\,\phi_{jn},\ j = 1, 2, \ldots N. \tag{58}$$

which after substitution of Eq. 57 becomes

$$u_{jn}(t) = -\frac{L_n}{M_n}\frac{\phi_{jn}}{\omega_{nD}}\int_0^t \ddot{u}_g(\tau)\exp[-\xi_n\omega_n(t-\tau)]\sin[\omega_{nD}(t-\tau)]d\tau \tag{59}$$

The deformation, or drift, in story j is given by the difference of displacements of the floors above and below:

$$\Delta_{jn}(t) = u_{jn}(t) - u_{j-1,n}(t) \tag{60}$$

The internal forces, such as story shears, moments, etc., associated with deformations of the multistory building can be conveniently determined by first introducing the concept of *equivalent lateral forces*. These are external forces $\mathbf{f}$ which, if applied as static forces, would cause structural displacements $\mathbf{u}$.* Thus, at any instant of time the equivalent lateral forces associated with modal displacements $\mathbf{u}_n(t)$ are

$$\mathbf{f}_n(t) = \mathbf{k}\mathbf{u}_n(t)$$

and after substituting Eq. 58 they become

$$\mathbf{f}_n(t) = \mathbf{k}\boldsymbol{\phi}_n Y_n(t)$$

It will be useful to express these forces in terms of the mass matrix, which can be achieved with the aid of Eq. 52. Thus

$$\mathbf{f}_n(t) = \omega_n^2 \mathbf{m}\boldsymbol{\phi}_n Y_n(t) \tag{61a}$$

from which the force at the jth floor (Fig. 31) is

$$f_{jn}(t) = \omega_n^2 m_j \phi_{jn} Y_n(t) \tag{61b}$$

which after substitution of Eq. 57 becomes

$$f_{jn}(t) = -\, m_j \phi_{jn} \frac{L_n}{M_n} \frac{\omega_n^2}{\omega_{nD}} \int_0^t \ddot{u}_g(\tau) \exp[-\xi_n \omega_n (t-\tau)] \sin[\omega_{nD}(t-\tau)] d\tau \tag{62}$$

Any internal force can be determined by static analysis of the structure subjected to the equivalent lateral forces. For example, the shear and moment at the base of the building are

$$V_{0n}(t) = \sum_{j=1}^{N} f_{jn}(t) \tag{63a}$$

$$M_{0n}(t) = \sum_{j=1}^{N} h_j f_{jn}(t) \tag{63b}$$

* To correspond with the earlier notation of f_S for the equivalent lateral force for a one-story structure, the force vector for a multistory building should have been denoted by $\mathbf{f}_S$. Some of the subsequent equations would have become cumbersome with this notation; hence the subscript s has been dropped.

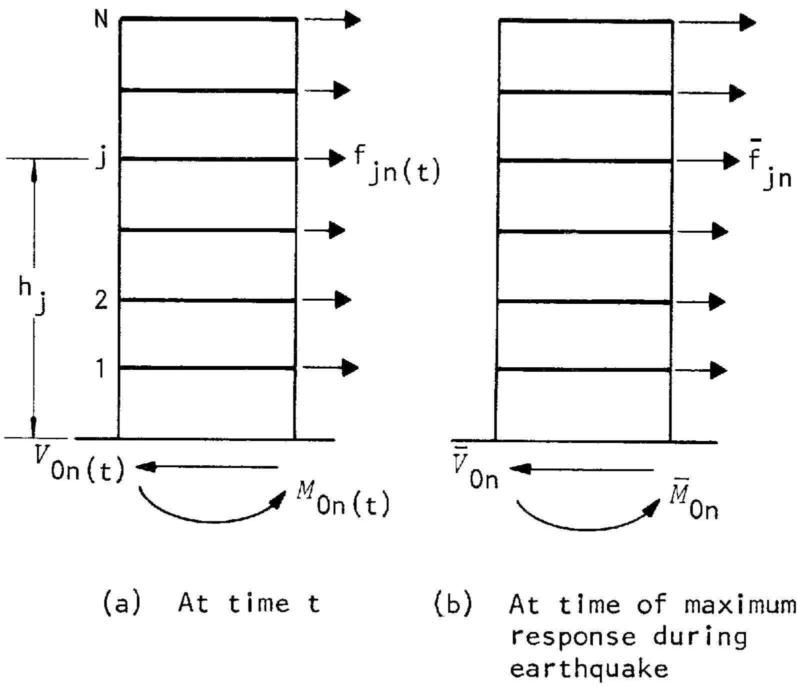


Figure 31. Equivalent lateral forces in nth vibration mode

where h_j is the height of the jth floor above the base (Fig. 31). After substituting Eq. 62, the base shear can be expressed as

$$V_{0n}(t) = -\frac{L_n^2}{M_n}\frac{\omega_n^2}{\omega_{nD}}\int_0^t \ddot{u}_g(\tau)\exp[-\xi_n\omega_n(t-\tau)]\sin[\omega_{nD}(t-\tau)]d\tau \tag{64a}$$

and the base moment as

$$M_{0n}(t) = -\frac{L_n}{M_n}\frac{\omega_n^2}{\omega_{nD}}\left(\sum_{j=1}^{N} h_j m_j \phi_{jn}\right)$$

$$\int_0^t \ddot{u}_g(\tau)\exp[-\xi_n\omega_n(t-\tau)]\sin[\omega_{nD}(t-\tau)]d\tau \tag{64b}$$

The earthquake response of the structure is obtained by combining the modal responses in all the modes of vibration. Thus the displacement at the jth floor, the equivalent lateral force at the jth floor, the base shear, and the base moment are given by

$$u_j(t) = \sum_{n=1}^{N} u_{jn}(t) \tag{65a}$$

$$f_j(t) = \sum_{n=1}^{N} f_{jn}(t) \tag{65b}$$

$$V_0(t) = \sum_{n=1}^{N} V_{0n}(t) \tag{65c}$$

$$M_0(t) = \sum_{n=1}^{N} M_{0n}(t) \tag{65d}$$

In general, the total value of any response $r(t)$ is the combination of the contributions of all the vibration modes to that response quantity:

$$r(t) = \sum_{n=1}^{N} r_n(t) \tag{66}$$

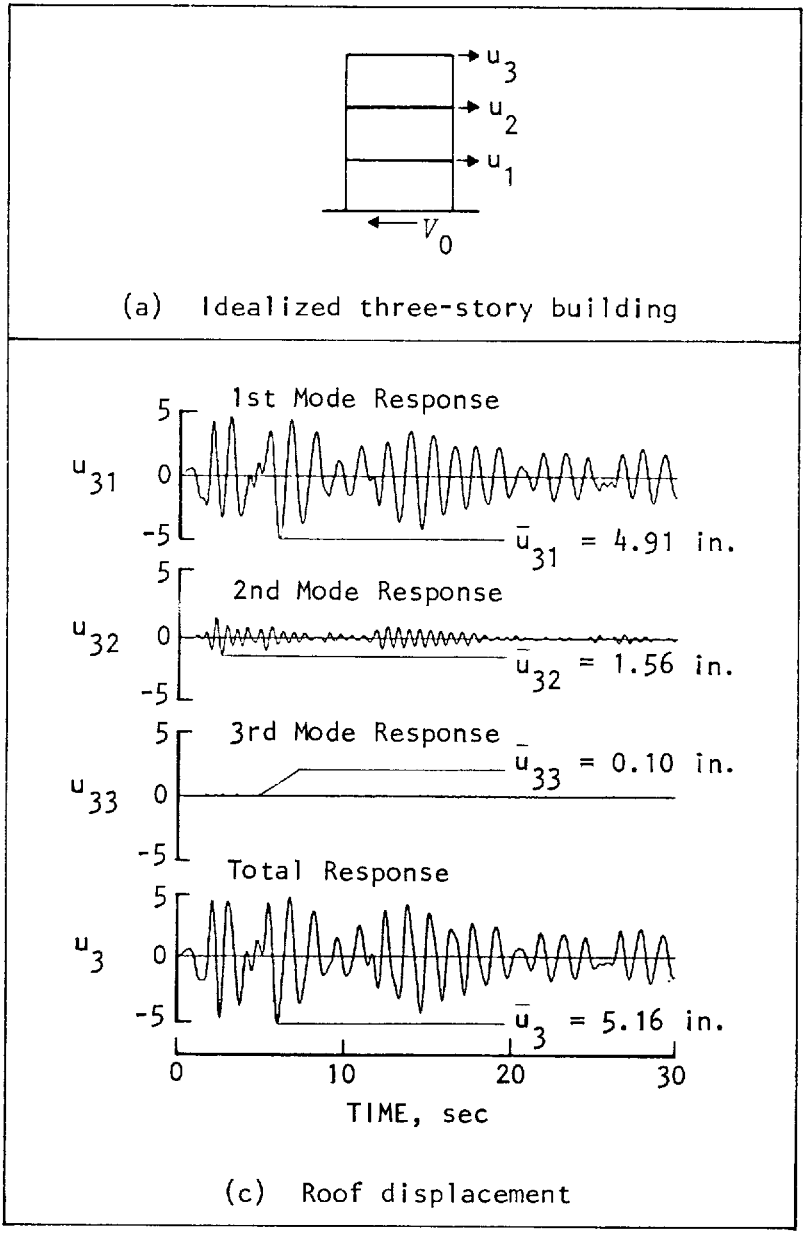
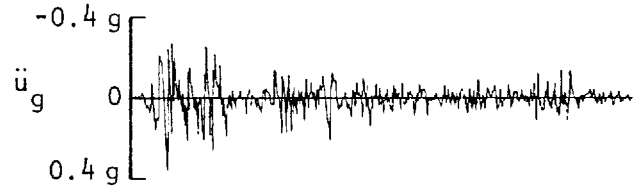
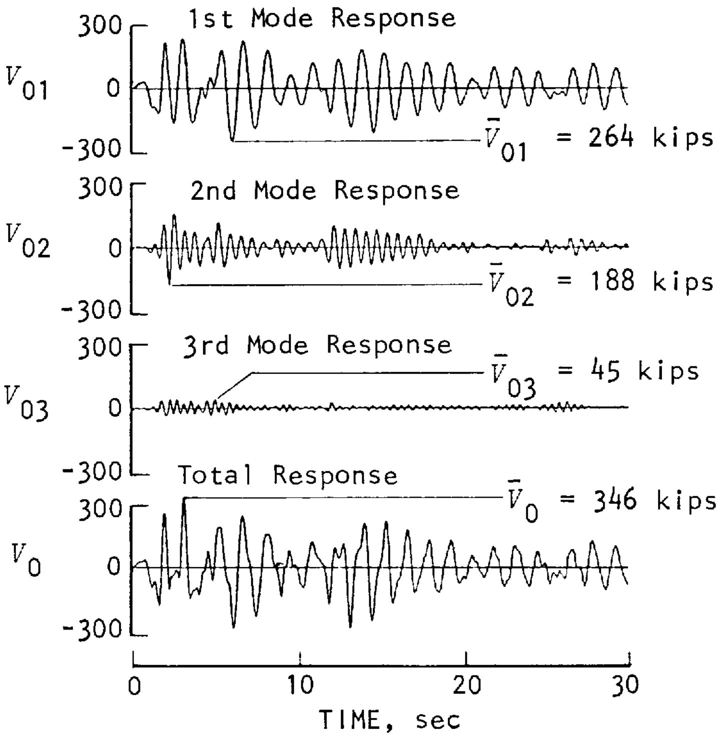


Figure 32. Earthquake response of a 3-story building



(b) El Centro ground motion — S00E Component
May 18, 1940



(d) Base shear

Figure 32. (concluded) Earthquake response of a 3-story building

Following the modal method, or mode superposition method, presented above, the response of an idealized three-story building of Fig. 32 to the ground motion shown in the same figure is determined with the aid of a computer program. A small portion of the response results, namely the roof displacement and base shear, is presented. Shown as a function of time during the earthquake are the contributions of the three natural modes of vibration to each response quantity along with the total responses.

A very attractive feature of the modal method is that independent analysis of SDF systems for each natural vibration can be made. Even more significant is the fact that, in general, the response need be determined only in the first few modes because response to earthquakes is primarily due to the lower modes of vibration (see for example the results of Fig. 32). Thus, only the first few natural frequencies and modes of vibration need be computed and the response computations need be repeated only for the first few vibration modes.

A complete modal analysis provides the history of response of a structure, including forces, displacements and deformations, to a specified ground acceleration history. From the response history the maximum response can readily be determined. Shown in Fig. 32 are the maximum values of the roof displacement $u_3(t)$ and base shear $V_0(t)$ for the three-story structure subjected to the selected ground motion.

Summary. The response of an idealized multistory building to earthquake ground motion can be computed by the following procedure:

1. Define the ground acceleration $\ddot{u}_g(t)$ by the numerical ordinates of the corrected accelerogram (Hudson, 1979)
2. Define structural properties
   (a) Compute mass and stiffness matrices $\mathbf{m}$ and $\mathbf{k}$
   (b) Estimate modal damping ratios ξ_n
3. Solve the eigen-problem of Eq. 52 to determine the natural frequencies ω_n (natural periods $T_n = 2\pi/\omega_n$) and modes ϕ_n of vibration

4. Compute the response in individual modes of vibration by repeating the following steps for each mode:
   (a) Compute the modal response $Y_n(t)$ by numerical evaluation of the Duhamel integral in Eq. 57 or by directly solving Eq. 56, the equation of motion for the SDF system with properties representative of the particular mode
   (b) Compute the floor displacements from Eq. 58
   (c) Compute story drifts from the floor displacements using Eq. 60
   (d) Compute equivalent lateral forces from Eq. 61
   (e) Compute internal forces—story shears and moments—by static analysis of the structure subjected to the equivalent lateral forces; in particular, the base shear and base moment can be calculated from Eq. 63
5. Determine the total value of response quantity $r(t)$ from Eq. 66 by combining the modal contributions $r_n(t)$ to the response quantity. In particular, floor displacements, equivalent lateral forces, base shear, and base moment can be determined from Eq. 65.

Response Spectrum Analysis

The complete response history is seldom needed for design of structures; the maximum values of response to the earthquake usually suffice. Because the response in each vibration mode can be modeled by the response of a SDF system, the maximum response in the mode can be directly computed from the earthquake response spectrum, and procedures for combining the modal maxima to obtain estimates (but not the *exact* value) of the maximum of total response are available.

Modal Response Maxima. The maximum response in the nth natural mode of vibration can be expressed in terms of S_{dn}, S_{vn}, and S_{an}, which are the ordinates of the deformation (or displacement), pseudo-velocity and pseudo-acceleration response spectra respectively, corresponding to the vibration period T_n (or vibration frequency ω_n) and damping ratio ξ_n of the mode. Based on the definitions of these response spectra and Eqs. 57, 59, 62 and 64, the maximum values of the various response quantities are given by Eqs. 67-72.

The maximum* modal displacement is

$$\bar{Y}_n = \frac{L_n}{M_n} S_{dn} ; \tag{67}$$

the maximum displacement at the jth floor is

$$\bar{u}_{jn} = \frac{L_n}{M_n} S_{dn}\phi_{jn}; \tag{68}$$

and the maximum deformation (or drift) in the jth story is

$$\bar{\Delta}_{jn} = \frac{L_n}{M_n} S_{dn}(\phi_{jn} - \phi_{j-1,n}) \tag{69}$$

The algebraic sign of L_n need not be retained in Eqs. 67-69. Furthermore, the algebraic sign of ϕ_{jn} (and $\phi_{j-1,n}$) can be dropped in Eq. 68, but it must be retained in Eq. 69 because the relative directions of displacements at floors above and below the story affect the story deformation.

The maximum value of the equivalent lateral force at the jth floor (Fig. 31) is

$$\bar{f}_{jn} = \frac{L_n}{M_n} S_{an} m_j \phi_{jn} \tag{70}$$

The maximum values of internal forces in the building—story shear and story moments—can be determined by static analysis of the building subjected to the maximum equivalent lateral forces $\bar{f}_{jn}$, j = 1, 2, . . . N. In applying these forces to the structure, the direction of forces is controlled by the algebraic sign of

* The maximum value without regard to algebraic sign of response $r(t)$ of a one-story structure was denoted by r_{max}. This notation becomes cumbersome in the equations describing the maximum responses of multistory buildings. Therefore, in the following equations the notation $\bar{r}$ is used instead of r_{max}. In some of these equations $\bar{r}$ also denotes the maximum value of $r(t)$ including an algebraic sign; this different usage should be apparent from its context.

ϕ_{jn}. Hence, the equivalent lateral forces for the fundamental mode will act in the same direction, but for the second and higher modes they will change direction as one moves up the structure. By static analysis, the maximum values of shear and moment at the base of the building are

$$\bar{V}_{0n} = \sum_{j=1}^{N} \bar{f}_{jn}$$

$$\bar{M}_{0n} = \sum_{j=1}^{N} h_j \bar{f}_{jn}$$

After substituting Eq. 70, these equations become

$$\bar{V}_{0n} = \frac{L_n^2}{M_n} S_{an} \tag{71}$$

$$\bar{M}_{0n} = \frac{L_n}{M_n} S_{an} \sum_{n=1}^{N} h_j m_j \phi_{jn} \tag{72}$$

In these equations, displacements are related to the deformation response spectrum and forces to the pseudo-acceleration response spectrum. However, S_{dn}, S_{vn}, and S_{an} are interrelated by the equations

$$S_{an} = \omega_n S_{vn} = \omega_n^2 S_{dn} \tag{73a}$$

or equivalently by the equations

$$S_{an} = \left(\frac{2\pi}{T_n}\right) S_{vn} = \left(\frac{2\pi}{T_n}\right)^2 S_{dn} \tag{73b}$$

Thus the displacements and deformations (Eqs. 67-69) can be expressed also in terms of S_{vn} or S_{an}, and the forces (Eqs. 70-72) in terms of S_{dn} or S_{vn}.

The form of Eqs. 68 and 70-72 is similar to that usually employed in standard references. Alternatively, these equations may be presented in a form that highlights the relationship between the modal analysis procedure and building code procedures, a topic discussed later. Equation 70 can be rewritten as

$$\bar{V}_{0n} = \frac{S_{an}}{g} W_n \tag{74}$$

in which g is the acceleration of gravity, and the effective weight W_n (or portion of the weight) of the building that participates in the nth mode of vibration is given by

$$W_n = \frac{\left[\sum_{j=1}^{N} w_j \phi_{jn}\right]^2}{\sum_{j=1}^{N} w_j \phi_{jn}^2} \tag{75}$$

where $w_j = m_j g$ is the weight at the jth floor level, ϕ_{jn} is the modal displacement of the jth floor, and N is the total number of floor levels. Comparison of Eqs. 75 and 33 indicates that the total weight of a one-story building is effective in producing the base shear, whereas only a portion of the weight of a multistory building is effective in producing the base shear due to the nth mode of vibration; the portion depends on the distribution of the weight over the height and the shape of the mode. Equation 75 will give values of W_n that are independent of how the modes are normalized. It can be analytically proven that the sum of the effective weights in all vibration modes of the building is equal to the total weight of the building; i.e.,

$$\sum_{n=1}^{N} W_n = \sum_{j=1}^{N} w_j$$

The maximum base moment due to the nth mode of vibration (Eq. 72) can be rewritten as

$$\bar{M}_{0n} = h_n \bar{V}_{0n} \tag{76}$$

where

$$h_n = \frac{\sum_{j=1}^{N} h_j w_j \phi_{jn}}{\sum_{j=1}^{N} w_j \phi_{jn}} \tag{77}$$

The base shear $\bar{V}_{0n}$ is equal to the resultant of the equivalent lateral forces $\bar{f}_{jn}$, and h_n may be interpreted as the height of the resultant force above the base. Because the equivalent lateral force is concentrated at the top of a one-story building (Fig. 18), the total height of the building is effective in producing the base moment (Eq. 35b). In a multistory building, however, the equivalent lateral forces are located at the various floors (Fig. 31) and the effective height h_n is less than the total height of the building; h_n depends on the distribution of the weight over the height and the shape of the mode. Eq. 77 will give values of h_n that are independent of how the modes are normalized.

For some of the vibration modes higher than the fundamental mode, the effective height computed from Eq. 77 may turn out to be negative. A negative value for h_n implies that at any instant of time, in particular at the time that modal responses attain their maxima, the base shear $V_{0n}(t)$ and base moment $M_{0n}(t)$ due to the nth vibration mode have opposite algebraic signs. If this distinction is of no concern, the negative sign in h_n may be ignored.

Starting with Eqs. 70 and 71, it can be shown that the lateral force $\bar{f}_{jn}$ at the jth floor in the nth mode of vibration is related to the base shear in the mode by the equation

$$\bar{f}_{jn} = \bar{V}_{0n} \frac{w_j \phi_{jn}}{\sum_{i=1}^{N} w_i \phi_{in}} \tag{78}$$

The floor displacements, or deflections, due to the lateral forces $\bar{f}_{jn}$ in the nth mode are proportional to the mode shape, and the two are related rather simply:

$$\bar{u}_{jn} = \frac{1}{\omega_n^2} \frac{g}{w_j} \bar{f}_{jn} \tag{79}$$

Combination of Modal Response Maxima. We have seen that a response $r(t)$ of the building to earthquake ground motion is the superposition of the contributions $r_n(t)$ of the natural modes of vibration to the response quantity, and the maximum response in individual modes of vibration can be determined directly from the earthquake response spectrum. Because, in general, the modal maxima $\bar{r}_n$ do not occur at the same time, they cannot be directly superimposed to obtain $\bar{r}$, the maximum of the combined response (cf. Fig. 32). This is apparent from the earthquake response of a three-story building presented in Fig. 32, wherein the maximum base shear due to each mode occurs at different time instants during the earthquake and the maximum of the total base shear occurs at yet a different time. The direct superposition of modal maxima, however, provides an upper bound to the maximum of total response:

$$\bar{r} \leq \sum_{n=1}^{N} \bar{r}_n \tag{80}$$

This estimate of total response is often too conservative and is therefore not popular in design applications. More commonly, the total response is estimated by combining the modal maxima according to the root-sum-square formula:

$$\bar{r} \approx \sqrt{\sum \bar{r}_n^2} \tag{81}$$

in which only the lower modes that contribute significantly to the total response need to be included in the summation. The root-sum-square formula is not always a conservative predictor of the earthquake response. However, it generally provides a good estimate of maximum response for systems with well separated natural periods of vibration, a property typically valid for the building idealization considered in the preceding sections, wherein only the lateral motion in one plane is considered. Improved formulas for combining maximum of modal responses are available for systems lacking this property (Newmark and Rosenblueth, 1971—Chapter 10).

Summary. The maximum response of an idealized multistory building to earthquake ground motion can be estimated from the response spectrum for the ground motion by the following procedure:

1. Determine the response spectrum for the ground motion if not already available
2. Define structural properties
   (a) Compute mass and stiffness matrices **m** and **k**
   (b) Estimate modal damping ratios ξ_n
3. Solve the eigen-problem of Eq. 52 to determine the few lower natural frequencies ω_n (natural periods $T_n = 2\pi/\omega_n$) and the modes ϕ_n of vibration
4. Compute the maximum response in individual modes of vibration by repeating the following steps for the lower modes of vibration:
   (a) Corresponding to period T_n and damping ratio ξ_n, read the ordinates S_{dn} and S_{an} of the deformation (or displacement) and pseudo-acceleration response spectra of the earthquake ground motion
   (b) Compute the floor displacements from Eq. 68
   (c) Compute story drifts from the floor displacements using Eq. 69
   (d) Compute equivalent lateral forces from Eq. 70 (or from Eq. 78 after computation of base shear by Eq. 74)
   (e) Compute internal forces (story shears and story moments) by static analysis of the structure subjected to the equivalent lateral forces; in particular, the base shear can be computed from Eq. 71 (or Eq. 74) and base moment from Eq. 72 (or Eq. 76)
5. Determine an estimate of the maximum $\bar{r}$ of any response (displacement of a floor, deformation in a story, shear or moment in a story, etc.) by combining the modal maxima $\bar{r}_n$ for the response quantity in accordance with Eq. 81.

Because the modal equations of motion need not be solved in obtaining estimates of maximum response directly from the earthquake response spectrum, the computational effort required in this approach is only a fraction of that required to obtain the complete response history.

APPROXIMATE ANALYSIS OF EARTHQUAKE RESPONSE

The distribution of lateral displacements and forces over the height of a building can be quite complex because a number of natural modes of vibration may contribute significantly to these responses. The contributions of the various vibration modes to the lateral displacements and forces and to the base shear depend on a number of factors, including shape of the earthquake response spectrum and natural vibration periods and mode shapes, which in turn depend on the mass and stiffness properties of the building. However, the contributions of the first, or fundamental, mode of vibration to these responses are generally larger than those of any other vibration mode. Thus, the best one-mode approximation to the maximum response during the earthquake is generally provided by Eqs. 74, 78 and 79, specialized for $n = 1$.

The resulting equations are useful in approximate analysis of forces and deformations for preliminary design of a building. They would be especially convenient if a simple procedure were available to compute the fundamental frequency and mode shape of vibration, even if only approximately. Such a procedure based on the principles of conservation of energy in free vibration of undamped structures was developed in the 19th century by Rayleigh and is presented in standard references (Thomson, 1965—Chapter 1; Newmark and Rosenblueth, 1971—Chapter 4; Clough and Penzien, 1975—Chapter 9).

If we assume that the vector ψ is an approximation to the fundamental natural mode of vibration ϕ_1, the following formula for estimating the fundamental frequency of vibration can be derived by Rayleigh's method.

$$\omega^2 = \frac{\psi^T \mathbf{k} \psi}{\psi^T \mathbf{m} \psi} \tag{82}$$

If the assumed shape coincides with the fundamental mode, the frequency obtained from Eq. 82 will coincide with the fundamental natural frequency. The effectiveness of Rayleigh's method lies in the fact that a useful estimate of the fundamental frequency can be obtained from even a relatively crude assumption for the fundamental mode shape.

Rather than directly estimating the shape vector ψ, it is better to define ψ as the deflected shape of the structure due to some selected set of lateral forces f_j ($j = 1, 2, \ldots N$) at the floor levels. If the resulting static deflections are u_j, then deflected shape ψ is defined by ψ_j equal to the ratio of u_j to the deflection at some reference location, say the top of the building; i.e. $\psi_j = u_j/u_N$, $j = 1, 2, \ldots N$.

The formula for estimating the fundamental frequency can then be expressed as

$$\omega^2 = \frac{g}{u_N} \frac{\sum_{j=1}^{N} f_j \psi_j}{\sum_{j=1}^{N} w_j \psi_j^2} \tag{83}$$

or directly in terms of the deflections as

$$\omega^2 = g \frac{\sum_{j=1}^{N} f_j u_j}{\sum_{j=1}^{N} w_j u_j^2} \tag{84}$$

Any reasonable distribution for f_j may be selected; for example $f_j = w_j$, the weight at floor j. The lateral forces computed from building code formulas, presented later, are especially convenient in application. These forces and resulting deflections would have been computed in the preliminary design of the building. They are then substituted in Eq. 84 to estimate the fundamental frequency.

The maximum earthquake response of a building can be estimated from Eqs. 74, 78 and 79, specialized for $n = 1$, wherein the fundamental frequency ω_1 is replaced by ω estimated from Eq. 84 and the fundamental mode shape ϕ_1 replaced by ψ, the estimated mode shape. Thus the base shear is

$$\bar{V}_0 = \frac{S_a}{g} W \tag{85}$$

where

$$W = \frac{\left[\sum_{j=1}^{N} w_j \psi_j\right]^2}{\sum_{j=1}^{N} w_j \psi_j^2} \tag{86}$$

The lateral forces are

$$\bar{f}_j = V_0 \frac{w_j \psi_j}{\sum_{i=1}^{N} w_i \psi_i} \tag{87}$$

and the floor displacements are

$$\bar{u}_j = \frac{1}{\omega^2} \frac{g}{w_j} \bar{f}_j \tag{88}$$

where S_a is the ordinate, corresponding to estimated frequency ω (or period $T = 2\pi/\omega$) and assumed damping ratio ξ, of the pseudo-acceleration response spectrum for the earthquake ground motion.

Equations 85, 87, and 88 provide approximate results for the earthquake response of a building in its fundamental natural mode of vibration; the approximation arises from use of estimated instead of *exact* values of frequency and mode shape. The contributions of the higher modes of vibration to building response are neglected in these equations.

The forces and deformations induced by earthquake ground motion in a tower or smokestack, idealized as a distributed mass system, can also be estimated by the procedure summarized above. For this purpose, Eqs. 83, 84, 86, 87, and 88 need to be generalized for distributed mass systems (Clough and Penzien, 1975—Chapters 9 and 27).

REFINED IDEALIZATION OF MULTISTORY BUILDINGS

Procedures were presented in the preceding sections for earthquake response analysis of the simplest idealization of multistory buildings. Rotation of beam-column joints and axial deformations in columns and in the floor systems were not permitted in this so-called shear building model; and only lateral motion in the direction of the ground motion was considered. The stiffness matrix for such an idealized structure was readily formulated from the story stiffnesses. As a result, the shear-building idealization was especially convenient in presenting an introduction to the dynamics of multi-degree-of-freedom systems. However, refined idealizations would generally be necessary to accurately determine the dynamic response of buildings.

Joint Rotations and Column Axial-Deformations

The effect of joint rotations and axial deformations in columns can be included in the analysis by considering two degrees-of-freedom (DF), joint rotation and vertical displacement, at each beam-column joint; and one DF, the lateral displacement, at each floor (Fig. 33). The floor systems are usually very stiff in their own plane so that it is reasonable to assume that all joints at any floor level undergo the same lateral displacement. Such a refined idealization for a 20-story building frame with five column lines would include 220 DF in contrast to 20 DF in a shear-building idealization. The effects of joint rotations on the first two vibration modes of a 5-story building are shown in Fig. 34.

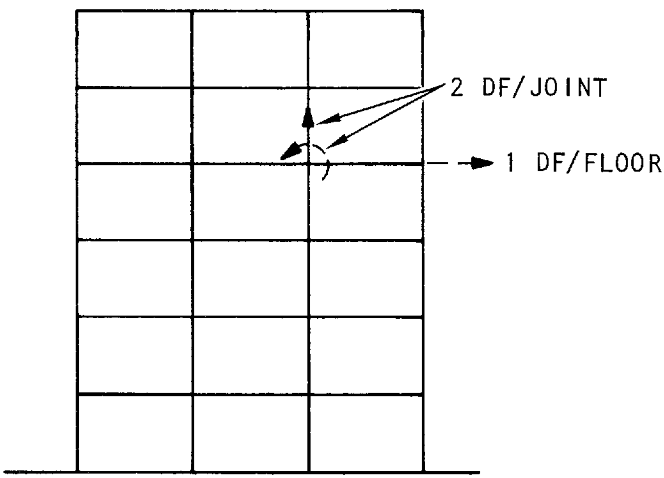
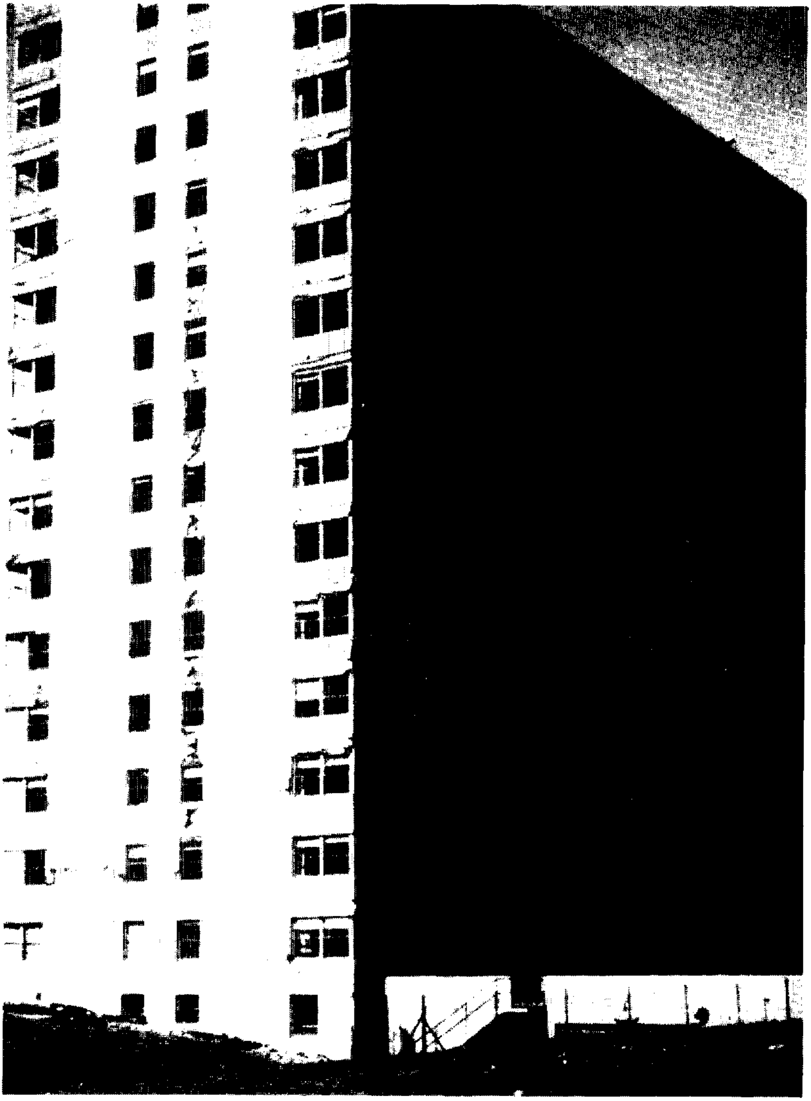


Figure 33. Degrees of freedom (DF) in a refined idealization of multistory buildings



Courtesy of V. V. Bertero

The 1200 L Street apartment building in Anchorage was damaged during the Alaska earthquake of 27 March 1964. The failure of the interior spandrel beams and also the failure of one of the shear walls can be seen on the south side of this 14-story apartment building. Extensive cracking of the walls was repaired with gunite concrete and with epoxy cement. The building was reoccupied a few years after the earthquake.

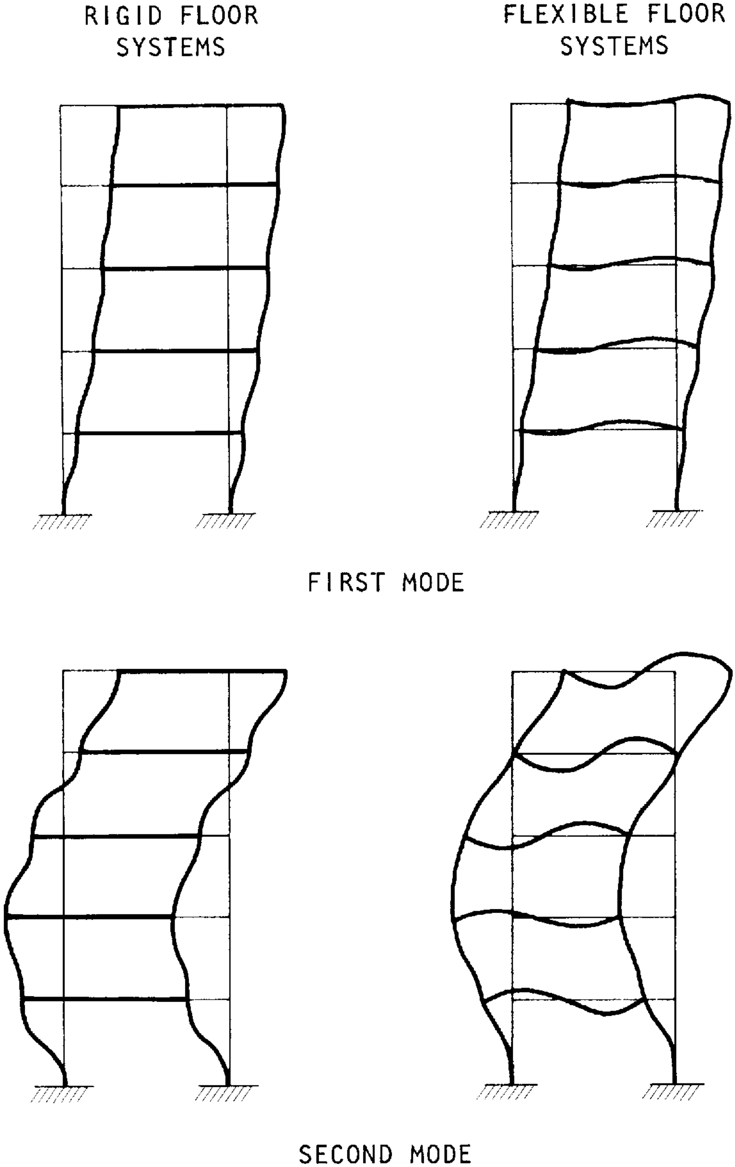


Figure 34. Influence of joint rotations on natural modes of vibration

Starting with the stiffness matrices of structural elements, the complete stiffness matrix for such a refined idealization can be determined by matrix structural analysis procedures. Because the effective earthquake forces act only in the lateral direction and the inertia effects associated with vertical and rotational motions of the joints are generally unimportant in the dynamic response of multistory buildings, the corresponding DF need not be included in the equations of motion. The vertical and rotational DF can be eliminated from the complete stiffness matrix, resulting in the *lateral stiffness matrix* of the building frame having one DF per story. This stiffness matrix is of the same size as that of a shear-building model but it contains the effects of vertical and rotational DF; in general, it is a full matrix, in contrast to the banded stiffness matrix of the shear-building idealization. When lateral stiffness matrices have been derived for each frame acting parallel with a given axis of the building, they may be superposed directly to obtain the total stiffness matrix $\tilde{\mathbf{k}}$ for the building in this direction.

The equations of motion for a multistory building including the effects of vertical and rotational DF can also be expressed as Eq. 49 provided $\mathbf{k}$ is replaced by $\tilde{\mathbf{k}}$. With this change, the procedures summarized in the section "Modal Analysis of Earthquake Response" are also applicable to earthquake response analysis of refined idealizations of multistory buildings, except that the internal forces of computational step 4c in the summary are now the forces in structural elements, beams and columns, instead of story shears and moments.

Lateral-Torsional Coupling

In dynamic analysis of building response to earthquake ground motion, it is usual to consider the above mentioned planar models of the structure in each of two orthogonal directions and to independently analyze the response of each model to the in-plane horizontal component of ground motion. Analysis on this basis is strictly valid only for buildings with coincident centers of mass and resistance.

The lateral, or translational, and torsional motions of the structure are coupled if the centers of mass and resistance do not coincide. The usual approach may be reasonable even for such torsionally-coupled buildings if the eccentricities of the centers of story resistance with respect to the centers of floor mass are small and the natural frequencies of the lower vibration modes are well separated. It is obvious that if the eccentricities are large, lateral and torsional motions will be strongly coupled. Less obvious perhaps but clearly displayed by forced vibration tests (Jennings, Matthiesen and Hoerner, 1972) is the strong coupling between lateral and torsional motions of buildings with close natural frequencies and nearly coincident centers of mass and resistance.

For such buildings, independent analyses for the two lateral directions may not suffice, and at least three DF per floor— two translational motions and one torsional—should be included in the idealization. The three DF for a one-story system are shown in Fig. 35. The modal method described earlier, with appropriate generalization of the concepts involved, can be applied to analysis of such buildings (Kan and Chopra, 1977). The mass and stiffness matrices now include the translational and torsional DF of each floor. Because most of the natural modes of vibration will show a combination of translational and torsional motions, it is necessary in determining the modal response to account for the facts that a given mode might be excited by both horizontal components of ground motion and that modes that are primarily torsional can be excited by translational components of ground motion. Because the natural vibration frequencies of a building with coupled lateral-torsional motions can be close to each other, the modal maxima should not be combined in accordance with the root-sum-square formula (Eq. 81); instead a more general formula should be employed (Newmark and Rosenblueth, 1971—Chapter 10).

Torsional motions may occur even in buildings with coincident centers of mass and resistance due to various causes, including wave propagation effects of horizontal ground motion and rotational component of ground motions about a vertical axis. Buildings with nominally coincident centers of mass and resistance may also respond in torsion because of various factors that cannot be explicitly considered in the analysis. These factors include unforeseeable differences between computed and actual stiffnesses, yield strengths, and dead-load masses; unforesee-

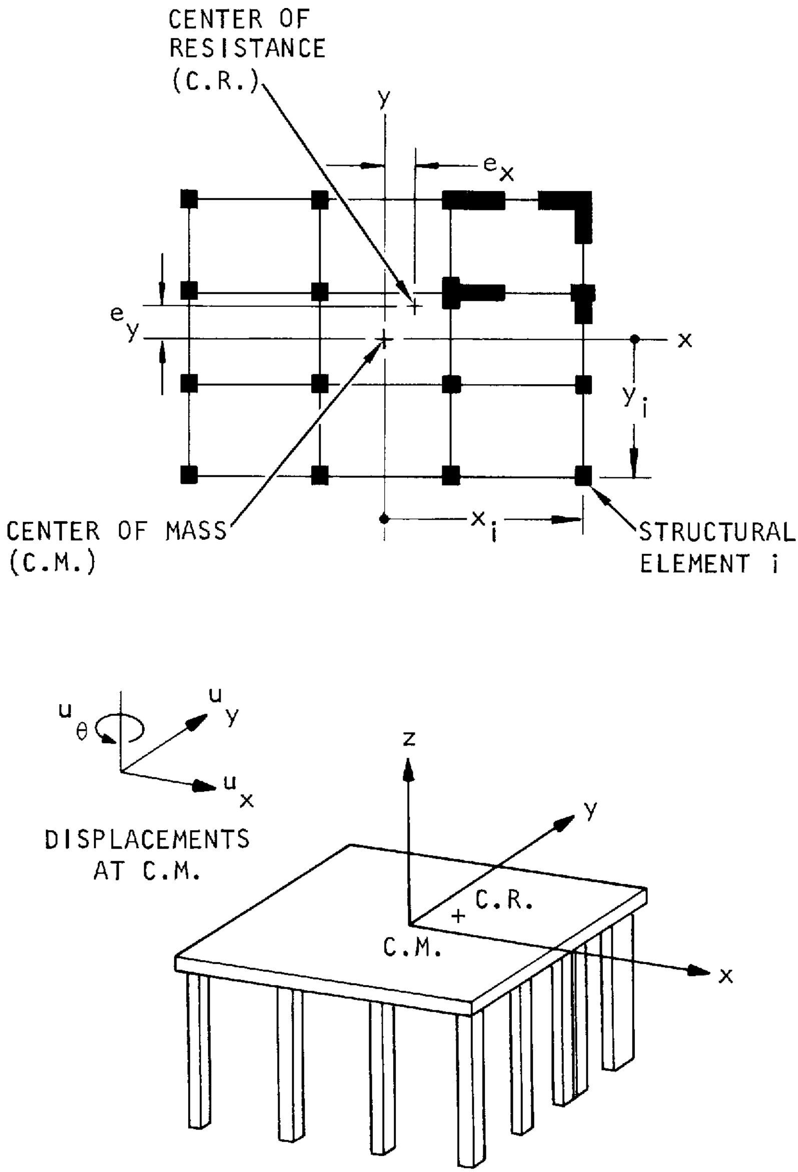


Figure 35. One-story building with eccentric centers of mass and resistance

able, unfavorable distributions of live load masses; and differences in coupling of the structural foundation with the supporting soil or rock. The so-called *accidental torsion* in building codes is intended to cover the effects of these causes and factors in design of buildings.

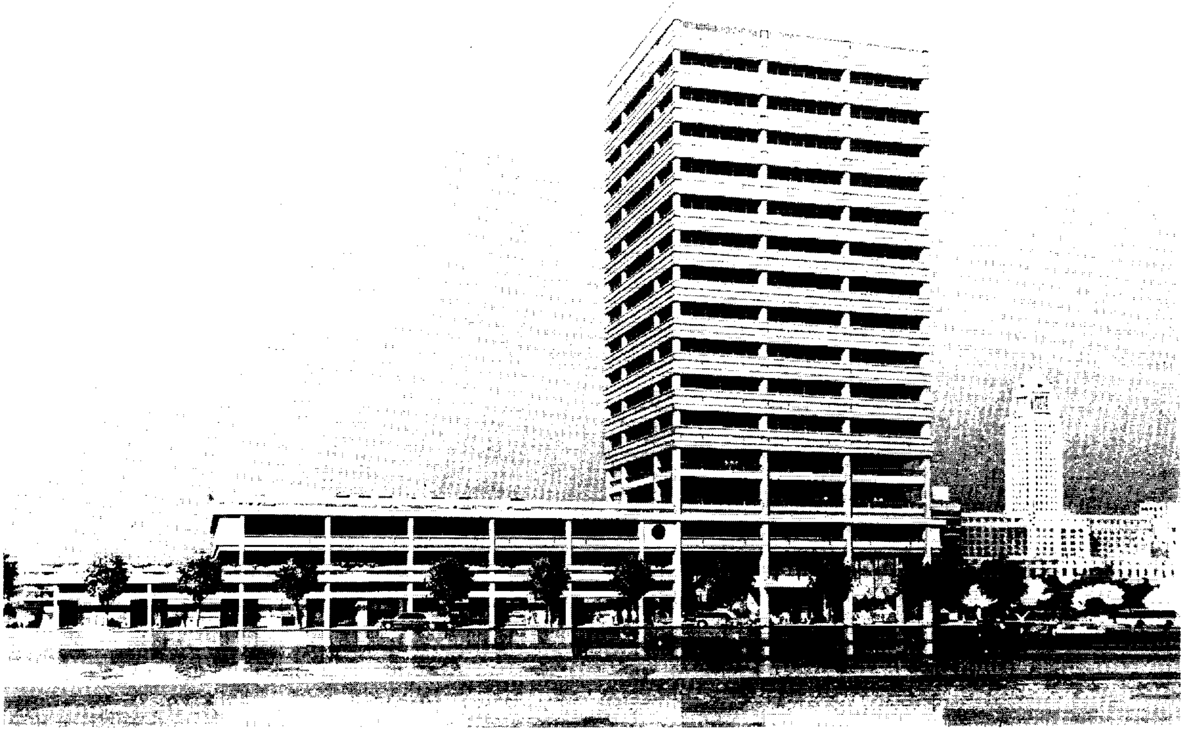
Additional Comment

In its most general form, the modal method is applicable to linear response analysis of arbitrary three-dimensional structural systems. Effects of floor-diaphragm flexibility, beam-column joint flexibility, etc., can be considered by including the appropriate DF in the building idealization.

BUILDING IDEALIZATION AND RECORDED EARTHQUAKE RESPONSE

With the development of earthquake response analysis procedures and the availability of modern digital computers, it is now possible to determine the response of even a refined idealization (mathematical model) of any building to prescribed ground motion. How well the computed response agrees with the response of the building during an actual earthquake depends primarily on the quality of the mathematical model. Responses of modern multistory buildings to recent earthquake motions have been recorded, including responses of more than 50 buildings recorded during the 1971 San Fernando earthquake. Most of these buildings were located in the city of Los Angeles and had accelerographs in the basement, at mid-height, and on the roof in accordance with building code requirements. These recorded motions permitted studies of modeling and analysis procedures, of the relationships between building response and damage, and of other important aspects of the response of structures to earthquake motions.

A brief description of six multistory buildings and their measured responses during the San Fernando earthquake is available in a convenient form (Foutch, Housner and Jennings, 1975). Dynamic properties, such as natural periods of vibration



Courtesy of G. W. Housner

Figure 36. Kajima International Building

and damping ratios, of these six buildings and others were determined from the measured responses during the earthquake (McVerry, 1979). As a sample of the results contained in these two references, the measured responses of one of these buildings (Kajima International building) is presented here, along with its dynamic properties estimated from the responses. The description of the building in the following two paragraphs is taken from Foutch, Housner and Jennings (1975).

The Kajima International building, designed in 1966, is located in downtown Los Angeles approximately 21 miles south of the center of the San Fernando earthquake. The 15-story office tower is a steel frame building that measures 66 x 96 feet in plan and stands 202 feet above grade. The basement, 1st and 2nd floor areas are used primarily for retail and banking space. The entire 15th floor and the portion of the roof enclosed by the penthouse contain mechanical equipment. Fig. 36 is a picture of the northwest elevation of the Kajima International building. The building sustained no significant damage during the San Fernando earthquake.

A three-dimensional moment-resisting steel frame provides the resistance to both lateral and vertical loads. Four moment-resisting frames are provided in both the transverse and longitudinal directions. Lightweight reinforced concrete floor slabs act as rigid diaphragms in the horizontal direction. Concrete encasement of the exterior columns of the frame, designed for fire protection, provide additional stiffness. Precast concrete spandrels, 6 feet deep, are used as part of the exterior facia of the building, and they also provide additional stiffness at low levels of vibration. The foundation system consists of spread footings combined in pairs. Although a seismic gap was provided between the main tower and an adjacent three-story parking facility, apparently impacting occurred during the earthquake. Figs. 37a and 37b are schematic drawings of a transverse section and a floor plan of the building.

Strong-motion accelerographs were installed on the basement, 8th floor, and roof of the building. Three components of acceleration (two horizontal and one vertical) were recorded by these accelerographs. Plots of the N36E component of these accelerations are presented in Fig. 38 and the N54W component in Fig.

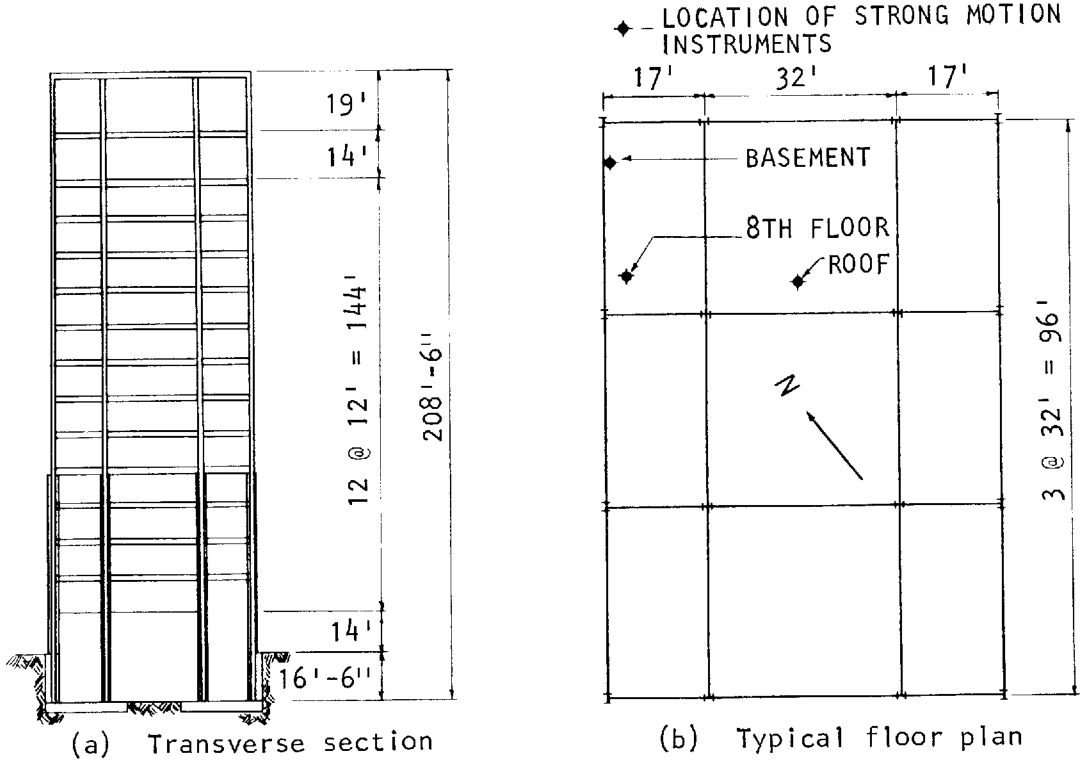


Figure 37. Schematic of Kajima International Building structural system (after Foutch, Housner, and Jennings, 1975)

39. These accelerations represent the total motion of the building, which is composed of the relative motions of the building with respect to the ground plus the motion of the ground. The total displacement of the building and the displacement of the ground were obtained by twice-integrating the recorded accelerations. The horizontal components of the relative displacements of the roof and 8th floor, determined by subtracting the ground displacement from the total displacement at those floors of the building, are presented in Figs. 40 and 41.

It can be seen that the horizontal accelerations at the upper levels of the buildings are larger and their time-variation is different from the ground (basement) accelerations. It is clearly seen in the horizontal displacement plots that the relative displacements at the floors are primarily due to the fundamental mode of vibration. The fundamental period in both directions was approximately 2.9 seconds during the earthquake.

Accurate values of the first three natural periods of vibration in the two horizontal directions determined by system identification procedures are presented in Table 1. Also included is the fundamental vibration period in each horizontal direction determined from ambient tests before and after the earthquake. The period of vibration 2.84 seconds in the N36E direction, determined from the earthquake response, is much longer than the period of 1.32 seconds found in the pre-earthquake ambient vibration study. The loss of stiffness indicated by this period change is believed to be the result of cracking and other types of degradation of the so-called non-structural elements during the higher level earthquake responses.

A nonlinear structural idealization (mathematical model) having stiffness properties varying with deformation level would be necessary to reproduce this period change and to describe the behavior of a building through the complete range of deformation amplitudes. However, if the building experiences only little or no structural damage, good estimates of its response during the earthquake can usually be computed from an equivalent linear model. Only those structural and non-structural elements that are effective at the amplitudes of motion expected during the earthquake should be included in this structural idealiza-

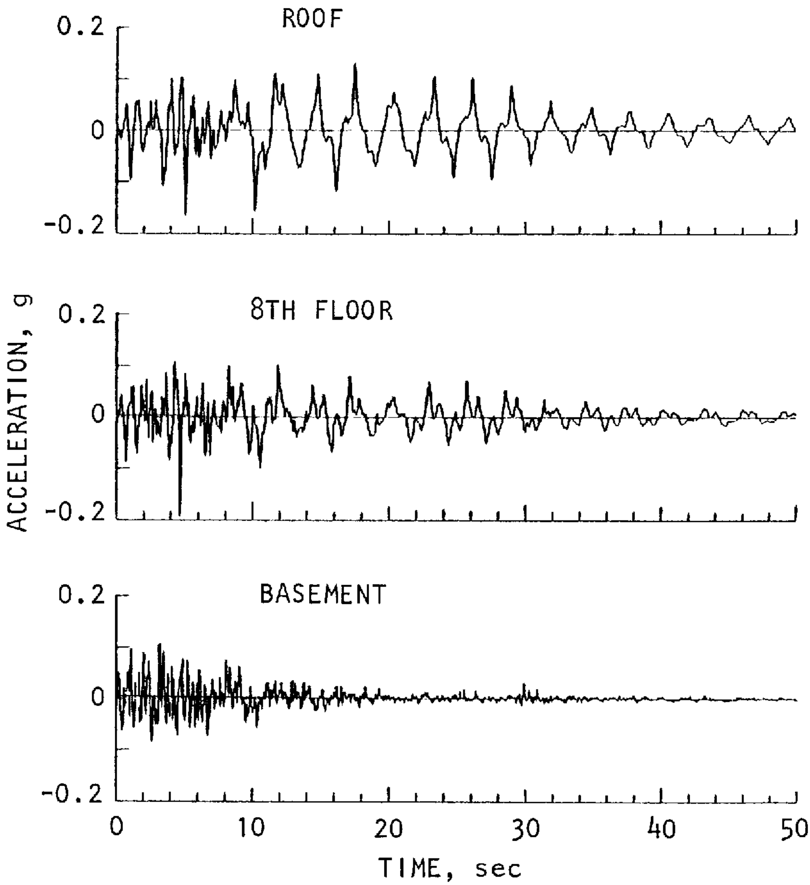


Figure 38. Accelerations (N36E component) recorded at basement, 8th floor, and roof of Kajima International Building during the San Fernando earthquake (adapted from Foutch, Housner, and Jennings, 1975)

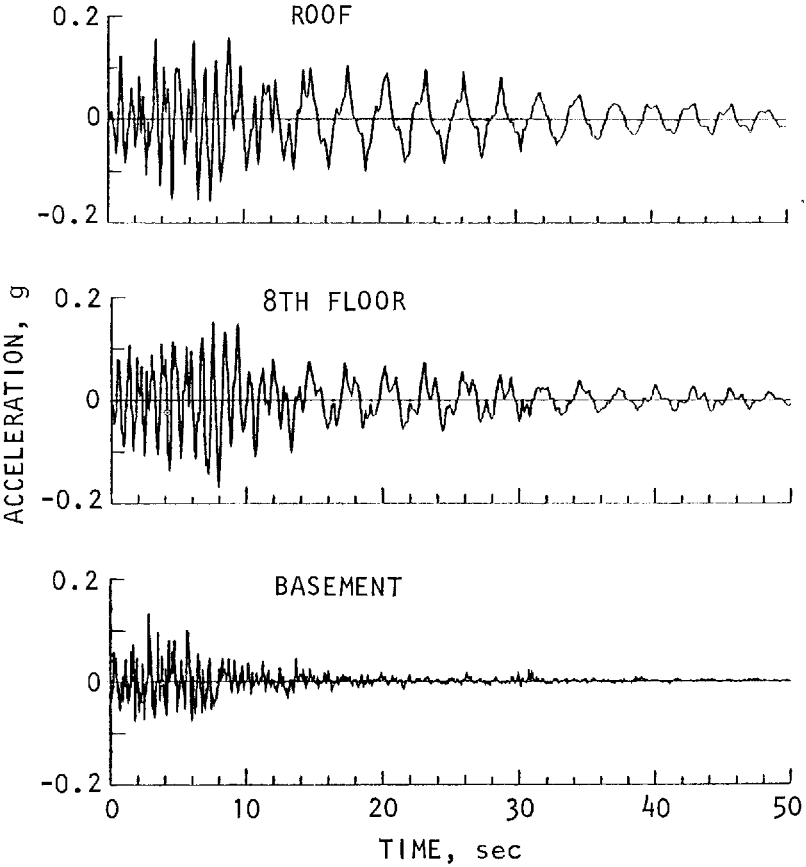


Figure 39. Accelerations (N54W component) recorded at basement, 8th floor, and roof of Kajima International Building during the San Fernando earthquake (adapted from Foutch, Housner, and Jennings, 1975)

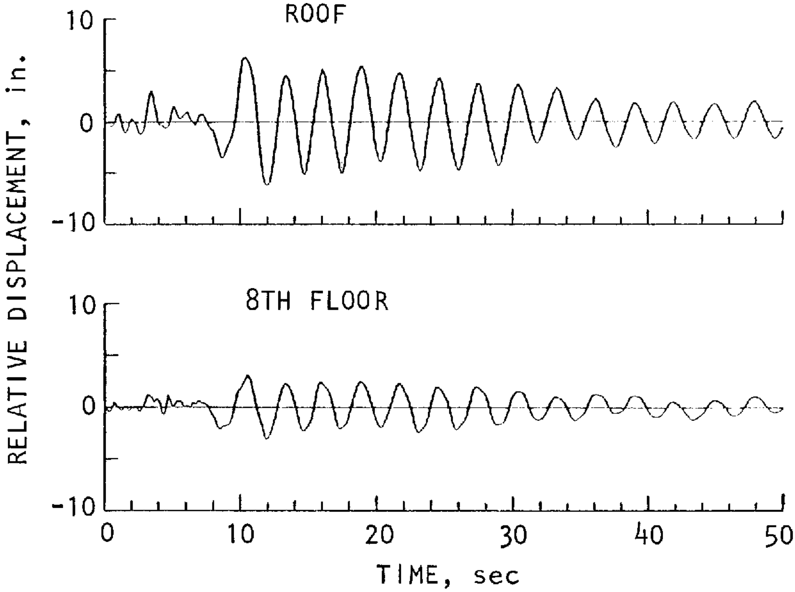


Figure 40. Displacements (N36E component) relative to ground, computed from accelerations recorded at Kajima International Building during the San Fernando earthquake (adapted from Foutch, Housner, and Jennings, 1975)

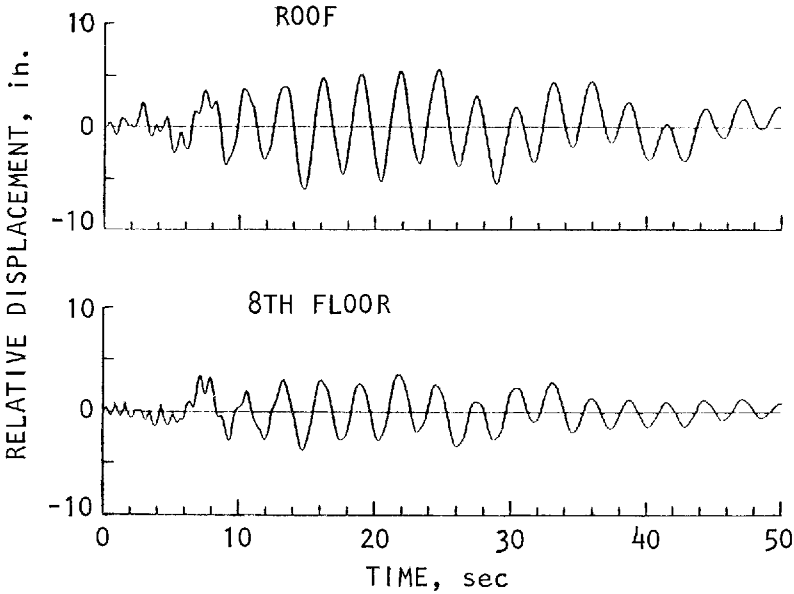


Figure 41. Displacements (N54W component) relative to ground, computed from accelerations recorded at Kajima International Building during the San Fernando earthquake (adapted from Foutch, Housner, and Jennings, 1975)

TABLE 1. MODAL PROPERTIES OF THE KAJIMA INTERNATIONAL BUILDING

(Adapted from McVerry, 1979)

Source	Vibration Periods (seconds)			Damping ratios (%)		
	T_1	T_2	T_3	ξ_1	ξ_2	ξ_3
N36°E DIRECTION						
Ambient Tests						
Pre-earthquake	1.32					
Post-earthquake	2.10					
San Fernando Earthquake	2.84	0.89	0.57	3.8	8.1	
N54°W DIRECTION						
Ambient Tests						
Pre-earthquake	1.88					
Post-earthquake	2.15					
San Fernando Earthquake	2.77	0.88	0.51	3.6	5.6	4.7

tion. The fundamental period after the earthquake was measured from an ambient test to be 2.10 seconds, which indicates that all of the stiffness lost during the earthquake was not recovered. A similar phenomenon was observed for the N54W motions.

The damping ratios determined from the measured earthquake responses by system identification procedures were 4 to 8 percent for the first two modes in the N36E direction and 3 to 5 percent for the N54W modes.

Using the periods and damping ratios determined from measured responses during the San Fernando earthquake and system identification procedures, the response of Kajima International building to the basement motion as input can be calculated by modal analysis. The agreement between computed and measured responses at the 8th floor and roof was essentially perfect (McVerry, 1979).

The usual situation, however, is different in that these vibration periods and mode shapes are computed from an idealization of the structure. For example, the fundamental vibration periods of the bare frame of the Kajima International building were computed to be 3.31 seconds and 3.19 seconds in the N36E and N54W directions, respectively (McVerry, 1979). These periods are longer than those obtained from measured earthquake response, indicating that the structural and non-structural elements that contribute to the mass and stiffness of the structure at the amplitudes of motion anticipated during an earthquake should be included in the structural idealization. Similarly, selection of damping values for a building should be based on available data from recorded earthquake response of similar buildings. Such data for several buildings is summarized in a convenient form by McVerry (1979).

Numerous analyses of recorded motions during earthquakes have shown that the use of modal analysis with viscously-damped single-degree-of-freedom systems describing the response of vibration modes is an accurate approximation for analysis of linear response. The accuracy of the results depends on how well the computed vibration periods and mode shapes and estimated damping ratios represent the properties of the structure during the earthquake.

3. Dynamic Analysis and Building Code Procedures

EARTHQUAKE FORCES IN THE UNIFORM BUILDING CODE

The key elements of the earthquake regulations in the Uniform Building Code (UBC) for design of buildings are the formulas for base shear and distribution of lateral forces over the height of the building (UBC, 1979). The design base shear is to be determined from the formula:

$$V = Z\ I\ K\ C\ S\ W \tag{89}$$

where

Z = Numerical coefficient depending on the seismic zone of the country; for seismic zones 1, 2, 3, and 4, Z = 3/16, 3/8, 3/4, and 1, respectively.

I = Occupancy Importance Factor

$$= \begin{cases} 1.5 \text{ for essential facilities} \\ 1.25 \text{ for any building used primarily for assembly of} \\ \quad \text{more than 300 persons in one room} \\ 1.0 \text{ for all other facilities} \end{cases}$$

K = 0.67 to 1.33, depending on the structural system

$C = \dfrac{1}{15\sqrt{T}}$, but need not exceed 0.12

T = Fundamental natural period of vibration of the building in secs

S = 1.0 to 1.5, depending on the values of T and T_s, the characteristic period of the site

W = Total dead load and appropriate portions of live load and snow load

The distribution of lateral forces over the height of the building is to be determined from the base shear in accordance with Eqs. 90–92. The base shear is the summation of the lateral forces:

$$V = F_t + \sum_{j=1}^{N} F_j \tag{90}$$

The additional force F_t at the top of the building depends on the vibration period T as follows:

$$F_t = \begin{cases} 0 & T \leq 0.7 \\ 0.07\ T\ V & 0.7 < T < 3.6 \\ 0.25\ V & T \geq 3.6 \end{cases} \tag{91}$$

The remaining portion of the base shear shall be distributed over the height of the structure to obtain the lateral forces at the various floor levels; the force at the jth floor is

$$F_j = (V - F_t) \frac{w_j h_j}{\sum_{i=1}^{N} w_i h_i} \tag{92}$$

where w_i is the weight at the ith floor and h_i is the height of the ith floor above the base.

The design shears and moments for the various stories of the building are determined from static analysis of the building subjected to the lateral forces computed from Eqs. 89 to 92.

Formerly many building codes and design recommendations, including the 1967 edition of the UBC, allowed large reduction in story moments relative to their values computed from lateral forces by statics. These reductions appeared to be excessive in light of damage to buildings during the 1967 Caracas earthquake, where a number of column failures were primarily due to the effects of overturning moment. In the 1973 and subsequent editions of the UBC, no reduction was allowed. However, no reduction at all is unjustified in light of the results of dynamic analysis. Moderate reduction, up to 20%, was therefore permitted in the Applied Technology Council recommendations (ATC, 1978).

COMPARISON OF CODE FORCES AND DYNAMIC ANALYSIS

The earthquake forces specified in the UBC are compared in this section with the equivalent lateral forces from modal analysis of earthquake response of buildings. In order to facilitate this comparison, the UBC formulas are rewritten in the notation employed in presenting modal analysis procedures. Thus the base shear of Eq. 89 becomes

$$\bar{V}_0 = \mathrm{Z\ I\ K\ C\ S\ W} \tag{93}$$

where, for purposes of the comparison that follows, it is appropriate to assume Z = 1, I = 1, and S = 1 leading to

$$\bar{V}_0 = \mathrm{K\ C\ W} \tag{94}$$

With change of notation, the lateral force at the jth floor (Eq. 92) can be expressed as

$$\bar{\mathrm{f}}_j = (\bar{V}_0 - \mathrm{f}_t)\frac{\mathrm{w}_j\mathrm{h}_j}{\sum_{i=1}^{N} \mathrm{w}_i\mathrm{h}_i} \tag{95}$$

and the additional force f_t at the top of the building (Eq. 91) as

$$\mathrm{f}_t = \begin{cases} 0 & \mathrm{T} \leq 0.7 \\ 0.07\ \mathrm{T}\ \bar{V}_0 & 0.7 < \mathrm{T} < 3.6 \\ 0.25\ \bar{V}_0 & \mathrm{T} \geq 3.6 \end{cases} \tag{96}$$

The lateral force at the top of the building is the sum of the force for j = N from Eq. 95 and f_t of Eq. 96. The lateral forces and base shear are shown in Fig. 42.

Considering only the dynamic response in the fundamental mode of vibration, the maximum base shear and equivalent lat-

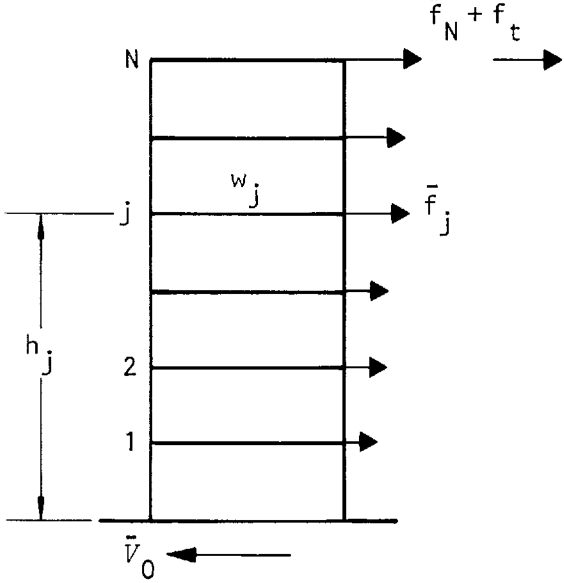


Figure 42. Earthquake forces in the Uniform Building Code for design of buildings

eral forces for a building are provided by Eqs. 74, 75, and 78 specialized for n = 1, resulting in the following equations:

$$\bar{V}_{01} = \frac{S_{a1}}{g} W_1 \tag{97}$$

$$W_1 = \frac{\left[\sum_{j=1}^{N} w_j \phi_{j1}\right]^2}{\sum_{j=1}^{N} w_j \phi_{j1}^2} \tag{98}$$

$$\bar{f}_{j1} = \bar{V}_{01} \frac{w_j \phi_{j1}}{\sum_{i=1}^{N} w_i \phi_{i1}} \tag{99}$$

Equations 95 and 99 will become identical if $\bar{V}_{01}$, the contribution of the fundamental mode to the base shear, is replaced by $\bar{V}_0 - f_t$, the portion of the base shear in the code formula (Eq. 95), and if the floor displacements in the fundamental vibration mode vary linearly with height (Fig. 43). Thus, except for the additional force f_t assigned to the top of the building, the lateral forces in the code formula are distributed under the assumption of linearly varying floor displacements in the fundamental mode shape.

Assignment of an additional force f_t at the top of the building is intended by the code to account for the contributions of the higher vibration modes to building response. The influence of vibration modes higher than the fundamental mode is small in earthquake response of short-period buildings, and the fundamental vibration mode of many buildings with regular distribution of stiffness and mass over height departs little from a straight line. Equation 95 with $f_t = 0$ is intended to approximate the lateral force distribution in short-period buildings (funda-

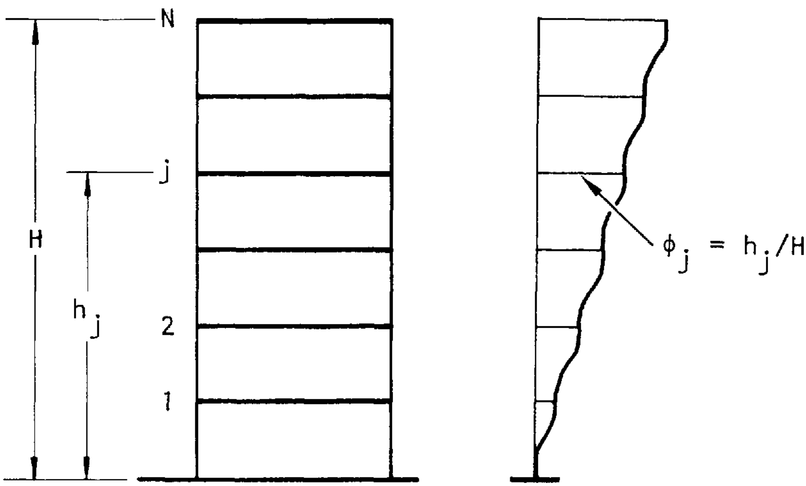


Figure 43. Floor displacements in fundamental mode of vibration increasing linearly with height

mental period of 0.7 sec or less). Although earthquake response of long-period buildings is primarily due to the fundamental mode of vibration, the influence of higher modes can be significant in the forces in the upper part of buildings. Assigning 25% of the base shear as an extra force at the top of the building and distributing the remaining 75% of the base shear in accordance with Eq. 95 are intended by the code to approximate the lateral force distribution in long-period buildings (fundamental period exceeding 3.6 secs). In the intermediate period range, linear increase of f_t with period T provides for increasing forces in the upper parts of the building to account for the increasing contributions of higher vibration modes as period increases.

If the effective weight W_1 for the fundamental mode is replaced with the total weight W, Eq. 97 will become identical with Eq. 94 provided the same value of the fundamental period is used to determine S_{a1} and C and the base shear coefficients S_{a1}/g and KC in the two equations have the same values. However, W_1 will always be smaller than W, with typical values for W_1 between 60 to 90% of W, depending on the distribution of weight over the height of the building and the shape of the first vibration mode. Eq. 94 would therefore provide a value for the base shear that is larger than the first mode value, even if S_{a1}/g and KC are the same; thus it indirectly and approximately accounts for the contributions of the higher modes of vibration to the base shear.

The base shear coefficient in the fundamental mode response (Eq. 97) is the pseudo-acceleration response spectrum, normalized with respect to acceleration of gravity, for the ground motion. In the code formula the base shear coefficient is the product of K and C. The two base shear coefficients are compared in Fig. 44. The pseudo-acceleration response spectrum presented is a smooth spectrum representative of ground motions similar in intensity and frequency characteristics to the El Centro ground motion (Fig. 17). The code base shear coefficient is presented as a function of fundamental period for three values of K = 0.67, 1.0 and 1.33. While the two base shear coefficients, S_{a1}/g and KC, vary similarly with vibration period, they differ greatly in value. Depending on the value of K and T, response spectrum ordinates are 3 to 6 times larger than the code coeffi-

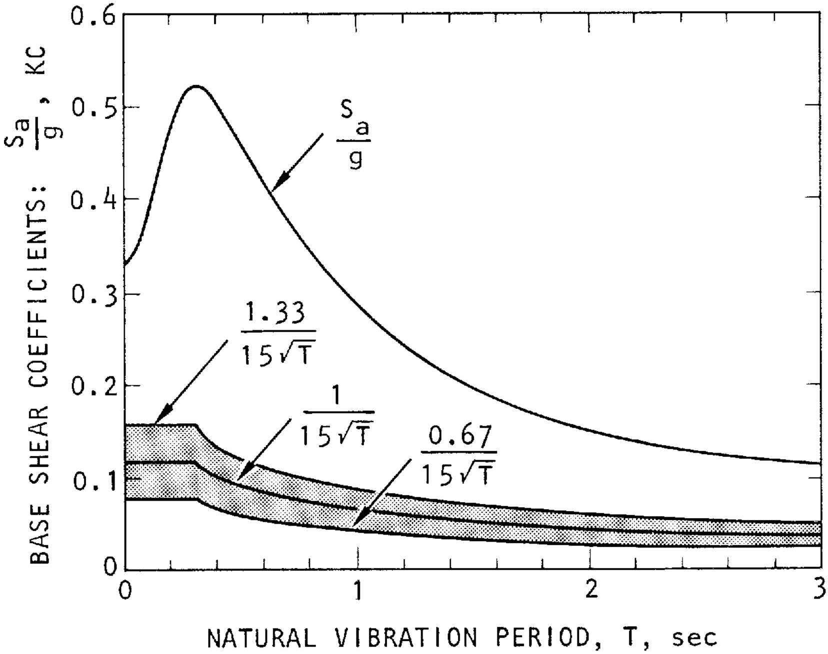
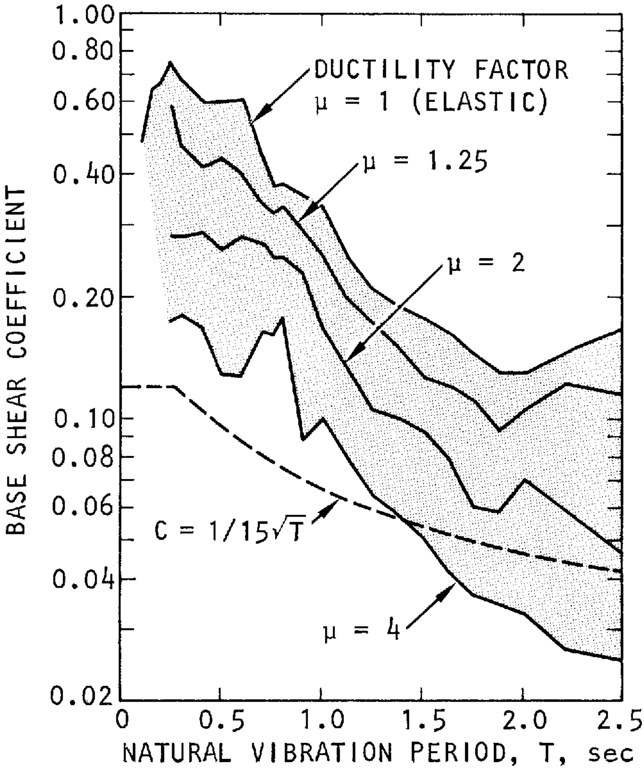


Figure 44. Comparison of base shear coefficients from (1) response spectrum for elastic systems (damping ratio = 5%) and (2) Uniform Building Code

Courtesy of G. W. Housner

During the 9 February 1971 San Fernando earthquake the new Olive View Hospital building was severely damaged. In response to very strong ground shaking, the building vibrated essentially as a large mass on relatively flexible columns. The spirally reinforced concrete columns were deformed far beyond the requirements of the building code.



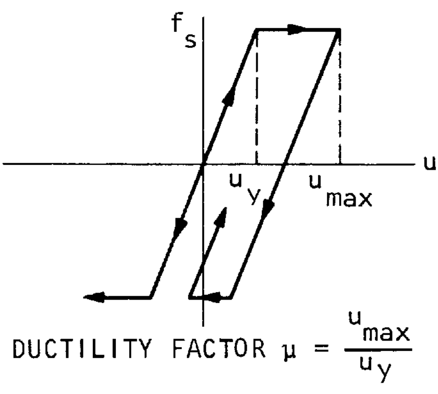


Figure 45. Base shear coefficients for 10% damped elasto-plastic systems subjected to El Centro (S00°E component) ground motion (after Veletsos and Newmark, 1960) compared with code values

cient. Thus, buildings designed to resist the code forces at working stress levels would experience yield stresses and deform beyond their linear range of behavior when subjected to intense ground shaking.

The important effects of yielding on the earthquake response of buildings can be readily displayed by considering the behavior of one-story structures with elastoplastic force-deformation relation. Response spectra showing the base shear coefficient (= base shear ÷ weight) as a function of natural vibration period (corresponding to stiffness in the linear range) for such systems are shown in Fig. 45 for values of ductility factor of 1.0 (corresponding to a linearly elastic structure), 1.25, 2 and 4 (Veletsos and Newmark, 1960). The effect of yielding is to reduce the value of the base shear coefficient below that for elastic behavior. Even a relatively small amount of yielding produces appreciable reductions in the value of the base shear coefficient.

Also included in Fig. 45 is the base shear coefficient in the UBC formula (with K = 1). It is apparent that the force requirements of the building code are adequate provided the structure is capable of developing enough ductility. For satisfactory performance during the ground motion considered in Fig. 45, long-period structures should have a ductility capacity of approximately 4, and short-period structures should possess even larger ductility capacity.

MODAL ANALYSIS BASED ON INELASTIC DESIGN SPECTRA

As seen in the preceding section, buildings designed for code forces are expected to deform significantly beyond the yield limit during moderate to very intense ground shaking. Strictly speaking, the modal method, which is applicable only to analysis of linear response, therefore cannot be used for calculation of the design forces for buildings. However, it is believed that for many buildings satisfactory approximations to the design forces and deformations can be obtained from the modal method by using the design spectrum for inelastic systems instead of the elastic response spectrum.

Based on results of the type presented in Fig. 45, procedures have been developed for constructing an inelastic design spectrum from the elastic design spectrum and the ductility factor considered allowable in the design of the building (Veletsos and Newmark, 1960; Veletsos, Newmark and Chelapati, 1965; Veletsos, 1969; Newmark and Hall, 1976).

The procedure to develop inelastic design spectra, starting with the estimated values of maximum acceleration, velocity, and displacement of the ground motion, and the application of modal analysis procedures to yielding structures have been summarized (Chopra and Newmark, 1980). The limitations of this approach to analysis of yielding structures were also discussed.

The recommendations prepared by the Applied Technology Council, California, for earthquake-resistant design of buildings were based on the above mentioned concepts (ATC, 1978). Inelastic design spectra were constructed by a simplified version of the above mentioned procedures. Based on the approximation that effects of yielding can be accounted for by linear analysis of the building using the inelastic design spectra, two methods of analysis were included: The modal analysis procedure and a simpler method referred to as the equivalent lateral force procedure.

References

Applied Technology Council, *Tentative Provisions for the Development of Seismic Regulations for Buildings*, NBS Special Publication 510, U.S. Government Printing Office, Washington, D.C., 1978.

Bathè, K-J. and E. L. Wilson, *Numerical Methods in Finite Element Analysis*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1976.

Chopra, A. K. and N. M. Newmark, "Analysis," Chapter 2 in *Design of Earthquake Resistant Structures* (E. Rosenblueth, ed.), Pentech Press Ltd., London, 1980.

Clough, R. W. and J. Penzien, *Dynamics of Structures*, McGraw-Hill Book Company, New York, 1975.

Foutch, D. A., G. W. Housner and P. C. Jennings, *Dynamic Responses of Six Multistory Buildings During the San Fernando Earthquake,* Report No. EERL 75-02, Earthquake Engineering Research Laboratory, California Institute of Technology, Pasadena, California, 1975.

Hudson, D. E., "Dynamic Tests of Full Scale Structures," in *Earthquake Engineering* (R. L. Wiegel, ed.), Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1970.

Hudson, D. E., *Reading and Interpreting Strong Motion Accelerograms*, Earthquake Engineering Research Institute, Berkeley, California, 1979.

Jennings, P. C., R. B. Matthiesen and J. B. Hoerner, "Forced Vibration of a Tall Steel-Frame Building," *Earthquake Engineering and Structural Dynamics*, Vol. 1, No. 2, pp. 107–132, 1972.

Kan, C. L. and A. K. Chopra, "Elastic Earthquake Analysis of Torsionally Coupled Buildings," *Earthquake Engineering and Structural Dynamics*, Vol. 5, No. 4, pp. 395–412, 1977.

McVerry, G. H., *Frequency Domain Identification of Structural Models from Earthquake Records*, Report No. EERL 79-02, Earthquake Engineering Research Laboratory, California Institute of Technology, Pasadena, California, 1979.

Newmark, N. M. and E. Rosenblueth, *Fundamentals of Earthquake Engineering*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

Newmark, N. M. and W. J. Hall, "Vibration of Structures Induced by Ground Motion," Chapter 29, Part I in *Shock and Vibration Handbook* (C. M. Harris and C. E. Crede, eds.), McGraw-Hill Book Company, New York, 1976.

Thomson, W. T., *Vibration Theory and Applications*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965.

Uniform Building Code—1979 Edition, International Conference of Building Officials, Whittier, California.

Veletsos, A. S. and N. M. Newmark, "Effect of Inelastic Behavior on the Response of Simple Systems to Earthquake Motions," *Proceedings of the Second World Conference on Earthquake Engineering*, Vol. II, pp. 895-912, Tokyo, Japan, 1960.

Veletsos, A. S., N. M. Newmark and C. V. Chelapati, "Deformation Spectra for Elastic and Elastoplastic Systems Subjected to Ground Shock and Earthquake Motions," *Proceedings of the Third World Conference on Earthquake Engineering*, Vol. II, pp. 663-682, New Zealand, 1965.

Veletsos, A. S., "Maximum Deformations of Certain Nonlinear Systems," *Proceedings of the Fourth World Conference on Earthquake Engineering*, Vol. 1, pp. 155-170, Santiago, Chile, 1969.

Notation

c = damping coefficient of a one-story structure

$\mathbf{c}$ = damping matrix of a multistory building

D = dimensionless response factor for a one-story building

D_j = dimensionless factor for response at the jth floor of a multistory building

f = natural cyclic frequency of vibration of an undamped one-story structure

f_D = natural cyclic frequency of vibration of a damped one-story structure

f_D = damping force in a one-story structure

f_I = inertia force in a one-story structure

f_S = elastic resisting force (or equivalent lateral force) in a one-story structure

f_{Dj} = damping force at the jth floor of a multistory building

f_{Ij} = inertia force at the jth floor of a multistory building

f_{Sj} = elastic resisting force at the jth floor of a multistory building

$f_{jn}(t)$ = contribution of nth mode to equivalent lateral force at jth floor

$\bar{f}_{jn}$ = maximum value of $f_{jn}(t)$; algebraic sign of $\bar{f}_{jn}$ is controlled by ϕ_{jn}

$\mathbf{f}_n(t)$ = vector of elements $f_{jn}(t)$, $j = 1, 2, \ldots N$.

$f_j(t)$ = total equivalent lateral force at jth floor g = acceleration of gravity

h = height of roof of one-story structure above its base

h_j = height of the jth floor above base

h_n = effective height in nth vibration mode of a multistory building

k = lateral stiffness of a one-story structure

k_j = lateral stiffness of the jth story of a multistory building

k = stiffness matrix of a multistory building

$L_n = \sum_{j=1}^{N} m_j \phi_{jn}$

m = mass of one-story structure

m = mass matrix of a multistory building

M_n = nth modal mass

$M_0(t)$ = base moment

$M_{0,max}$ = maximum value of the base moment $M_0(t)$ in a one-story structure without regard to algebraic sign

$\bar{M}_0$ = maximum value of the base moment $M_0(t)$ in a multistory building without regard to algebraic sign

$M_{0n}(t)$ = contribution of nth mode to moment at base of multistory building

$\bar{M}_{0n}$ = maximum value of $M_{0n}(t)$ without regard to algebraic sign

N = total number of floors in a multistory building

p(t) = external dynamic force on a one-story structure

p(t) = vector of external dynamic forces

$p_j(t)$ = external dynamic force at the jth floor of a multistory building

p_0 = amplitude or maximum value of external harmonic force

p_{0j} = amplitude of external harmonic force at the jth floor

$r_n(t)$ = contribution of nth mode to response $r(t)$

S_a = ordinate of the pseudo-acceleration response spectrum

S_d = ordinate of the deformation (or displacement) response spectrum

S_v = ordinate of the pseudo-velocity response spectrum

S_{dn} = ordinate of the deformation response spectrum corresponding to T_n (or ω_n) and ξ_n

S_{vn} = ordinate of the pseudo-velocity response spectrum corresponding to T_n (or ω_n) and ξ_n

S_{an} = ordinate of the pseudo-acceleration response spectrum corresponding to T_n (or ω_n) and ξ_n

t = time

T = natural period of vibration of an undamped one-story structure

T_D = natural period of vibration of a damped one-story structure

T_n = nth natural period of vibration of an undamped multistory building

T_1 = first or fundamental natural period of vibration of an undamped multistory building

T_{nD} = nth natural period of vibration of a damped multistory building

$\bar{T}$ = forcing period *or* period of harmonic load

u = interfloor (or relative) displacement of a one-story structure

u^t = total displacement of the roof of a one-story structure

$\dot{u}$ = interfloor or relative velocity of a one-story structure

$\ddot{u}$ = acceleration of mass in a one-story structure

$\ddot{u}^t$ = total acceleration of the mass of a one-story structure

$u(0)$ = initial displacement of a one-story structure

$\dot{u}(0)$ = initial velocity of a one-story structure

u_j = displacement of the jth floor of a multistory building relative to the ground

$u_j(0)$ = initial displacement of the jth floor of a multistory building

$\dot{u}_j(0)$ = initial velocity of the jth floor of a multistory building

$\mathbf{u}$ = vector of displacements u_j (j=1,2, . . . N) in a multistory building

$\dot{\mathbf{u}}$ = vector of velocities in a multistory building

$\ddot{u}_j$ = acceleration of the jth floor of a multistory building relative to the ground

$\ddot{\mathbf{u}}$ = vector of accelerations in a multistory building

$\ddot{u}_j^t$ = total acceleration of the jth floor of a multistory building

$u_{jn}(t)$ = contribution of nth mode to displacement at the jth floor

$\bar{u}_{jn}$ = maximum value of $u_{jn}(t)$; algebraic sign of $\bar{u}_{jn}$ is controlled by ϕ_{jn}

u_{st} = static displacement in a one-story structure due to load p_0

$u_{j,st}$ = static displacement at jth floor of a multistory building due to external forces p_{0j}

u_{max} = maximum value of u(t) without regards to algebraic sign

$u_g(t)$ = ground displacement

$\dot{u}_g(t)$ = ground velocity

$\ddot{u}_g(t)$ = ground acceleration

$V_0(t)$ = base shear

$V_{0,max}$ = maximum value of the base shear $V_0(t)$ in a one-story structure without regards to algebraic sign

$\bar{V}_0$ = maximum value of the base shear $V_0(t)$ in a multistory building without regards to algebraic sign

w = weight of the one-story structure

w_j = weight lumped at jth floor of a multistory building

W_n = effective weight in nth vibration mode of a multistory building

W = effective weight corresponding to assumed shape ψ (Eq. 86)

$Y_n(t)$ = nth modal displacement

$\bar{Y}_n$ = maximum value of $Y_n(t)$ without regards to algebraic sign

$\mathbf{1}$ = vector with each element equal to unity

β = $\bar{\omega}/\omega$ = ratio of the forcing frequency to the natural frequency

δ = logarithmic decrement

$\Delta_{jn}(t)$ = contribution of nth mode to deformation or drift in story j

$\bar{\Delta}_{jn}$ = maximum value of $\Delta_{jn}(t)$ without regards to algebraic sign

θ = phase angle or angular phase shift

ξ = damping ratio *or* fraction of critical damping coefficient

ξ_n = damping ratio in the nth natural mode of vibration of a multistory building

ϕ_n = nth natural mode of vibration of a multistory building

ϕ_{jn} = jth element of ϕ_n

ϕ_1 = first or fundamental natural mode of vibration of a multistory building

ψ_j = jth element of ψ

ψ = estimated mode shape; an approximation to ϕ_1

ω = natural circular frequency of vibration of an undamped one-story structure

ω = estimate of the fundamental frequency of multistory building

ω_D = natural circular frequency of vibration of a damped one-story structure

ω_n = nth natural circular frequency of vibration of an undamped multistory building

ω_{nD} = nth natural circular frequency of vibration of a damped multistory building

$\bar{\omega}$ = forcing frequency *or* circular frequency of harmonic load